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ON A NECESSARY CONDITION FOR THE WELLPOSEDNESS OF THE CAUCHY PROBLEM FOR SOME DEGENERATE PARABOLIC EQUATIONS

In memory of Sigeru Mizohata

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§1. Introduction

Let $P(x, \partial_t, D_x)$ be

$$P(x, \partial_t, D_x) = \partial_t + \sum_{j=1}^{m} \mu_j (D_j^2 + x_j^2 D_n^2) + \sum_{j=m+1}^{m+\ell} D_j^2 + \sum_{j=1}^{n} b_j(x) D_j + c(x) \qquad (1)$$

in a neighbourhood of the origin and all coefficients are smooth and bounded on $\mathbb{R}^n$, where $0 < m + \ell < n$, ∂_t, D_j stand for $\partial/\partial t$, $(1/i)(\partial/\partial x_j)$ $(j = 1, 2, \ldots, n)$ respectively. Here μ_j $(j = 1, 2, \ldots, m)$ are positive constants. We note that P is degenerate at $(0, e_n)$ $(e_n = (0, \ldots, 0, 1))$.

We consider the Cauchy problem for $P(x, \partial_t, D_x)$ and give a necessary condition for the Cauchy problem to be H^∞ well-posed.

Concerning this problem, K. Igari [3] treated the evolution equations of the 1st order in t. Recently H. Honda [2] studied the degenerate heat equation in the analytic clases. This is an extention of the interesting example given by P. D'Ancona and S. Spagnolo [1]. In the case that the operators are degenerate in t, M. Miyake [6] treated the evolution equations of the 1st order in t and T. Sadamatsu [8] studied a p-evolution equation. The general cases are studied by K. Kitagawa [5]. For the sufficiency, we only mention the result in Gevrey clases of K. Kajitani and M. Mikami [4].

§2. Statement of Theorem

Our aim is to show

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THEOREM.

In order that the Cauchy problem for P to be H^∞ well-posed, it is necessary that

$$|Re\ b_n(0)| \leq \sum_{j=1}^{m} \mu_j. \tag{2}$$

In the proof of the theorem, we assume that

$$Re\ b_n(0) < -\sum_{j=1}^{m} \mu_j \tag{3}$$

if not, we change ξ_n to $-\xi_n$, and we derive a contradiction.

Here we give the definition of H^∞ well-posedness. We say that the Cauchy problem for P is H^∞ well-posed, if there exists a positive constant δ and for any fixed τ $(0 < \tau \leq \delta)$ and for any f in $C^0([-\delta, 0]; H^\infty)$ such that $f = 0$ in $t \leq -\tau$, there exists a unique solution u in $C^1([-\delta, 0]; H^\infty)$ with $u = 0$ in $t \leq -\tau$ of the equation $Pu = f$ and for any $p \in N$ there exist $q \in N$ and a constant C which is independent of τ.

$$\|u(t, \cdot)\|_p \leq C \int_{-\tau}^{t} \|f(s, \cdot)\|_q\, ds \quad (-\tau < t \leq 0) \tag{4}$$

holds, where $\|\cdot\|_p$ denotes the H^p-norm.

In particular, we obtain

$$\|u(0, \cdot)\|_0 \leq C \int_{-\tau}^{0} \|(Pu)(t, \cdot)\|_q\, dt. \tag{5}$$

In the proof of the theorem, we use the change of variable of the following type:

$$t = \rho^{-\nu_0} y_0, \quad x_i = \rho^{-\nu_i} y_i \quad (i = 1, 2, \ldots, n)$$

and if we set

$$v(y_0, y) = u(t, x)|_{t=\rho^{-\nu_0} y_0,\, x=\rho^{-\nu} y}$$

and

$$P_\rho(y, \partial_0, D_y) v(y) = P(x, \partial_t, D_x) u(t, x)|_{t=\rho^{-\nu_0} y_0,\, x=\rho^{-\nu} y},$$

then we have from the above inequality

$$\|v(0, \cdot)\|_0 \leq C\rho^{\bar{\nu}q} \int_{-\infty}^{0} \|(P_\rho v)(y_0, \cdot)\|_q\, dy_0 \tag{6}$$

where $\bar{\nu} = \max_{0 \leq i \leq n}\{\nu_i\}$. In order to prove the theorem we use this inequality.

