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ON THE VERSCHIEBUNG MORPHISM FOR $\mathscr{G}^{(\lambda)}$

By

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Introduction

In the theory of algebraic and formal groups over a field of positive characteristic, the Frobenius morphisms and the Verschiebung morphisms play very important role. The definition of the Frobenius is not difficult, but it is somewhat subtle to define the Verschiebung morphism. Roughly speaking, the Verschiebung is dual to the Frobenius. In fact, the Frobenius and the Verschiebung change the places in the Cartier duality for finite flat commutative group schemes.

In this paper, we consider the Verschiebung morphism for the group scheme

$$\mathscr{G}_A^{(\lambda)} = \mathrm{Spec}\, A[T, 1/(1+\lambda T)]$$

and

$$\hat{\mathscr{G}}_A^{(\lambda)} = \mathrm{Spf}\, A[[T]]$$

the formal completion of $\mathscr{G}_A^{(\lambda)}$ along the unit section (For the definition, see the section 1).

The main theorem of this paper is stated as follows:

THEOREM (Th. 2.1 and Cor. 2.1.1). *Let A be an $\boldsymbol{F}_p$-algebra and $\lambda \in A$. Then the Verschiebung morphism*

$$V : \mathscr{G}_A^{(\lambda^p)} = \mathrm{Spec}\, A[T, 1/(1+\lambda^p T)] \to \mathscr{G}_A^{(\lambda)} = \mathrm{Spec}\, A[T, 1/(1+\lambda T)]$$

$$(\text{resp. } V : \hat{\mathscr{G}}_A^{(\lambda^p)} = \mathrm{Spf}\, A[[T]] \to \hat{\mathscr{G}}_A^{(\lambda)} = \mathrm{Spf}\, A[[T]])$$

is given by the homomorphism

$$T \mapsto \lambda^{p-1} T : A[T, 1/(1+\lambda T)] \to A[T, 1/(1+\lambda^p T)]$$

$$(\text{resp. } T \mapsto \lambda^{p-1} T : A[[T]] \to A[[T]]).$$

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Furthermore, we verify a duality between the Frobenius morphism and the Verschiebung morphism. Sekiguchi and Suwa [3] constructed the isomorphisms

$$\begin{aligned} &\mathrm{Ker}[F^{(\mu)} - [\lambda^{p-1}] : W^{(\mu)}(A) \to W^{(\mu)}(A)] \xrightarrow{\sim} \mathrm{Hom}_{A\text{-gr}}(\hat{\mathscr{G}}_A^{(\lambda)}, \hat{\mathscr{G}}_A^{(\mu)}), \\ &\mathrm{Coker}[F^{(\mu)} - [\lambda^{p-1}] : W^{(\mu)}(A) \to W^{(\mu)}(A)] \xrightarrow{\sim} H_0^2(\hat{\mathscr{G}}_A^{(\lambda)}, \hat{\mathscr{G}}_A^{(\mu)}), \end{aligned} \tag{1}$$

where $\lambda, \mu \in A$. Moreover, if λ is nilpotent, [3] established also the isomorphisms

$$\begin{aligned} &\mathrm{Ker}[F^{(\mu)} - [\lambda^{p-1}] : \hat{W}^{(\mu)}(A) \to \hat{W}^{(\mu)}(A)] \xrightarrow{\sim} \mathrm{Hom}_{A\text{-gr}}(\mathscr{G}_A^{(\lambda)}, \mathscr{G}_A^{(\mu)}), \\ &\mathrm{Coker}[F^{(\mu)} - [\lambda^{p-1}] : \hat{W}^{(\mu)}(A) \to \hat{W}^{(\mu)}(A)] \xrightarrow{\sim} H_0^2(\mathscr{G}_A^{(\lambda)}, \mathscr{G}_A^{(\mu)}). \end{aligned} \tag{2}$$

Here $H_0^2(\hat{\mathscr{G}}_A^{(\lambda)}, \hat{\mathscr{G}}_A^{(\mu)})$ and $H_0^2(\mathscr{G}_A^{(\lambda)}, \mathscr{G}_A^{(\mu)})$ denote the Hochschild cohomology.

The second result is stated as follows:

THEOREM (Th. 3.6 and Cor. 3.7). *Let A be an $\mathbf{F}_p$-algebra and $\lambda, \mu \in A$.*

(i) *Under the identification* (1), *the homomorphisms*

$$V^* : \mathrm{Hom}_{A\text{-gr}}(\hat{\mathscr{G}}_A^{(\lambda)}, \hat{\mathscr{G}}_A^{(\mu)}) \to \mathrm{Hom}_{A\text{-gr}}(\hat{\mathscr{G}}_A^{(\lambda^p)}, \hat{\mathscr{G}}_A^{(\mu)}),$$

$$V^* : H_0^2(\hat{\mathscr{G}}_A^{(\lambda)}, \hat{\mathscr{G}}_A^{(\mu)}) \to H_0^2(\hat{\mathscr{G}}_A^{(\lambda^p)}, \hat{\mathscr{G}}_A^{(\mu)})$$

$$(\text{resp. } F^* : \mathrm{Hom}_{A\text{-gr}}(\hat{\mathscr{G}}_A^{(\lambda^p)}, \hat{\mathscr{G}}_A^{(\mu)}) \to \mathrm{Hom}_{A\text{-gr}}(\hat{\mathscr{G}}_A^{(\lambda)}, \hat{\mathscr{G}}_A^{(\mu)}),$$

$$F^* : H_0^2(\hat{\mathscr{G}}_A^{(\lambda^p)}, \hat{\mathscr{G}}_A^{(\mu)}) \to H_0^2(\hat{\mathscr{G}}_A^{(\lambda)}, \hat{\mathscr{G}}_A^{(\mu)}))$$

are given by $\boldsymbol{a} \mapsto F^{(\mu)}\boldsymbol{a}$ (resp. $\boldsymbol{a} \mapsto V\boldsymbol{a}$),

(ii) *Assume that λ is nilpotent in A. Under the identification* (2), *the homomorphisms*

$$V^* : \mathrm{Hom}_{A\text{-gr}}(\mathscr{G}_A^{(\lambda)}, \mathscr{G}_A^{(\mu)}) \to \mathrm{Hom}_{A\text{-gr}}(\mathscr{G}_A^{(\lambda^p)}, \mathscr{G}_A^{(\mu)}),$$

$$V^* : H_0^2(\mathscr{G}_A^{(\lambda)}, \mathscr{G}_A^{(\mu)}) \to H_0^2(\mathscr{G}_A^{(\lambda^p)}, \mathscr{G}_A^{(\mu)})$$

$$(\text{resp. } F^* : \mathrm{Hom}_{A\text{-gr}}(\mathscr{G}_A^{(\lambda^p)}, \mathscr{G}_A^{(\mu)}) \to \mathrm{Hom}_{A\text{-gr}}(\mathscr{G}_A^{(\lambda)}, \mathscr{G}_A^{(\mu)}),$$

$$F^* : H_0^2(\mathscr{G}_A^{(\lambda^p)}, \mathscr{G}_A^{(\mu)}) \to H_0^2(\mathscr{G}_A^{(\lambda)}, \mathscr{G}_A^{(\mu)}))$$

are given by $\boldsymbol{a} \mapsto F^{(\mu)}\boldsymbol{a}$ (resp. $\boldsymbol{a} \mapsto V\boldsymbol{a}$).

In the section 1, we recall the definition of the Verschiebung morphism for an affine commutative group scheme over a ring of characteristic $p > 0$, and the main theorem is proved in the section 2. After reviewing some result on extensions of group schemes by Sekiguchi and Suwa [3], we state our second theorem in the section 3. A proof of the second theorem is given in the section 4.

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NOTATIONS

Throughout this paper, p denotes a fixed prime number.

$\boldsymbol{G}_{a,A}$ denotes the additive group scheme over A.

$\boldsymbol{G}_{m,A}$ denotes the multiplicative group scheme over A.

For any group scheme $G, \hat{G}$ denotes the formal completion of G along the unit section.

$H_0^2(G,H) = Z^2(G,H)/B^2(G,H)$ denotes the Hochschild cohomology group consisting of symmetric 2-cocycles of G with coefficients in a trivial G-module H.

If A is an $\boldsymbol{F}_p$-algebra, we define a ring homomorphism $\sigma : A \to A$ by $a \mapsto a^p$. We also denote by σ the morphism $\operatorname{Spec} A \to \operatorname{Spec} A$ induced by $\sigma : A \to A$. If X is a scheme over A, we have the following cartesian diagram:

$$\begin{array}{ccc} X \times_\sigma \operatorname{Spec} A & \longrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} A & \xrightarrow{\sigma} & \operatorname{Spec} A. \end{array}$$

Therefore, we define a scheme $X^{(p)}$ over A by $X^{(p)} = X \times_\sigma \operatorname{Spec} A$. If $X = \operatorname{Spec} R$, we have $X^{(p)} = \operatorname{Spec}(R \otimes_\sigma A)$. Here $R \otimes_\sigma A$ is a commutative ring satisfying $(\alpha x) \otimes_\sigma a = x \otimes_\sigma (\alpha^p a)$ for all $\alpha, a \in A$, $x \in R$. Moreover, $R \otimes_\sigma A$ is an A-algebra with the ring homomorphism

$$a \mapsto 1 \otimes_\sigma a : A \to R \otimes_\sigma A.$$

1. Some Preparations

In this section, we review the definition of the Verschiebung morphism for affine group schemes (cf. Demazure and Gabriel [1, IV, §3, n°4]) and that of the group scheme $\mathscr{G}_A^{(\lambda)}$.

1.1. Let A be an $\boldsymbol{F}_p$-algebra and $G = \operatorname{Spec} R$ an affine flat commutative group scheme over A. We denote by Δ the comultiplication of R.

The A-module $R^{\otimes n}$ consisting of the tensors of rank n is an A-algebra with the multiplication of each component. We denote by $S^n R$ the sub-A-algebra consisting of symmetric tensors of rank n.