§3. Proof of Theorem

In this section we prove the theorem under the condition (3)

1. Dilation and E_ρ.

Let s be a large integer and we put

$$t=\rho^{-s-1}y_0, \quad x'=\rho^{-s/2}y', \quad x''=\rho^{-s/2+1}y'', \quad x_n=\rho^{-s}y_n$$

here $x=(x',x'',x_n)=(x_1,\dots,x_m,x_{m+1},\dots,x_{n-1},x_n)$ and

$$P_\rho(y,\partial_0,D)=P(x,\partial_t,D_x)|_{t=\rho^{-s-1}y_0,x'=\rho^{-s/2}y',x''=\rho^{-s/2+1}y'',x_n=\rho^{-s}y_n}$$

$$=\rho^{s+1}\partial_0+\rho^s\left[\sum_{j=1}^m\mu_j(D_j^2+y_j^2D_n^2)+b_n(0)D_n\right]$$

$$+\rho^{s-2}\sum_{k=m+1}^{m+\ell}D_k^2+\rho^{s/2}\sum_{j=1}^m b_j(0)D_j+\rho^{s/2-1}\sum_{k=m+1}^{n-1}b_k(0)D_k$$

$$+\ (\textit{lower order terms in }\rho),$$

here $\partial_0=\partial/\partial y_0$, $D_j=(1/i)(\partial/\partial y_j)$ $(j=1,2,\dots,n)$.

Let $E_\rho=\exp(i\rho^2(y_n+(i/2)(y_1^2+\cdots+y_m^2))+i\rho\varphi(y))$, then

$$E_\rho^{-1}P_\rho E_\rho=\rho^{s+3}\left[\sum_{j=1}^m\mu_j(2iy_j\varphi'_{y_j}+2y_j^2\varphi'_{y_n})\right]$$

$$+\rho^{s+2}\left[i\varphi'_{y_0}+\sum_{j=1}^m\mu_j(2iy_jD_j+2y_j^2D_n+(\varphi'_{y_j})^2+y_j^2(\varphi'_{y_n})^2+1)+b_n(0)\right]$$

$$+\rho^{s+1}\left[\partial_0+\sum_{j=1}^m\mu_j(2\varphi'_{y_j}D_j+2y_j^2\varphi'_{y_n}D_n+(D_j\varphi'_{y_j})\right.$$

$$\left.+\ y_j^2(D_n\varphi'_{y_n}))+b_n(0)\varphi'_{y_n}\right]$$

$$+\ (\text{lower order terms in }\rho),$$

where $\varphi'_{y_j}=\partial\varphi/\partial y_j$ $(j=0,1,\dots,n)$.

2. Construction of φ and estimates of E_ρ.

We take γ which satisfies

$$i\gamma+\sum_{j=1}^m\mu_j+b_n(0)=0. \tag{7}$$

So we note $\operatorname{Im}\gamma < 0$ by (3). When $m > 0$, let $\psi(y_0, x, y_{m+1}, \dots, y_n)$ be a solution of the Cauchy problem

$$2i\frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y_n} = 0$$

with the initial data $\psi(y_0, 0, y_{m+1}, \dots, y_n) = \gamma y_0 + i(y_{m+1}^2 + \cdots + y_n^2)$, then $\varphi(y) = \psi(y_0, y_1^2 + \cdots + y_m^2, y_{m+1}, \dots, y_n)$ satisfies

$$\begin{cases} \sum_{j=1}^m \mu_j \left(2iy_j \frac{\partial\varphi}{\partial y_j} + 2y_j^2 \frac{\partial\varphi}{\partial y_n}\right) = 0, \\ \varphi(y_0, 0, y_{m+1}, \dots, y_n) = \gamma y_0 + i(y_{m+1}^2 + \cdots + y_n^2). \end{cases} \tag{8}$$

When $m = 0$, we take $\varphi(y) = \gamma y_0 + i(y_1^2 + \cdots + y_n^2)$ so that the coefficient of ρ^{s+2} vanishes.

The definition of E_ρ and the function φ yield the following estimates of E_ρ, that is, there exists positive constants c_0, c_1 such that

$$|E_\rho| \le \exp[-c_0(\rho^2(y_1^2 + \cdots + y_m^2) + \rho((\operatorname{Im}\gamma)y_0 + y_{m+1}^2 + \cdots + y_n^2))]$$

and

$$|E_\rho| \ge \exp[-c_1(\rho^2(y_1^2 + \cdots + y_m^2) + \rho((\operatorname{Im}\gamma)y_0 + y_{m+1}^2 + \cdots + y_n^2))].$$

3. Asymptotic solution

In this subsection we construct the asymptotic solution $v_\rho^N = v_0 + (1/\rho)v_1 + \cdots + (1/\rho^N)v_N$ of the equation $E_\rho^{-1}P_\rho E_\rho v = 0$.