We define a homomorphism $s_n : R^{\otimes n} \to S^n R$ of A-modules by

$$x_1 \otimes \cdots \otimes x_n \mapsto \sum_{\tau \in \mathfrak{S}_n} x_{\tau(1)} \otimes \cdots \otimes x_{\tau(n)},$$

where $\mathfrak{S}_n$ denotes the symmetric group of degree n. Then $s_n(R^{\otimes n})$ is an ideal of $S^n R$. Especially, consider the case of $n = p$. Then the correspondence

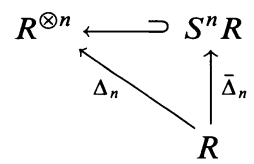
$$x \otimes_\sigma a \mapsto [a(x \otimes \cdots \otimes x)] : R \otimes_\sigma A \to S^p R / s_p(R^{\otimes p})$$

is an A-isomorphism ([1, IV, §3, 4.1]). We denote by i_R the surjection $S^p R \to R \otimes_\sigma A$ induced by the isomorphism $R \otimes_\sigma A \simeq S^p R / s_p(R^{\otimes p})$.

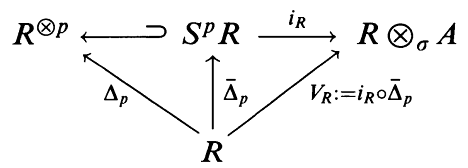
For any $n \geq 2$, we define an A-homomorphism $\Delta_n : R \to R^{\otimes n}$ inductively by

$$\Delta_2 = \Delta,$$
$$\Delta_{n+1} = (\Delta \otimes \mathrm{id}_R^{\otimes(n-1)}) \circ \Delta_n.$$

Since G is a commutative group scheme, Δ_n passes through $S^n R$, that is, we have the following commutative diagram:



Consider the case of $n = p$. Putting $V_R = i_R \circ \bar{\Delta}_p$, we have the following commutative diagram:



Put $G^p = \mathrm{Spec}(R^{\otimes p})$, $S^p G = \mathrm{Spec}(S^p R)$ and $G^{(p)} = \mathrm{Spec}(R \otimes_\sigma A)$. We denote by V_G the homomorphism $G^{(p)} \to G$ of group schemes defined by V_R, which is called the Verschiebung morphism for G.

EXAMPLE 1.1.1. (i) Consider the multiplicative group scheme $\boldsymbol{G}_{m,A} = \mathrm{Spec}\, A[T, 1/T]$. Put $R = A[T, 1/T]$. Since Δ_p is given by $T \mapsto T^{\otimes p}$, we have

$$V_R(T) = i_R \circ \bar{\Delta}_p(T) = i_R(T^{\otimes p}) = T \otimes_\sigma 1.$$

Hence $V_{\boldsymbol{G}_{m,A}} = \mathrm{id}_{\boldsymbol{G}_{m,A}}$, identifying R to $R \otimes_\sigma A$ by $aT \mapsto T \otimes_\sigma a$.

(ii) Consider the additive group scheme $\boldsymbol{G}_{a,A} = \operatorname{Spec} A[T]$. Put $R = A[T]$. Then Δ_p is given by

$$T \mapsto \sum_{i=1}^{p} 1 \otimes \cdots \otimes 1 \otimes \overset{i}{\check{T}} \otimes 1 \otimes \cdots \otimes 1.$$

Noting that

$$\sum_{i=1}^{p} 1 \otimes \cdots \otimes 1 \otimes \overset{i}{\check{T}} \otimes 1 \otimes \cdots \otimes 1 = \frac{1}{(p-1)!} s_p(T \otimes 1 \otimes \cdots \otimes 1)$$

$$\equiv -s_p(T \otimes 1 \otimes \cdots \otimes 1) \mod p,$$

we have

$$V_R(T) = i_R \circ \overline{\Delta}_p(T) = i_R(-s_p(T \otimes 1 \otimes \cdots \otimes 1)) = 0.$$

Hence $V_{\boldsymbol{G}_{a,A}} = 0$, identifying R to $R \otimes_\sigma A$ by $aT \mapsto T \otimes_\sigma a$.

1.2. Let A be a ring and $\lambda \in A$. We define an affine flat commutative group scheme $\mathscr{G}_A^{(\lambda)}$ over A by $\mathscr{G}_A^{(\lambda)} = \operatorname{Spec} A[T, 1/(1+\lambda T)]$ with

(i) the comultiplication: $\Delta : T \mapsto \lambda T \otimes T + T \otimes 1 + 1 \otimes T$,

(ii) the counit: $\varepsilon : T \mapsto 0$,

(iii) the coinverse: $S : T \mapsto -T/(1+\lambda T)$.

If λ is invertible in A, the correspondence

$$U \mapsto 1 + \lambda T : A[U, 1/U] \to A[T, 1/(1+\lambda T)]$$

is an isomorphism of A-algebras, and hence $\mathscr{G}_A^{(\lambda)}$ is isomorphic to the multiplicative group scheme $\boldsymbol{G}_{m,A}$. On the other hand, if $\lambda = 0$, $\mathscr{G}_A^{(\lambda)}$ is nothing but the additive group scheme $\boldsymbol{G}_{a,A}$.

Let A be an $\boldsymbol{F}_p$-algebra and $\lambda \in A$. Since the correspondence $aT \mapsto T \otimes_\sigma a$ gives rise to an A-isomorphism

$$A[T, 1/(1+\lambda^p T)] \xrightarrow{\sim} A[T, 1/(1+\lambda T)] \otimes_\sigma A,$$

we have the isomorphism of group A-schemes

$$(\mathscr{G}_A^{(\lambda)})^{(p)} = \operatorname{Spec}(A[T, 1/(1+\lambda T)] \otimes_\sigma A) \xrightarrow{\sim} \mathscr{G}_A^{(\lambda^p)} = \operatorname{Spec} A[T, 1/(1+\lambda^p T)].$$

Moreover, the Frobenius morphism

$$F : \mathscr{G}_A^{(\lambda)} = \operatorname{Spec} A[T, 1/(1+\lambda T)] \to \mathscr{G}_A^{(\lambda^p)} = \operatorname{Spec} A[T, 1/(1+\lambda^p T)]$$

is given by

$$T \mapsto T^p : A[T, 1/(1+\lambda^p T)] \to A[T, 1/(1+\lambda T)].$$

2. The Verschiebung Morphism for $\mathscr{G}_A^{(\lambda)}$

THEOREM 2.1. *Let A be an $\boldsymbol{F}_p$-algebra and $\lambda \in A$. Then the Verschiebung morphism*

$$V : \mathscr{G}_A^{(\lambda^p)} = \operatorname{Spec} A[T, 1/(1+\lambda^p T)] \to \mathscr{G}_A^{(\lambda)} = \operatorname{Spec} A[T, 1/(1+\lambda T)]$$

is given by $T \mapsto \lambda^{p-1} T : A[T, 1/(1+\lambda T)] \to A[T, 1/(1+\lambda^p T)]$.

COROLLARY 2.1.1. *Let A be an $\boldsymbol{F}_p$-algebra and $\lambda \in A$. Then the Verschiebung morphism*

$$V : \hat{\mathscr{G}}_A^{(\lambda^p)} = \operatorname{Spf} A[[T]] \to \hat{\mathscr{G}}_A^{(\lambda)} = \operatorname{Spf} A[[T]]$$

is given by $T \mapsto \lambda^{p-1} T : A[[T]] \to A[[T]]$.

REMARK 2.2. Applying 2.1 to $\lambda = 1$ (resp. $\lambda = 0$), we obtain immediately $V_{\boldsymbol{G}_{m,A}} = \mathrm{id}_{\boldsymbol{G}_{m,A}}$ (resp. $V_{\boldsymbol{G}_{a,A}} = 0$).

Hereafter we prove the theorem. We may assume that $A = \boldsymbol{F}_p[\Lambda]$ and $\lambda = \Lambda$.

LEMMA 2.3. *Let $R = \boldsymbol{F}_p[\Lambda, T, 1/(1+\Lambda T)]$. Then we have equalities*

$$(\Delta \otimes \mathrm{id}_R^{\otimes(n-1)})\left(\frac{s_n(T \otimes 1^{\otimes(n-1)})}{(n-1)!}\right)$$
$$= \frac{s_{n+1}(T \otimes 1^{\otimes n})}{n!} + \Lambda \frac{T \otimes T \otimes s_{n-1}(T \otimes 1^{\otimes(n-1)})}{(n-1)!}$$

in $R^{\otimes(n+1)}$ for all n with $2 \le n \le p-1$. Moreover, we have equalities

$$(\Delta \otimes \mathrm{id}_R^{\otimes(n-1)})\left(\Lambda^{i-1} \frac{s_n(T^{\otimes i} \otimes 1^{\otimes(n-i)})}{i!(n-i)!}\right)$$
$$= \Lambda^{i-1}\left\{\frac{s_{n+1}(T^{\otimes i} \otimes 1^{\otimes(n-i+1)})}{i!(n-i+1)!} - \frac{T \otimes T \otimes s_{n-1}(T^{\otimes(i-2)} \otimes 1^{\otimes(n-i+1)})}{(i-2)!(n-i+1)!}\right\}$$
$$+ \Lambda^i \frac{T \otimes T \otimes s_{n-1}(T^{\otimes(i-1)} \otimes 1^{\otimes(n-i)})}{(i-1)!(n-i)!}$$

in $R^{\otimes(n+1)}$ for all n with $2 \le n \le p-1$ and all i with $2 \le i \le n$.