Taking account of the choice of φ and γ, when $m = 0$ we put

$$E_\rho^{-1}P_\rho E_\rho = \rho^{s+1}(\partial_0 + b_n(0)\varphi'_{y_n}) + \rho^s R_0 + \rho^{s-1}R_1 + \cdots$$

and therefore

$$\begin{aligned} E_\rho^{-1}P_\rho E_\rho v_\rho^N &= \rho^{s+1}(\partial_0 + b_n(0)\varphi'_{y_n})v_0 + \rho^s[(\partial_0 + b_n(0)\varphi'_{y_n})v_1 + R_0 v_0] + \cdots \\ &\quad + \rho^{s+1-N}[(\partial_0 + b_n(0)\varphi'_{y_n})v_N + R_0 v_{N-1} + \cdots + R_{N-1}v_0] + \cdots. \end{aligned}$$

we take v_0 is the solution of the Cauchy problem:

$$(\partial_0 + b_n(0)\varphi'_{y_n})v_0 = 0, \quad v_0(0, y_1, \dots, y_n) = 1,$$

and v_i $(i = 1, \dots, N)$ are the solutions of the Cauchy problem:

$$(\partial_0 + b_n(0)\varphi'_{y_n})v_i + R_0 v_{i-1} + \cdots + R_{i-1}v_0 = 0, \quad v_i(0, y_1, \dots, y_n) = 0.$$

Since it is easy to treat the case $m = 0$, we omit this case from this time.

When $m > 0$, then

$$E_\rho^{-1} P_\rho E_\rho = \rho^{s+2}\left[i\varphi'_{y_0} + \sum_{j=1}^m \mu_j\left(2y_j\frac{\partial}{\partial y_j} - 2iy_j^2\frac{\partial}{\partial y_n} + \varphi'^{\,2}_{y_j} + y_j^2\varphi'^{\,2}_{y_n} + 1\right) + b_n(0)\right]$$

$$+\rho^{s+1}R + \rho^s R_0 + \rho^{s-1}R_1 + \cdots,$$

where $R = \partial_0 + \sum_{j=1}^m \mu_j(2\varphi'_{y_j}D_j + 2y_j^2\varphi'_{y_n}D_n + (D_j\varphi'_{y_j}) + y_j^2(D_n\varphi'_{y_n})) + b_n(0)\varphi'_{y_n}$, and by the choice of φ, we have

$$i\varphi'_{y_0} + \sum_{j=1}^m \mu_j\left(2y_j\frac{\partial}{\partial y_j} - 2iy_j^2\frac{\partial}{\partial y_n} + \varphi'^{\,2}_{y_j} + y_j^2\varphi'^{\,2}_{y_n} + 1\right) + b_n(0)$$

$$= \sum_{j=1}^m \mu_j\left(2y_j\frac{\partial}{\partial y_j} - 2iy_j^2\frac{\partial}{\partial y_n}\right) + i\varphi'_{y_0} + \sum_{j=1}^m \mu_j + b_n(0).$$

Hereafter we put $\Lambda = \sum_{j=1}^m \mu_j(2y_j(\partial/\partial y_j) - 2iy_j^2(\partial/\partial y_n))$ and $c_0(y_0, y) = i\varphi'_{y_0} + \sum_{j=1}^m \mu_j + b_n(0)$, then

$$E_\rho^{-1}P_\rho E_\rho v_\rho^N = \rho^{s+2}(\Lambda + c_0)v_0 + \rho^{s+1}((\Lambda + c_0)v_1 + Rv_o) + \cdots$$

$$+\rho^{s+2-N}((\Lambda + c_0)v_N + Rv_{N-1} + \cdots + R_{N-2}v_0) + \cdots. \tag{9}$$

Now following Nishitani, T. [7], we shall solve the equation $(\Lambda + c_0)v = f$ in formal series in y.

First we treat the equation $(\Lambda + c_0)v_0 = 0$. Since Λ is degenerate on $y' = 0$, the equation $(\Lambda + c_0)v_0 = 0$ is solvable in formal series in y' if and only if $c_0(y_0, 0, y'') = 0$, here $y = (y', y'') = (y_1, \ldots, y_m, y_{m+1}, \ldots, y_n)$ and this codition is satisfied by the choice of γ that is (7) and φ. Taking account of this condition, we put $c_0(y_0, y) = \sum_{\beta'>0} c^{\beta'}(y_0, y'')y'^{\beta'}$ and solve the equation in the form $v_0(y_0, y) = \sum_{\alpha'} v^{\alpha'}(y_0, y'')y'^{\alpha'}$.