PROOF. We have

$$(\Delta \otimes \mathrm{id}_R^{\otimes(n-1)})\left(\Lambda^{i-1}\frac{s_n(T^{\otimes i} \otimes 1^{\otimes(n-i)})}{i!(n-i)!}\right)$$

$$= i\Lambda^{i-1}\frac{(\Lambda T \otimes T + T \otimes 1 + 1 \otimes T) \otimes s_{n-1}(T^{\otimes(i-1)} \otimes 1^{\otimes(n-i)})}{i!(n-i)!}$$

$$+ (n-i)\Lambda^{i-1}\frac{1 \otimes 1 \otimes s_{n-1}(T^{\otimes i} \otimes 1^{\otimes(n-i-1)})}{i!(n-i)!}$$

$$= \Lambda^{i-1}\left[\frac{(T \otimes 1 + 1 \otimes T) \otimes s_{n-1}(T^{\otimes(i-1)} \otimes 1^{\otimes(n-i)})}{(i-1)!(n-i)!} + \frac{1 \otimes 1 \otimes s_{n-1}(T^{\otimes i} \otimes 1^{\otimes(n-i-1)})}{i!(n-i-1)!}\right]$$

$$+ \Lambda^{i}\frac{T \otimes T \otimes s_{n-1}(T^{\otimes(i-1)} \otimes 1^{\otimes(n-i)})}{(i-1)!(n-i)!}$$

for all i with $1 \le i \le n$. Substituting 1 for i, we easily obtain the first formula. If $i \ge 2$, noting that

$$\frac{s_{n+1}(T^{\otimes i} \otimes 1^{\otimes(n-i+1)})}{i!(n-i+1)!} = \frac{T \otimes T \otimes s_{n-1}(T^{\otimes(i-2)} \otimes 1^{\otimes(n-i+1)})}{(i-2)!(n-i+1)!} + \frac{(T \otimes 1 + 1 \otimes T) \otimes s_{n-1}(T^{\otimes(i-1)} \otimes 1^{\otimes(n-i)})}{(i-1)!(n-i)!} + \frac{1 \otimes 1 \otimes s_{n-1}(T^{\otimes i} \otimes 1^{\otimes(n-i-1)})}{i!(n-i-1)!},$$

we obtain the second formula. □

LEMMA 2.4. *Let $R = F_p[\Lambda, T, 1/(1+\Lambda T)]$. Then we have equalities*

$$\Delta_n(T) = \sum_{i=1}^{n-1}\Lambda^{i-1}\frac{s_n(T^{\otimes i} \otimes 1^{\otimes(n-i)})}{i!(n-i)!} + \Lambda^{n-1}T^{\otimes n}$$

in $R^{\otimes n}$ for all n with $2 \le n \le p$.

PROOF. We prove the assertion by induction on n. It is easily seen that the case for $n = 2$. Assume that the case for n $(\le p-1)$. Since

$$\Lambda^{n-1}T^{\otimes n} = \Lambda^{n-1}\frac{s_n(T^{\otimes n})}{n!}$$

for all n with $2 \leq n \leq p-1$, we have

$$\Delta_n(T) = \sum_{i=1}^{n} \Lambda^{i-1} \frac{s_n(T^{\otimes i} \otimes 1^{\otimes(n-i)})}{i!(n-i)!}.$$

Then, by 2.3,

$$\begin{aligned}
\Delta_{n+1}(T) &= (\Delta \otimes \mathrm{id}_R^{\otimes(n-1)}) \left(\sum_{i=1}^{n} \Lambda^{i-1} \frac{s_n(T^{\otimes i} \otimes 1^{\otimes(n-i)})}{i!(n-i)!} \right) \\
&= \frac{s_{n+1}(T \otimes 1^{\otimes n})}{n!} + \Lambda \frac{T \otimes T \otimes s_{n-1}(T \otimes 1^{\otimes(n-1)})}{(n-1)!} \\
&\quad + \sum_{i=2}^{n} \left[\Lambda^{i-1} \left\{ \frac{s_{n+1}(T^{\otimes i} \otimes 1^{\otimes(n-i+1)})}{i!(n-i+1)!} \right. \right. \\
&\qquad\qquad \left. - \frac{T \otimes T \otimes s_{n-1}(T^{\otimes(i-2)} \otimes 1^{\otimes(n-i+1)})}{(i-2)!(n-i+1)!} \right\} \\
&\qquad\qquad \left. + \Lambda^{i} \frac{T \otimes T \otimes s_{n-1}(T^{\otimes(i-1)} \otimes 1^{\otimes(n-i)})}{(i-1)!(n-i)!} \right] \\
&= \sum_{i=1}^{n} \Lambda^{i-1} \frac{s_{n+1}(T^{\otimes i} \otimes 1^{\otimes(n-i+1)})}{i!(n-i+1)!} \\
&\quad + \sum_{i=1}^{n} \Lambda^{i} \frac{T \otimes T \otimes s_{n-1}(T^{\otimes(i-1)} \otimes 1^{\otimes(n-i)})}{(i-1)!(n-i)!} \\
&\quad - \sum_{i=2}^{n} \Lambda^{i-1} \frac{T \otimes T \otimes s_{n-1}(T^{\otimes(i-2)} \otimes 1^{\otimes(n-i+1)})}{(i-2)!(n-i+1)!} \\
&= \sum_{i=1}^{n} \Lambda^{i-1} \frac{s_{n+1}(T^{\otimes i} \otimes 1^{\otimes(n-i+1)})}{i!(n-i+1)!} + \Lambda^{n} \frac{T \otimes T \otimes s_{n-1}(T^{\otimes(n-1)})}{(n-1)!} \\
&= \sum_{i=1}^{n} \Lambda^{i-1} \frac{s_{n+1}(T^{\otimes i} \otimes 1^{\otimes(n-i+1)})}{i!(n-i+1)!} + \Lambda^{n} T^{\otimes(n+1)}.
\end{aligned}$$

This completes the proof of 2.4. □

2.5. Now we finish the proof of 2.1. Let $R = \boldsymbol{F}_p[\Lambda, T, 1/(1+\Lambda T)]$. By 2.4, we have

$$\bar{\Delta}_p(T) = \sum_{i=1}^{p-1} \Lambda^{i-1} \frac{s_p(T^{\otimes i} \otimes 1^{\otimes(p-i)})}{i!(p-i)!} + \Lambda^{p-1} T^{\otimes p}.$$

Since

$$\sum_{i=1}^{p-1} \Lambda^{i-1} \frac{s_p(T^{\otimes i} \otimes 1^{\otimes(p-i)})}{i!(p-i)!} \in s_p(R^{\otimes p}),$$

we have

$$\begin{aligned} V_R(T) &= i_R \circ \bar{\Delta}_p(T) \\ &= i_R\left(\sum_{i=1}^{p-1} \Lambda^{i-1} \frac{s_p(T^{\otimes i} \otimes 1^{\otimes(p-i)})}{i!(p-i)!} + \Lambda^{p-1} T^{\otimes p}\right) \\ &= T \otimes_\sigma \Lambda^{p-1}. \end{aligned}$$

Hence $V_R(T) = \Lambda^{p-1}T$, identifying $\boldsymbol{F}_p[\Lambda, T, 1/(1+\Lambda^p T)]$ to $R \otimes_\sigma \boldsymbol{F}_p[\Lambda]$ by $aT \mapsto T \otimes_\sigma a$. This completes the proof of 2.1.

3. A Duality between The Frobenius and The Verschiebung

First, we review the main result of Sekiguchi and Suwa [2] and [3]. We start with reviewing necessary facts on Witt vectors and some formal power series.

3.1. Sekiguchi and Suwa [3] define polynomials $\Phi_r^{(M)}(\boldsymbol{T})$ in $\boldsymbol{Z}[M][T_0, \dots, T_r]$ by

$$\begin{aligned} \Phi_r^{(M)}(\boldsymbol{T}) &= \frac{1}{M}\Phi_n(MT_0, MT_1, \dots, MT_r) \\ &= M^{p^r-1}T_0^{p^r} + pM^{p^{r-1}-1}T_1^{p^{r-1}} + \cdots + p^{r-1}M^{p-1}T_{r-1}^p + p^r T_r \end{aligned}$$

paraphrasing Witt's argument. Furthermore, the polynomials

$$S_r^{(M)}(\boldsymbol{X}, \boldsymbol{Y}) = S_r^{(M)}(X_0, \dots, X_r, Y_0, \dots, Y_r) \in \boldsymbol{Z}[M][X_0, \dots, X_r, Y_0, \dots, Y_r],$$

$$P_r^{(M)}(\boldsymbol{X}, \boldsymbol{Y}) = P_r^{(M)}(X_0, \dots, X_r, Y_0, \dots, Y_r) \in \boldsymbol{Z}[M][X_0, \dots, X_r, Y_0, \dots, Y_r],$$

$$F_r^{(M)}(\boldsymbol{T}) = F_r^{(M)}(T_0, \dots, T_{r+1}) \in \boldsymbol{Z}[M][T_0, \dots, T_{r+1}]$$

are defined by

$$S_r^{(M)}(\boldsymbol{X}, \boldsymbol{Y}) = \frac{1}{M}S_r(MX_0, \dots, MX_r, MY_0, \dots, MY_r),$$

$$P_r^{(M)}(\boldsymbol{X}, \boldsymbol{Y}) = \frac{1}{M}P_r(X_0, \dots, X_r, MY_0, \dots, MY_r),$$

$$F_r^{(M)}(\boldsymbol{T}) = \frac{1}{M}F_r(MT_0, \dots, MT_{r+1}),$$

respectively ([3, 1.4]). Substituting M for 1, we obtain the Witt polynomials

$$\Phi_r(\boldsymbol{T}) = T_0^{p^r} + pT_1^{p^{r-1}} + \cdots + p^r T_r \in \boldsymbol{Z}[T_0, \ldots, T_r]$$

and

$$S_r(\boldsymbol{X}, \boldsymbol{Y}) = S_r(X_0, \ldots, X_r, Y_0, \ldots, Y_r) \in \boldsymbol{Z}[X_0, \ldots, X_r, Y_0, \ldots, Y_r],$$
$$P_r(\boldsymbol{X}, \boldsymbol{Y}) = P_r(X_0, \ldots, X_r, Y_0, \ldots, Y_r) \in \boldsymbol{Z}[X_0, \ldots, X_r, Y_0, \ldots, Y_r],$$
$$F_r(\boldsymbol{T}) = F_r(T_0, \ldots, T_{r+1}) \in \boldsymbol{Z}[T_0, \ldots, T_{r+1}]$$

for the Witt vectors.

A commutative group scheme $W^{(M)}$ over $\boldsymbol{Z}[M]$ is defined by

$$W^{(M)} = \operatorname{Spec} \boldsymbol{Z}[M][T_0, T_1, T_2, \ldots]$$

with the addition

$$T_0 \mapsto S_0^{(M)}(\boldsymbol{T} \otimes 1, 1 \otimes \boldsymbol{T}), \quad T_1 \mapsto S_1^{(M)}(\boldsymbol{T} \otimes 1, 1 \otimes \boldsymbol{T}),$$
$$T_2 \mapsto S_2^{(M)}(\boldsymbol{T} \otimes 1, 1 \otimes \boldsymbol{T}), \ldots.$$

Furthermore, $W^{(M)}$ is a $W_{\boldsymbol{Z}[M]}$-module with a morphism $W_{\boldsymbol{Z}[M]} \times_{\boldsymbol{Z}[M]} W^{(M)} \to W^{(M)}$ defined by

$$T_0 \mapsto P_0^{(M)}(\boldsymbol{U} \otimes 1, 1 \otimes \boldsymbol{T}), \quad T_1 \mapsto P_1^{(M)}(\boldsymbol{U} \otimes 1, 1 \otimes \boldsymbol{T}),$$
$$T_2 \mapsto P_2^{(M)}(\boldsymbol{U} \otimes 1, 1 \otimes \boldsymbol{T}), \ldots$$

([3, 1.5]). We define a morphism $\alpha^{(M)} : W^{(M)} \to W_{\boldsymbol{Z}[M]}$ by

$$T_0 \mapsto MT_0, \quad T_1 \mapsto MT_1, \quad T_2 \mapsto MT_2, \ldots.$$

3.2. Sekiguchi and Suwa [2, 2.3] define a formal power series $E_p(U, \Lambda; T) \in \boldsymbol{Q}[\Lambda, U][[T]]$ by

$$E_p(U, \Lambda; T) = (1 + \Lambda T)^{U/\Lambda} \prod_{k=1}^{\infty} (1 + \Lambda^{p^k} T^{p^k})^{(1/p^k)\{(U/\Lambda)^{p^k} - (U/\Lambda)^{p^{k-1}}\}}.$$

Recall now the definition of the Artin-Hasse exponential series

$$E_p(T) = \exp\left(\sum_{r \geq 0} \frac{T^{p^r}}{p^r}\right) \in \boldsymbol{Z}_{(p)}[[T]].$$

Then it is proved in [2, 2.5] that $E_p(U, \Lambda; T) \in \boldsymbol{Z}_{(p)}[\Lambda, U][[T]]$.

Let $\boldsymbol{U} = (U_0, U_1, U_2, \ldots)$. Sekiguchi and Suwa [2, 2.7] define a formal power series

$$E_p(\boldsymbol{U}, \Lambda; T) \in \boldsymbol{Z}_{(p)}[\Lambda, U_0, U_1, U_2, \ldots][[T]]$$

by

$$E_p(\boldsymbol{U}, \Lambda; T) = \prod_{k=0}^{\infty} E_p(U_k, \Lambda^{p^k}; T^{p^k}).$$

Then the equality

$$E_p(\boldsymbol{U}, \Lambda; T) = (1 + \Lambda T)^{\Phi_0(\boldsymbol{U})/\Lambda} \prod_{k=1}^{\infty} (1 + \Lambda^{p^k} T^{p^k})^{(1/p^k \Lambda^{p^k})\{\Phi_k(\boldsymbol{U}) - \Lambda^{p^{k-1}(p-1)} \Phi_{k-1}(\boldsymbol{U})\}}$$

is verified ([2, 2.8]). Furthermore, Sekiguchi and Suwa [3, 2.3] define a formal power series

$$E_p^{(M)}(\boldsymbol{U}, \Lambda; T) \in \boldsymbol{Z}_{(p)}[\Lambda, M, U_0, U_1, U_2, \ldots][[T]]$$

by

$$E_p^{(M)}(\boldsymbol{U}, \Lambda; T) = \frac{1}{M}[E_p(\alpha^{(M)} \boldsymbol{U}, \Lambda; T) - 1],$$

where $\alpha^{(M)} \boldsymbol{U} = (MU_0, MU_1, MU_2, \ldots)$.

3.3. Let $\boldsymbol{U} = (U_0, U_1, U_2, \ldots)$. A formal power series $F_p(\boldsymbol{U}, \Lambda; X, Y)$ in $\boldsymbol{Q}[\Lambda, U_0, U_1, U_2, \ldots][[X, Y]]$ is defined by

$$F_p(\boldsymbol{U}, \Lambda; X, Y) = \prod_{k=1}^{\infty} \left[\frac{(1 + \Lambda^{p^k} X^{p^k})(1 + \Lambda^{p^k} Y^{p^k})}{1 + \Lambda^{p^k} (X + Y + \Lambda XY)^{p^k}} \right]^{\Phi_{k-1}(\boldsymbol{U})/p^k \Lambda^{p^k}}$$

([2, 2.15]). Then it is proved in [2, 2.16] that $F_p(\boldsymbol{U}, \Lambda; X, Y) \in \boldsymbol{Z}_{(p)}[\Lambda, U_0, U_1, U_2, \ldots][[X, Y]]$. Furthermore, Sekiguchi and Suwa [3, 2.7] define a formal power series

$$F_p^{(M)}(\boldsymbol{U}, \Lambda; X, Y) \in \boldsymbol{Z}_{(p)}[\Lambda, M, U_0, U_1, U_2, \ldots][[X, Y]]$$

by

$$F_p^{(M)}(\boldsymbol{U}, \Lambda; X, Y) = \frac{1}{M}[F_p(\alpha^{(M)} \boldsymbol{U}, \Lambda; X, Y) - 1].$$

3.4. Let A be a $\boldsymbol{Z}_{(p)}[\Lambda, M]$-algebra. There are defined homomorphisms $\xi^0 : W^{(M)}(A) \to A[[T]]^{\times}$ and $\xi^1 : W^{(M)}(A) \to Z^2(\hat{\mathscr{G}}_A^{(\Lambda)}, \hat{\mathscr{G}}_A^{(M)})$ by $\boldsymbol{a} \mapsto E_p^{(M)}(\boldsymbol{a}, \Lambda; T)$

and $\boldsymbol{a} \mapsto F_p^{(M)}(\boldsymbol{a}, \Lambda; X, Y)$, respectively ([3, 3.5]). On the other hand, we define a boundary map $\partial : A[[T]]^{\times} \to Z^2(\hat{\mathscr{G}}_A^{(\Lambda)}, \hat{\mathscr{G}}_A^{(M)})$ by

$$f(T) \mapsto \frac{f(X) + f(Y) + Mf(X)f(Y) - f(X + Y + \Lambda XY)}{1 + Mf(X + Y + \Lambda XY)}.$$

Then the diagram

$$\begin{array}{ccc} W^{(M)}(A) & \xrightarrow{\xi^0} & A[[T]]^{\times} \\ {\scriptstyle F^{(M)} - [\Lambda^{p-1}]} \downarrow & & \downarrow {\scriptstyle \partial} \\ W^{(M)}(A) & \xrightarrow{\xi^1} & Z^2(\hat{\mathscr{G}}_A^{(\Lambda)}, \hat{\mathscr{G}}_A^{(M)}) \end{array}$$

commutes, where $[\Lambda^{p-1}] = (\Lambda^{p-1}, 0, 0, \ldots) \in W(A)$. Moreover, ξ^0 and ξ^1 induce homomorphisms

$$\begin{aligned} \xi^0 : \mathrm{Ker}[F^{(M)} - [\Lambda^{p-1}] : W^{(M)}(A) \to W^{(M)}(A)] &\to \mathrm{Hom}_{A\text{-gr}}(\hat{\mathscr{G}}_A^{(\Lambda)}, \hat{\mathscr{G}}_A^{(M)}) \\ \boldsymbol{a} &\mapsto E_p^{(M)}(\boldsymbol{a}, \Lambda; T) \end{aligned} \tag{3}$$

and

$$\begin{aligned} \xi^1 : \mathrm{Coker}[F^{(M)} - [\Lambda^{p-1}] : W^{(M)}(A) \to W^{(M)}(A)] &\to H_0^2(\hat{\mathscr{G}}_A^{(\Lambda)}, \hat{\mathscr{G}}_A^{(M)}) \\ \boldsymbol{a} &\mapsto F_p^{(M)}(\boldsymbol{a}, \Lambda; X, Y). \end{aligned} \tag{4}$$

It is proved that the homomorphisms ξ^0 and ξ^1 are bijective ([3, Th. 3.5.1]). Moreover, if Λ is nilpotent in A, homomorphisms

$$\begin{aligned} \xi^0 : \mathrm{Ker}[F^{(M)} - [\Lambda^{p-1}] : \hat{W}^{(M)}(A) \to \hat{W}^{(M)}(A)] &\to \mathrm{Hom}_{A\text{-gr}}(\mathscr{G}_A^{(\Lambda)}, \mathscr{G}_A^{(M)}) \\ \boldsymbol{a} &\mapsto E_p^{(M)}(\boldsymbol{a}, \Lambda; T) \end{aligned} \tag{5}$$

and

$$\begin{aligned} \xi^1 : \mathrm{Coker}[F^{(M)} - [\Lambda^{p-1}] : \hat{W}^{(M)}(A) \to \hat{W}^{(M)}(A)] &\to H_0^2(\mathscr{G}_A^{(\Lambda)}, \mathscr{G}_A^{(M)}) \\ \boldsymbol{a} &\mapsto F_p^{(M)}(\boldsymbol{a}, \Lambda; X, Y) \end{aligned} \tag{6}$$

are bijective ([3, 3.12]). Here $\hat{W}^{(M)}(A)$ denotes the functor defined by

$$\hat{W}^{(M)}(A) = \left\{ (a_0, a_1, a_2, \ldots) \in W^{(M)}(A); \begin{array}{l} Ma_i \text{ is nilpotent for all } i \text{ and} \\ a_i = 0 \text{ for all but a finite number of } i \end{array} \right\}.$$