We substitute this in the above equation, then

$$(\Lambda + c_0)v_0 = \sum_{\alpha'}\left[\sum_{j=1}^m \mu_j(2\alpha_j v^{\alpha'}(y_0, y'') - 2i\partial_{y_n}v^{\alpha'-2e'_j}(y_0, y''))\right.$$

$$\left. + \sum_{\beta'+\gamma'=\alpha', \beta'>0} c^{\beta'}(y_0, y'')v^{\gamma'}(y_0, y'')\right]y'^{\alpha'}. \tag{10}$$

In order to determine $\{v^{\alpha'}(y_0, y'')\}$ successively, we put the coefficients of $y^{\alpha'}$ vanish, that is,

$$\left(\sum_{j=1}^{m} 2\mu_j\alpha_j\right) v^{\alpha'}(y_0, y'') - \sum_{j=1}^{m} 2i\mu_j\partial_{y_n} v^{\alpha'-2e'_j}(y_0, y'')$$

$$+ \sum_{\beta'+\gamma'=\alpha', \beta'>0} c^{\beta'}(y_0, y'') v^{\gamma'}(y_0, y'') = 0 \quad |\alpha'| \geq 0 \tag{11}$$

here we note that we can take $v^{0'}(y_0, y'')$ free.

Next we put $c^{\beta'}(y_0, y'') = \sum_{\beta''} c^{\beta}(y_0) y''^{\beta''}$ and $v^{\gamma'}(y_0, y'') = \sum_{\gamma''} v^{\gamma}(y_0) y''^{\gamma''}$ and subsutitute this in the above formula, then

$$(\Lambda + c_0)v_0 = \sum_{\alpha} \left[\left(\sum_{j=1}^{m} 2\mu_j\alpha_j\right) v^{\alpha}(y_0) - \sum_{j=1}^{m} 2i\mu_j\alpha_n v^{\alpha-2e_j+e_n}(y_0) \right.$$

$$\left. + \sum_{\beta+\gamma=\alpha, \beta'>0} c^{\beta}(y_0) v^{\gamma}(y_0) \right] y^{\alpha}$$

and we determine $v^{\alpha}(y_0)$ ($|\alpha| < N'$) successively for any integer N' such that

$$\left(\sum_{j=1}^{m} 2\mu_j\alpha_j\right) v^{\alpha}(y_0) - \sum_{j=1}^{m} 2i\mu_j\alpha_n v^{\alpha-2e_j+e_n}(y_0) + \sum_{\beta+\gamma=\alpha, \beta'>0} c^{\beta}(y_0) v^{\gamma}(y_0) = 0.$$

Therefore $(\Lambda + c_0)v_0 = O(|y|^{N'})$ and we denote this fact by $(\Lambda + c_0)v_0 \sim 0$.

Secondly we shall determine $v_1, v_2, \ldots, v_N$ which satisfy

$$\begin{cases} (\Lambda + c_0)v_1 + Rv_0 = O(|y|^{N'}), \\ (\Lambda + c_0)v_2 + Rv_1 + R_0v_0 = O(|y|^{N'}), \\ \ldots, \\ (\Lambda + c_0)v_N + Rv_{N-1} + \cdots + R_{N-2}v_0 = O(|y|^{N'}) \end{cases}$$

by the induction. Let f be a formal power series in y, then the equation

$$(\Lambda + c_0)v = f$$

is solvable in the formal power series in y if and only if $f(y_0, 0, y'') = 0$ and we call this the solvability condition.

Assume that we have obtained $v_0, v_1, \ldots, v_\ell$ such that

$$\begin{cases} (\Lambda + c_0)v_0 = O(|y|^{N'}) \\ (\Lambda + c_0)v_1 + Rv_0 = O(|y|^{N'}) \\ \cdots \\ (\Lambda + c_0)v_\ell + Rv_{\ell-1} + \cdots + R_{\ell-2}v_0 = O(|y|^{N'}). \end{cases}$$

In order to determine $v_{\ell+1}$ such that

$$(\Lambda + c_0)v_{\ell+1} + Rv_\ell + \cdots + R_{\ell-1}v_0 = O(|y|^{N'}),$$

we modify v_ℓ slightly to satisfy the solvability condition.