REMARK 3.5. We now apply the result of 3.4 to $M = 1$. Notice that $U \mapsto 1 + T : A[[U]] \to A[[T]]$ (resp. $U \mapsto 1 + T : A[U, 1/U] \to A[T, 1/(1+T)]$) gives an isomorphism

$$\hat{\mathscr{G}}_A^{(1)} = \mathrm{Spf}\ A[[T]] \xrightarrow{\sim} \hat{\boldsymbol{G}}_{m,A} = \mathrm{Spf}\ A[[U-1]]$$

$$(\text{resp. } \mathscr{G}_A^{(1)} = \mathrm{Spec}\ A[T, 1/(1+\Lambda T)] \xrightarrow{\sim} \boldsymbol{G}_{m,A} = \mathrm{Spec}\ A[U, 1/U]).$$

Then it follows by the equation

$$E_p^{(1)}(\boldsymbol{U}, \Lambda; T) = E_p(\boldsymbol{U}, \Lambda; T) - 1$$

that the homomorphisms

$$\xi^0 : \mathrm{Ker}[F - [\Lambda^{p-1}] : W(A) \to W(A)] \to \mathrm{Hom}_{A\text{-gr}}(\hat{\mathscr{G}}_A^{(\Lambda)}, \hat{\boldsymbol{G}}_{m,A}); \quad \boldsymbol{a} \mapsto E_p(\boldsymbol{a}, \Lambda; T) \tag{7}$$

and

$$\xi^1 : \mathrm{Coker}[F - [\Lambda^{p-1}] : W(A) \to W(A)] \to H_0^2(\hat{\mathscr{G}}_A^{(\Lambda)}, \hat{\boldsymbol{G}}_{m,A}); \quad \boldsymbol{a} \mapsto F_p(\boldsymbol{a}, \Lambda; X, Y) \tag{8}$$

are bijective ([2, Th. 2.19.1]). Moreover, if Λ is nilpotent in A, the homomorphisms

$$\xi^0 : \mathrm{Ker}[F - [\Lambda^{p-1}] : \hat{W}(A) \to \hat{W}(A)] \to \mathrm{Hom}_{A\text{-gr}}(\mathscr{G}_A^{(\Lambda)}, \boldsymbol{G}_{m,A}); \quad \boldsymbol{a} \mapsto E_p(\boldsymbol{a}, \Lambda; T) \tag{9}$$

and

$$\xi^1 : \mathrm{Coker}[F - [\Lambda^{p-1}] : \hat{W}(A) \to \hat{W}(A)] \to H_0^2(\mathscr{G}_A^{(\Lambda)}, \boldsymbol{G}_{m,A}); \quad \boldsymbol{a} \mapsto F_p(\boldsymbol{a}, \Lambda; X, Y) \tag{10}$$

are bijective ([2, Th. 2.19.1]).

Now we can state the second main result.

THEOREM 3.6. *Let A be an $\boldsymbol{F}_p[\Lambda]$-algebra.*
(i) *Under the identifications* (7) *and* (8), *the homomorphisms*

$$V^* : \mathrm{Hom}_{A\text{-gr}}(\hat{\mathscr{G}}_A^{(\Lambda)}, \hat{\boldsymbol{G}}_{m,A}) \to \mathrm{Hom}_{A\text{-gr}}(\hat{\mathscr{G}}_A^{(\Lambda^p)}, \hat{\boldsymbol{G}}_{m,A}),$$

$$V^* : H_0^2(\hat{\mathscr{G}}_A^{(\Lambda)}, \hat{\boldsymbol{G}}_{m,A}) \to H_0^2(\hat{\mathscr{G}}_A^{(\Lambda^p)}, \hat{\boldsymbol{G}}_{m,A})$$

$$\text{(resp. } F^* : \mathrm{Hom}_{A\text{-gr}}(\hat{\mathscr{G}}_A^{(\Lambda^p)}, \hat{\boldsymbol{G}}_{m,A}) \to \mathrm{Hom}_{A\text{-gr}}(\hat{\mathscr{G}}_A^{(\Lambda)}, \hat{\boldsymbol{G}}_{m,A}),$$

$$F^* : H_0^2(\hat{\mathscr{G}}_A^{(\Lambda^p)}, \hat{\boldsymbol{G}}_{m,A}) \to H_0^2(\hat{\mathscr{G}}_A^{(\Lambda)}, \hat{\boldsymbol{G}}_{m,A}))$$

are given by $\boldsymbol{a} \mapsto F\boldsymbol{a}$ *(resp.* $\boldsymbol{a} \mapsto V\boldsymbol{a}$*).*

(ii) *Assume that* Λ *is nilpotent in* A*. Under the identifications* (9) *and* (10), *the homomorphisms*

$$V^* : \mathrm{Hom}_{A\text{-gr}}(\mathscr{G}_A^{(\Lambda)}, \boldsymbol{G}_{m,A}) \to \mathrm{Hom}_{A\text{-gr}}(\mathscr{G}_A^{(\Lambda^p)}, \boldsymbol{G}_{m,A}),$$

$$V^* : H_0^2(\mathscr{G}_A^{(\Lambda)}, \boldsymbol{G}_{m,A}) \to H_0^2(\mathscr{G}_A^{(\Lambda^p)}, \boldsymbol{G}_{m,A})$$

$$\text{(resp. } F^* : \mathrm{Hom}_{A\text{-gr}}(\mathscr{G}_A^{(\Lambda^p)}, \boldsymbol{G}_{m,A}) \to \mathrm{Hom}_{A\text{-gr}}(\mathscr{G}_A^{(\Lambda)}, \boldsymbol{G}_{m,A}),$$

$$F^* : H_0^2(\mathscr{G}_A^{(\Lambda^p)}, \boldsymbol{G}_{m,A}) \to H_0^2(\mathscr{G}_A^{(\Lambda)}, \boldsymbol{G}_{m,A}))$$

are given by $\boldsymbol{a} \mapsto F\boldsymbol{a}$ *(resp.* $\boldsymbol{a} \mapsto V\boldsymbol{a}$*).*

COROLLARY 3.7. *Let* A *be an* $\boldsymbol{F}_p[\Lambda, M]$*-algebra.*

(i) *Under the identifications* (3) *and* (4), *the homomorphisms*

$$V^* : \mathrm{Hom}_{A\text{-gr}}(\hat{\mathscr{G}}_A^{(\Lambda)}, \hat{\mathscr{G}}_A^{(M)}) \to \mathrm{Hom}_{A\text{-gr}}(\hat{\mathscr{G}}_A^{(\Lambda^p)}, \hat{\mathscr{G}}_A^{(M)}),$$

$$V^* : H_0^2(\hat{\mathscr{G}}_A^{(\Lambda)}, \hat{\mathscr{G}}_A^{(M)}) \to H_0^2(\hat{\mathscr{G}}_A^{(\Lambda^p)}, \hat{\mathscr{G}}_A^{(M)})$$

$$\text{(resp. } F^* : \mathrm{Hom}_{A\text{-gr}}(\hat{\mathscr{G}}_A^{(\Lambda^p)}, \hat{\mathscr{G}}_A^{(M)}) \to \mathrm{Hom}_{A\text{-gr}}(\hat{\mathscr{G}}_A^{(\Lambda)}, \hat{\mathscr{G}}_A^{(M)}),$$

$$F^* : H_0^2(\hat{\mathscr{G}}_A^{(\Lambda^p)}, \hat{\mathscr{G}}_A^{(M)}) \to H_0^2(\hat{\mathscr{G}}_A^{(\Lambda)}, \hat{\mathscr{G}}_A^{(M)}))$$

are given by $\boldsymbol{a} \mapsto F^{(M)}\boldsymbol{a}$ *(resp.* $\boldsymbol{a} \mapsto V\boldsymbol{a}$*).*

(ii) *Assume that* Λ *is nilpotent in* A*. Under the identifications* (5) *and* (6), *the homomorphisms*

$$V^* : \mathrm{Hom}_{A\text{-gr}}(\mathscr{G}_A^{(\Lambda)}, \mathscr{G}_A^{(M)}) \to \mathrm{Hom}_{A\text{-gr}}(\mathscr{G}_A^{(\Lambda^p)}, \mathscr{G}_A^{(M)}),$$

$$V^* : H_0^2(\mathscr{G}_A^{(\Lambda)}, \mathscr{G}_A^{(M)}) \to H_0^2(\mathscr{G}_A^{(\Lambda^p)}, \mathscr{G}_A^{(M)})$$

$$\text{(resp. } F^* : \mathrm{Hom}_{A\text{-gr}}(\mathscr{G}_A^{(\Lambda^p)}, \mathscr{G}_A^{(M)}) \to \mathrm{Hom}_{A\text{-gr}}(\mathscr{G}_A^{(\Lambda)}, \mathscr{G}_A^{(M)}),$$

$$F^* : H_0^2(\mathscr{G}_A^{(\Lambda^p)}, \mathscr{G}_A^{(M)}) \to H_0^2(\mathscr{G}_A^{(\Lambda)}, \mathscr{G}_A^{(M)}))$$

are given by $\boldsymbol{a} \mapsto F^{(M)}\boldsymbol{a}$ *(resp.* $\boldsymbol{a} \mapsto V\boldsymbol{a}$*).*

4. Proof of The Theorem

First, we give a proof of 3.6. We devote 4.1, 4.2, 4.5 and 4.6 to giving some formulas on the formal power series $E_p(\boldsymbol{U},\Lambda;T)$ and $F_p(\boldsymbol{U},\Lambda;X,Y)$, and then complete the proof by 4.3, 4.4, 4.7 and 4.8.

LEMMA 4.1. *Let* $\boldsymbol{U}=(U_0,U_1,U_2,\dots)$ *and put* $\boldsymbol{V}=(\Lambda^{p-1}U_0,\Lambda^{p(p-1)}U_1,\Lambda^{p^2(p-1)}U_2,\dots)$. *Then we have the equation*

$$E_p(\boldsymbol{U},\Lambda;\Lambda^{p-1}T)=E_p(\boldsymbol{V},\Lambda^p;T).$$

PROOF. Noting

$$\Phi_k(\boldsymbol{V})=\Phi_k(\Lambda^{p-1}U_0,\Lambda^{p(p-1)}U_1,\Lambda^{p^2(p-1)}U_2,\dots,\Lambda^{p^k(p-1)}U_k)=\Lambda^{p^k(p-1)}\Phi_k(\boldsymbol{U})$$

for all $k\geq 0$, we see easily that

$$\frac{\Phi_0(\boldsymbol{V})}{\Lambda^p}=\frac{\Phi_0(\boldsymbol{U})}{\Lambda}$$

and

$$\frac{1}{p^k\Lambda^{p^{k+1}}}\{\Phi_k(\boldsymbol{V})-\Lambda^{p^k(p-1)}\Phi_{k-1}(\boldsymbol{V})\}=\frac{1}{p^k\Lambda^{p^k}}\{\Phi_k(\boldsymbol{U})-\Lambda^{p^{k-1}(p-1)}\Phi_{k-1}(\boldsymbol{U})\}$$

for all $k\geq 1$. Hence we obtain

$$\begin{aligned}
&E_p(\boldsymbol{U},\Lambda;\Lambda^{p-1}T)\\
&=(1+\Lambda^pT)^{\Phi_0(\boldsymbol{U})/\Lambda}\prod_{k=1}^{\infty}(1+\Lambda^{p^{k+1}}T^{p^k})^{(1/p^k\Lambda^{p^k})\{\Phi_k(\boldsymbol{U})-\Lambda^{p^{k-1}(p-1)}\Phi_{k-1}(\boldsymbol{U})\}}\\
&=(1+\Lambda^pT)^{\Phi_0(\boldsymbol{V})/\Lambda^p}\prod_{k=1}^{\infty}(1+\Lambda^{p^{k+1}}T^{p^k})^{(1/p^k\Lambda^{p^{k+1}})\{\Phi_k(\boldsymbol{V})-\Lambda^{p^k(p-1)}\Phi_{k-1}(\boldsymbol{V})\}}\\
&=E_p(\boldsymbol{V},\Lambda^p;T).
\end{aligned}$$

□

LEMMA 4.2. *Let* $\boldsymbol{U}=(U_0,U_1,U_2,\dots)$ *and put* $\boldsymbol{V}=(0,U_0,U_1,U_2,\dots)$. *Then we have the equation*

$$E_p(\boldsymbol{U},\Lambda^p;T^p)=E_p(\boldsymbol{V},\Lambda;T).$$

PROOF. Noting

$$\Phi_k(\boldsymbol{V})=\Phi_k(0,U_0,U_1,U_2,\dots,U_{k-1})=p\Phi_{k-1}(\boldsymbol{U})$$

for all $k\geq 1$, we see easily that

$$\frac{\Phi_1(\boldsymbol{V})}{p\Lambda^p}=\frac{\Phi_0(\boldsymbol{U})}{\Lambda^p}$$

and

$$\frac{1}{p^{k+1}\Lambda^{p^{k+1}}}\{\Phi_{k+1}(\boldsymbol{V}) - \Lambda^{p^k(p-1)}\Phi_k(\boldsymbol{V})\} = \frac{1}{p^k\Lambda^{p^{k+1}}}\{\Phi_k(\boldsymbol{U}) - \Lambda^{p^k(p-1)}\Phi_{k-1}(\boldsymbol{U})\}$$

for all $k \geq 1$. Hence we obtain

$$\begin{aligned}
&E_p(\boldsymbol{U}, \Lambda^p; T^p) \\
&\quad = (1 + \Lambda^p T^p)^{\Phi_0(\boldsymbol{U})/\Lambda^p} \prod_{k=1}^{\infty}(1 + \Lambda^{p^{k+1}} T^{p^{k+1}})^{(1/p^k\Lambda^{p^{k+1}})\{\Phi_k(\boldsymbol{U}) - \Lambda^{p^k(p-1)}\Phi_{k-1}(\boldsymbol{U})\}} \\
&\quad = \prod_{k=1}^{\infty}(1 + \Lambda^{p^k} T^{p^k})^{(1/p^k\Lambda^{p^k})\{\Phi_k(\boldsymbol{V}) - \Lambda^{p^{k-1}(p-1)}\Phi_{k-1}(\boldsymbol{V})\}} \\
&\quad = E_p(\boldsymbol{V}, \Lambda; T). \qquad \square
\end{aligned}$$

4.3. Let A be an $\boldsymbol{F}_p[\Lambda]$-algebra, where Λ is an arbitrary element (resp. a nilpotent element) in A, and let $\boldsymbol{a} = (a_0, a_1, a_2, \ldots) \in \mathrm{Ker}[F - [\Lambda^{p-1}] : W(A) \to W(A)]$ (resp. $\mathrm{Ker}[F - [\Lambda^{p-1}] : \hat{W}(A) \to \hat{W}(A)]$). Noting $a_k^p = \Lambda^{p^k(p-1)}a_k$ for all $k \geq 0$, we see that

$$F^2\boldsymbol{a} = (a_0^{p^2}, a_1^{p^2}, a_2^{p^2}, \ldots) = (\Lambda^{p(p-1)}a_0^p, \Lambda^{p^2(p-1)}a_1^p, \Lambda^{p^3(p-1)}a_2^p, \ldots) = [\Lambda^{p(p-1)}]F\boldsymbol{a},$$

so we have

$$\begin{gathered}
F\boldsymbol{a} = (a_0^p, a_2^p, a_2^p, \ldots) \in \mathrm{Ker}[F - [\Lambda^{p(p-1)}] : W(A) \to W(A)] \\
(\text{resp. } \mathrm{Ker}[F - [\Lambda^{p(p-1)}] : \hat{W}(A) \to \hat{W}(A)]).
\end{gathered}$$

Since we have

$$E_p(\boldsymbol{a}, \Lambda; \Lambda^{p-1}T) = E_p(F\boldsymbol{a}, \Lambda^p; T)$$

by 4.1, we conclude that the homomorphism

$$\begin{aligned}
&V^* : \mathrm{Ker}[F - [\Lambda^{p-1}] : W(A) \to W(A)] \to \mathrm{Ker}[F - [\Lambda^{p(p-1)}] : W(A) \to W(A)] \\
&\qquad (\text{resp. } V^* : \mathrm{Ker}[F - [\Lambda^{p-1}] : \hat{W}(A) \to \hat{W}(A)] \\
&\qquad\quad \to \mathrm{Ker}[F - [\Lambda^{p(p-1)}] : \hat{W}(A) \to \hat{W}(A)])
\end{aligned}$$

is given by $\boldsymbol{a} \mapsto F\boldsymbol{a}$.

4.4. Let A be an $F_p[\Lambda]$-algebra, where Λ is an arbitrary element (resp. a nilpotent element) in A, and let $\boldsymbol{a} = (a_0, a_1, a_2, \dots) \in \mathrm{Ker}[F - [\Lambda^{p(p-1)}] : W(A) \to W(A)]$ (resp. $\mathrm{Ker}[F - [\Lambda^{p(p-1)}] : \hat{W}(A) \to \hat{W}(A)]$). Noting $a_k^p = \Lambda^{p^{k+1}(p-1)} a_k$ for all $k \geq 0$, we see that

$$FV\boldsymbol{a} = (0, a_0^p, a_1^p, a_2^p, \dots) = (0, \Lambda^{p(p-1)} a_0, \Lambda^{p^2(p-1)} a_1, \Lambda^{p^3(p-1)} a_2, \dots) = [\Lambda^{p-1}] V\boldsymbol{a},$$

so we have

$$V\boldsymbol{a} = (0, a_0, a_1, a_2, \dots) \in \mathrm{Ker}[F - [\Lambda^{p-1}] : W(A) \to W(A)]$$
$$(\text{resp. } \mathrm{Ker}[F - [\Lambda^{p-1}] : W(A) \to W(A)]).$$

Since we have

$$E_p(\boldsymbol{a}, \Lambda^p; T^p) = E_p(V\boldsymbol{a}, \Lambda; T)$$

by 4.2, we conclude that the homomorphism

$$F^* : \mathrm{Ker}[F - [\Lambda^{p(p-1)}] : W(A) \to W(A)] \to \mathrm{Ker}[F - [\Lambda^{p-1}] : W(A) \to W(A)]$$
$$(\text{resp. } F^* : \mathrm{Ker}[F - [\Lambda^{p(p-1)}] : \hat{W}(A) \to \hat{W}(A)]$$
$$\to \mathrm{Ker}[F - [\Lambda^{p-1}] : \hat{W}(A) \to \hat{W}(A)])$$

is given by $\boldsymbol{a} \mapsto V\boldsymbol{a}$.

Lemma 4.5. *Let $\boldsymbol{U} = (U_0, U_1, U_2, \dots)$ and put $\boldsymbol{V} = (\Lambda^{p(p-1)} U_0, \Lambda^{p^2(p-1)} U_1, \Lambda^{p^3(p-1)} U_2, \dots)$. Then we have the equation*

$$F_p(\boldsymbol{U}, \Lambda; \Lambda^{p-1} X, \Lambda^{p-1} Y) = F_p(\boldsymbol{V}, \Lambda^p; X, Y).$$

Proof. Noting

$$\Phi_k(\boldsymbol{V}) = \Phi_k(\Lambda^{p(p-1)} U_0, \Lambda^{p^2(p-1)} U_1, \Lambda^{p^3(p-1)} U_2, \dots, \Lambda^{p^{k+1}(p-1)} U_k)$$
$$= \Lambda^{p^{k+1}(p-1)} \Phi_k(\boldsymbol{U})$$

for all $k \geq 0$, we see easily that

$$\frac{\Phi_{k-1}(\boldsymbol{V})}{p^k \Lambda^{p^{k+1}}} = \frac{\Phi_{k-1}(\boldsymbol{U})}{p^k \Lambda^{p^k}}$$

for all $k \geq 1$. Hence we obtain

$$F_p(U,\Lambda;\Lambda^{p-1}X,\Lambda^{p-1}Y) = \prod_{k=1}^{\infty}\left[\frac{(1+\Lambda^{p^{k+1}}X^{p^k})(1+\Lambda^{p^{k+1}}Y^{p^k})}{1+\Lambda^{p^{k+1}}(X+Y+\Lambda^p XY)^{p^{k+1}}}\right]^{\Phi_{k-1}(U)/p^k\Lambda^{p^k}}$$

$$= \prod_{k=1}^{\infty}\left[\frac{(1+\Lambda^{p^{k+1}}X^{p^k})(1+\Lambda^{p^{k+1}}Y^{p^k})}{1+\Lambda^{p^{k+1}}(X+Y+\Lambda^p XY)^{p^{k+1}}}\right]^{\Phi_{k-1}(V)/p^k\Lambda^{p^{k+1}}}$$

$$= F_p(V,\Lambda^p;X,Y). \qquad \square$$

LEMMA 4.6. *Let* $U=(U_0,U_1,U_2,\dots)$ *and put* $V=(0,U_0,U_1,U_2,\dots)$. *Then we have*

$$F_p(U,\Lambda^p;X^p,Y^p) \equiv F_p(V,\Lambda;X,Y) \mod p.$$

PROOF. Noting $\Phi_0(V)=0$ and $\Phi_k(V)=p\Phi_{k-1}(U)$ for all $k\geq 1$, we obtain

$$F_p(U,\Lambda^p;X^p,Y^p) = \prod_{k=1}^{\infty}\left[\frac{(1+\Lambda^{p^{k+1}}X^{p^{k+1}})(1+\Lambda^{p^{k+1}}Y^{p^{k+1}})}{1+\Lambda^{p^{k+1}}(X^p+Y^p+\Lambda^p X^p Y^p)^{p^{k+1}}}\right]^{\Phi_{k-1}(U)/p^k\Lambda^{p^{k+1}}}$$

$$\equiv \prod_{k=2}^{\infty}\left[\frac{(1+\Lambda^{p^k}X^{p^k})(1+\Lambda^{p^k}Y^{p^k})}{1+\Lambda^{p^k}(X+Y+\Lambda XY)^{p^{k+1}}}\right]^{\Phi_{k-2}(U)/p^{k-1}\Lambda^{p^k}}$$

$$= \prod_{k=1}^{\infty}\left[\frac{(1+\Lambda^{p^k}X^{p^k})(1+\Lambda^{p^k}Y^{p^k})}{1+\Lambda^{p^k}(X+Y+\Lambda XY)^{p^{k+1}}}\right]^{\Phi_{k-1}(V)/p^k\Lambda^{p^k}}$$

$$= F_p(V,\Lambda;X,Y) \mod p. \qquad \square$$

4.7. Let A be an $F_p[\Lambda]$-algebra, where Λ is an arbitrary element (resp. a nilpotent element) in A, and let $a=(a_0,a_1,a_2,\dots)\in W(A)$ (resp. $\hat{W}(A)$). Noting

$$Fa - [\Lambda^{p(p-1)}]a \in \operatorname{Im}[F-[\Lambda^{p(p-1)}] : W(A)\to W(A)]$$

$$(\text{resp. } \operatorname{Im}[F-[\Lambda^{p(p-1)}] : \hat{W}(A)\to \hat{W}(A)]),$$

we see that

$$F_p(Fa,\Lambda^p;X,Y) \equiv F_p([\Lambda^{p(p-1)}]a,\Lambda^p;X,Y) \mod B^2(\hat{\mathscr{G}}_A^{(\Lambda^p)},\hat{G}_{m,A})$$

$$(\text{resp. } B^2(\mathscr{G}_A^{(\Lambda^p)},G_{m,A})).$$

Since we have

$$F_p(a,\Lambda;\Lambda^{p-1}X,\Lambda^{p-1}Y) = F_p([\Lambda^{p(p-1)}]a,\Lambda^p;X,Y)$$

by 4.5, we see that

$$F_p(\boldsymbol{a}, \Lambda; \Lambda^{p-1}X, \Lambda^{p-1}Y) \equiv F_p(F\boldsymbol{a}, \Lambda^p; X, Y) \mod B^2(\hat{\mathscr{G}}_A^{(\Lambda^p)}, \hat{\boldsymbol{G}}_{m,A})$$

$$(\text{resp. } B^2(\mathscr{G}_A^{(\Lambda^p)}, \boldsymbol{G}_{m,A})).$$

Hence we conclude that the homomorphism

$$V^* : \mathrm{Coker}[F - [\Lambda^{p-1}] : W(A) \to W(A)]$$
$$\to \mathrm{Coker}[F - [\Lambda^{p(p-1)}] : W(A) \to W(A)]$$
$$(\text{resp. } V^* : \mathrm{Coker}[F - [\Lambda^{p-1}] : \hat{W}(A) \to \hat{W}(A)]$$
$$\to \mathrm{Coker}[F - [\Lambda^{p(p-1)}] : \hat{W}(A) \to \hat{W}(A)])$$

is given by $\boldsymbol{a} \mapsto F\boldsymbol{a}$.

4.8. Let A be an $\boldsymbol{F}_p[\Lambda]$-algebra, where Λ is an arbitrary element (resp. a nilpotent element) in A, and let $\boldsymbol{a} = (a_0, a_1, a_2, \dots) \in W(A)$ (resp. $\hat{W}(A)$). We see easily that if $\boldsymbol{a} \in \mathrm{Im}[F - [\Lambda^{p(p-1)}] : W(A) \to W(A)]$ (resp. $\mathrm{Im}[F - [\Lambda^{p(p-1)}] : \hat{W}(A) \to \hat{W}(A)]$) then $V\boldsymbol{a} \in \mathrm{Im}[F - [\Lambda^{p-1}] : W(A) \to W(A)]$ (resp. $\mathrm{Im}[F - [\Lambda^{p-1}] : \hat{W}(A) \to \hat{W}(A)]$). Since we have

$$F_p(\boldsymbol{a}, \Lambda^p; X^p, Y^p) = F_p(V\boldsymbol{a}, \Lambda; X, Y)$$

by 4.6, we conclude that the homomorphism

$$F^* : \mathrm{Coker}[F - [\Lambda^{p(p-1)}] : W(A) \to W(A)]$$
$$\to \mathrm{Coker}[F - [\Lambda^{p-1}] : W(A) \to W(A)]$$
$$(\text{resp. } F^* : \mathrm{Coker}[F - [\Lambda^{p(p-1)}] : \hat{W}(A) \to \hat{W}(A)]$$
$$\to \mathrm{Coker}[F - [\Lambda^{p-1}] : \hat{W}(A) \to \hat{W}(A)])$$

is given by $\boldsymbol{a} \mapsto V\boldsymbol{a}$.

Now we give a proof of 3.7.

4.9. Let A be an $\boldsymbol{F}_p[\Lambda, M]$-algebra and $B = A[t]/(t^2 - Mt)$, and let ε denote the image of t in B. Then we have $\varepsilon^2 = Mt$. Then the correspondence $(a_0, a_1, a_2, \dots) \mapsto (\varepsilon a_0, \varepsilon a_1, \varepsilon a_2, \dots)$ gives rise to a $W(A)$-isomorphism

$$\phi : W^{(M)}(A) \xrightarrow{\sim} \mathrm{Ker}[\varepsilon \mapsto 0 : W(B) \to W(A)].$$

Moreover, $F : W(B) \to W(B)$ (resp. $V : W(B) \to W(B)$) induces $F^{(M)}$ (resp. V) on $W^{(M)}(A)$ ([3, 1.21]). Hence we see readily that the isomorphism ϕ induces injections

$$\mathrm{Ker}[F^{(M)} - [\Lambda^{p-1}] : W^{(M)}(A) \to W^{(M)}(A)] \hookrightarrow \mathrm{Ker}[F - [\Lambda^{p-1}] : W(B) \to W(B)]$$

and

$$\mathrm{Coker}[F^{(M)} - [\Lambda^{p-1}] : W^{(M)}(A) \to W^{(M)}(A)]$$
$$\hookrightarrow \mathrm{Coker}[F - [\Lambda^{p-1}] : W(B) \to W(B)].$$

Furthermore, we see that the homomorphisms

$$\boldsymbol{a} \mapsto F\boldsymbol{a} : \mathrm{Ker}[F - [\Lambda^{p-1}] : W(B) \to W(B)]$$
$$\to \mathrm{Ker}[F - [\Lambda^{p(p-1)}] : W(B) \to W(B)],$$
$$\boldsymbol{a} \mapsto V\boldsymbol{a} : \mathrm{Ker}[F - [\Lambda^{p(p-1)}] : W(B) \to W(B)]$$
$$\to \mathrm{Ker}[F - [\Lambda^{p-1}] : W(B) \to W(B)],$$
$$\boldsymbol{a} \mapsto F\boldsymbol{a} : \mathrm{Coker}[F - [\Lambda^{p-1}] : W(B) \to W(B)]$$
$$\to \mathrm{Coker}[F - [\Lambda^{p(p-1)}] : W(B) \to W(B)],$$
$$\boldsymbol{a} \mapsto V\boldsymbol{a} : \mathrm{Coker}[F - [\Lambda^{p(p-1)}] : W(B) \to W(B)]$$
$$\to \mathrm{Coker}[F - [\Lambda^{p-1}] : W(B) \to W(B)]$$

induce homomorphisms

$$\boldsymbol{a} \mapsto F^{(M)}\boldsymbol{a} : \mathrm{Ker}[F^{(M)} - [\Lambda^{p-1}] : W^{(M)}(A) \to W^{(M)}(A)]$$
$$\to \mathrm{Ker}[F^{(M)} - [\Lambda^{p(p-1)}] : W^{(M)}(A) \to W^{(M)}(A)],$$
$$\boldsymbol{a} \mapsto V\boldsymbol{a} : \mathrm{Ker}[F^{(M)} - [\Lambda^{p(p-1)}] : W^{(M)}(A) \to W^{(M)}(A)]$$
$$\to \mathrm{Ker}[F^{(M)} - [\Lambda^{p-1}] : W^{(M)}(A) \to W^{(M)}(A)],$$
$$\boldsymbol{a} \mapsto F^{(M)}\boldsymbol{a} : \mathrm{Coker}[F^{(M)} - [\Lambda^{p-1}] : W^{(M)}(A) \to W^{(M)}(A)]$$
$$\to \mathrm{Coker}[F^{(M)} - [\Lambda^{p(p-1)}] : W^{(M)}(A) \to W^{(M)}(A)],$$
$$\boldsymbol{a} \mapsto V\boldsymbol{a} : \mathrm{Coker}[F^{(M)} - [\Lambda^{p(p-1)}] : W^{(M)}(A) \to W^{(M)}(A)]$$
$$\to \mathrm{Coker}[F^{(M)} - [\Lambda^{p-1}] : W^{(M)}(A) \to W^{(M)}(A)],$$

respectively. This completes the proof of 3.7.

We have already attained our goal, but we finish this paper after determining corresponding vectors for $V_{\hat{\mathscr{G}}_A^{(\Lambda)}}, V_{\mathscr{G}_A^{(\Lambda)}}, F_{\hat{\mathscr{G}}_A^{(\Lambda)}}$ and $F_{\mathscr{G}_A^{(\Lambda)}}$ via identifications (3), (4), (5) and (6) in 3.4.

Lemma 4.10. *We have the equation*

$$E_p(V^r[\Lambda^{p^r}], \Lambda; T) = 1 + \Lambda^{p^r} T^{p^r}.$$

Proof. We will first prove in case by case the equation

$$\frac{1}{p^k \Lambda^{p^k}} \{\Phi_k(V^r[\Lambda^{p^r}]) - \Lambda^{p^{k-1}(p-1)} \Phi_{k-1}(V^r[\Lambda^{p^r}])\} = \begin{cases} 1 & (k = r), \\ 0 & (\text{otherwise}). \end{cases}$$

Case $1 \leq k \leq r$:

$$\frac{1}{p^k \Lambda^{p^k}} \{\Phi_k(V^r[\Lambda^{p^r}]) - \Lambda^{p^{k-1}(p-1)} \Phi_{k-1}(V^r[\Lambda^{p^r}])\} = 0.$$

Case $k = r$:

$$\frac{1}{p^k \Lambda^{p^k}} \{\Phi_k(V^r[\Lambda^{p^r}]) - \Lambda^{p^{k-1}(p-1)} \Phi_{k-1}(V^r[\Lambda^{p^r}])\} = \frac{1}{p^r \Lambda^{p^r}} p^r \Lambda^{p^r} = 1.$$

Case $k > r$:

$$\begin{aligned} &\frac{1}{p^k \Lambda^{p^k}} \{\Phi_k(V^r[\Lambda^{p^r}]) = \Lambda^{p^{k-1}(p-1)} \Phi_{k-1}(V^r[\Lambda^{p^r}])\} \\ &= \frac{1}{p^k \Lambda^{p^k}} \{p^r (\Lambda^{p^r})^{p^{k-r}} - \Lambda^{p^{k-1}(p-1)} p^r (\Lambda^{p^r})^{p^{k-1-r}}\} \\ &= \frac{1}{p^k \Lambda^{p^k}} \{p^r \Lambda^{p^k} - p^r \Lambda^{p^k}\} = 0. \end{aligned}$$

Hence

$$\begin{aligned} &E_p(V^r[\Lambda^{p^r}], \Lambda; T) \\ &= (1 + \Lambda T)^{\Phi_0(V^r[\Lambda^{p^r}])/\Lambda} \prod_{k=1}^{\infty} (1 + \Lambda^{p^k} T^{p^k})^{(1/p^k \Lambda^{p^k})\{\Phi_k(V^r[\Lambda^{p^r}]) - \Lambda^{p^{k-1}(p-1)} \Phi_{k-1}(V^r[\Lambda^{p^r}])\}} \\ &= 1 + \Lambda^{p^r} T^{p^r}. \end{aligned}$$

□

Corollary 4.11. *We have the following equations*:
(i) $E_p^{(\Lambda)}([\Lambda^{p-1}]_\Lambda, \Lambda^p; T) = \Lambda^{p-1} T$,
(ii) $E_p^{(\Lambda^p)}((0, 1, 0, 0, \ldots), \Lambda; T) = T^p$.

PROOF. (i) Applying 4.10 to $r=0$ and $\Lambda=\Lambda^p$, we have $E_p([\Lambda^p],\Lambda^p;T)=1+\Lambda^pT$. Hence

$$E_p^{(\Lambda)}([\Lambda^{p-1}]_\Lambda,\Lambda^p;T)=\frac{1}{\Lambda}[E_p([\Lambda^p],\Lambda^p;T)-1]=\frac{1}{\Lambda}[(1+\Lambda^pT)-1]=\Lambda^{p-1}T.$$

(ii) Applying 4.10 to $r=1$, we have $E_p(V[\Lambda^p],\Lambda;T)=1+\Lambda^pT^p$. Hence

$$E_p^{(\Lambda^p)}((0,1,0,0,\dots),\Lambda;T)=\frac{1}{\Lambda^p}[E_p(V[\Lambda^p],\Lambda;T)-1]=\frac{1}{\Lambda^p}[(1+\Lambda^pT^p)-1]=T^p.$$

□

LEMMA 4.12. *Let A be an $\boldsymbol{F}_p[\Lambda]$-algebra. Then we have the followings*:
(i) $[\Lambda^{p-1}]_\Lambda\in\mathrm{Ker}[F^{(\Lambda)}-[\Lambda^{p(p-1)}]:W^{(\Lambda)}(A)\to W^{(\Lambda)}(A)]$,
(ii) $(0,1,0,0,\dots)\in\mathrm{Ker}[F^{(\Lambda^p)}-[\Lambda^{p-1}]:W^{(\Lambda^p)}(A)\to W^{(\Lambda^p)}(A)]$.

PROOF. (i) We see easily that $F^{(\Lambda)}([a]_\Lambda)=[\Lambda^{p-1}a^p]_\Lambda$ for all $a\in A$. In fact, for $r\geq 0$,

$$\begin{aligned}\Phi_r^{(\Lambda)}(F^{(\Lambda)}([T]_\Lambda))&=\Phi_{r+1}^{(\Lambda)}([T]_\Lambda)=\Lambda^{p^{r+1}-1}T^{p^{r+1}}=\Lambda^{p^r-1}(\Lambda^{p-1}T^p)^{p^r}\\&=\Phi_r^{(\Lambda)}([\Lambda^{p-1}T^p]_\Lambda).\end{aligned}$$

Hence we have

$$\begin{aligned}&(F^{(\Lambda)}-[\Lambda^{p(p-1)}])([\Lambda^{p-1}]_\Lambda)\\&\quad=(\Lambda^{p-1}(\Lambda^{p-1})^p,0,0,\dots)-[\Lambda^{p^2-1}]_\Lambda=[\Lambda^{p^2-1}]_\Lambda-[\Lambda^{p^2-1}]_\Lambda\\&\quad=(0,0,0,\dots).\end{aligned}$$

(ii) We see easily that

$$F_r^{(M)}(T_0,T_1,T_2,\dots)\equiv M^{p-1}T_r^p \mod p$$

for all $r\geq 0$. Hence we have

$$\begin{aligned}&(F^{(\Lambda^p)}-[\Lambda^{p-1}])((0,1,0,0,\dots))\\&\quad=(0,\Lambda^{p(p-1)},0,0,\dots)-(0,\Lambda^{p(p-1)},0,0,\dots)=(0,0,0,\dots).\end{aligned}$$ □

COROLLARY 4.13. *Let A be an $\boldsymbol{F}_p[\Lambda]$-algebra such that Λ is nilpotent in A. Then we have the followings*:
(i) $[\Lambda^{p-1}]_\Lambda\in\mathrm{Ker}[F^{(\Lambda)}-[\Lambda^{p(p-1)}]:\hat{W}^{(\Lambda)}(A)\to\hat{W}^{(\Lambda)}(A)]$,
(ii) $(0,1,0,0,\dots)\in\mathrm{Ker}[F^{(\Lambda^p)}-[\Lambda^{p-1}]:\hat{W}^{(\Lambda^p)}(A)\to\hat{W}^{(\Lambda^p)}(A)]$.

PROOF. Note that Λ^p is nilpotent in A. □

4.14. Let A be an $\boldsymbol{F}_p[\Lambda]$-algebra, where Λ is an arbitrary element (resp. a nilpotent element) in A. By 4.11 and 4.12 (resp. 4.11 and 4.13), we see that the Verschiebung morphism for $\hat{\mathscr{G}}_A^{(\Lambda)}$ (resp. $\mathscr{G}_A^{(\Lambda)}$), which is defined by $T \mapsto \Lambda^{p-1}T$, corresponds to the vector $[\Lambda^{p-1}]_\Lambda$ via the isomorphism

$$\mathrm{Ker}[F^{(\Lambda)} - [\Lambda^{p(p-1)}] : W^{(\Lambda)}(A) \to W^{(\Lambda)}(A)] \simeq \mathrm{Hom}_{A\text{-gr}}(\hat{\mathscr{G}}_A^{(\Lambda^p)}, \hat{\mathscr{G}}_A^{(\Lambda)})$$

$$(\text{resp. } \mathrm{Ker}[F^{(\Lambda)} - [\Lambda^{p(p-1)}] : \hat{W}^{(\Lambda)}(A) \to \hat{W}^{(\Lambda)}(A)] \simeq \mathrm{Hom}_{A\text{-gr}}(\mathscr{G}_A^{(\Lambda^p)}, \mathscr{G}_A^{(\Lambda)})).$$

Similarly, the Frobenius morphism for $\hat{\mathscr{G}}_A^{(\Lambda)}$ (resp. $\mathscr{G}_A^{(\Lambda)}$), which is defined by $T \mapsto T^p$, corresponds to the vector $(0, 1, 0, 0, \dots)$ via the isomorphism

$$\mathrm{Ker}[F^{(\Lambda^p)} - [\Lambda^{p-1}] : W^{(\Lambda^p)}(A) \to W^{(\Lambda^p)}(A)] \simeq \mathrm{Hom}_{A\text{-gr}}(\hat{\mathscr{G}}_A^{(\Lambda)}, \hat{\mathscr{G}}_A^{(\Lambda^p)})$$

$$(\text{resp. } \mathrm{Ker}[F^{(\Lambda^p)} - [\Lambda^{p-1}] : \hat{W}^{(\Lambda^p)}(A) \to \hat{W}^{(\Lambda^p)}(A)] \simeq \mathrm{Hom}_{A\text{-gr}}(\mathscr{G}_A^{(\Lambda)}, \mathscr{G}_A^{(\Lambda^p)})).$$

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