In fact, let w be $(\Lambda + c_0)w \sim 0$, then $v_0, v_1, \ldots, v_\ell + w$ satisfy the above assumptions, and the solvability condition is

$$R(v_\ell + w) + R_0 v_{\ell-1} + \cdots + R_{\ell-1}v_0 = 0 \quad on\ y' = 0$$

and R can be rewritten

$$R = \partial_0 + a(y_0, y'') + \sum_{j=1}^{m} y_j r_j(y, \partial_y).$$

Now we choose w such that $v_0, v_1, \ldots, v_\ell + w$ satisfy the solvability condition. Let set $w(y) = \sum_{\alpha'} w^{\alpha'}(y_0, y'')y'^{\alpha'}$, then

$$Rw|_{y'=0} = \left(\partial_0 + a(y_0, y'') + \sum_{j=1}^{m} y_j r_j(y, \partial_y)\right) \sum_{\alpha'} w^{\alpha'}(y_0, y'')y'^{\alpha'}|_{y'=0}$$

$$= \partial_0 w^{0'}(y_0, y'') + a(y_0, y'')w^{0'}(y_0, y'').$$

On the other hand, the solvability condition is

$$Rw|_{y'=0} = -\left(Rv_\ell + \sum_{j=0}^{\ell-1} R_j v_{\ell-j-1}\right)\Bigg|_{y'=0}$$

and set $Rw = -(Rv_\ell + \sum_{j=0}^{\ell-1} R_j v_{\ell-j-1}) = g(y_0, y'') + O(|y'|)$. Then the solvability condition becomes

$$(\partial_0 + a(y_0, y''))w^{0'}(y_0, y'') = g(y_0, y'').$$

Let $w^*(y_0, y'')$ be a solution of this equation, then we can obtain the solution of the equation $(\Lambda + c_o)w \sim 0$ in the form $w(y_0, y) = w^*(y_0, y'') + \sum_{\alpha'>0} w^{\alpha'}(y_0, y'')y'^{\alpha'}$. That is, this $w(y_0, y)$ satisfies the solvability condition with $v_0, \ldots, v_\ell$ and $(\Lambda + c_0)w = O(|y|^{N'})$.

Hence we can determine $v_{\ell+1}$ so that

$$(\Lambda + c_0)v_{\ell+1} + Rv_\ell + \cdots + R_{\ell-1}v_0 = O(|y|^{N'}).$$

4. Estimates

We set

$$\begin{aligned}E_\rho^{-1}P_\rho E_\rho v_\rho^N &= \rho^{s+2}(\Lambda+c_0)v_0+\rho^{s+1}((\Lambda+c_0)v_1+Rv_o)+\cdots\\&\quad+\rho^{s+2-N}((\Lambda+c_0)v_N+Rv_{N-1}+\cdots+R_{N-2}v_0)\\&\quad+\rho^{s+1-N}(Rv_N+R_0v_{N-1}+\cdots)+\cdots.\\&=\rho^{s+2}f_0(y_0,y)+\rho^{s+1}f_1(y_0,y)+\cdots\\&\quad+\rho^{s+2-N}f_N(y_0,y)+\rho^{s+1-N}f(y_0,y;\rho)\end{aligned}$$

and take C^∞ function with small support χ whose value is 1 near the origin. Then

$$\begin{aligned}P_\rho\chi E_\rho v_\rho^N &= \chi E_\rho[\rho^{s+2}f_0(y_0,y)+\rho^{s+1}f_1(y_0,y)+\cdots+\rho^{s+2-N}f_N(y_0,y)\\&\quad+\rho^{s+1-N}f(y_0,y;\rho)]+[P_\rho,\chi]E_\rho v_\rho^N,\end{aligned}$$

where $[P_\rho,\chi]$ denotes the commutator of P_ρ and χ.

Now we estimate $\|(\chi E_\rho v_\rho^N)(0,\cdot)\|_0$ from below. We have

$$\|(\chi E_\rho v_\rho^N)(0,\cdot)\|_0\geq\delta\rho^{-(m+n)/4}\quad\textit{for large }\rho.$$

Concerning the right hand side, we have

$$\int_{-\infty}^0\|\chi E_\rho f_i(y_0,\cdot)\|_q\,dy_0\leq\frac{C}{(-\mathrm{Im}\,\gamma)}\rho^{2q-N'/2}\quad(i=1,2,\dots,N)$$

and

$$\int_{-\infty}^0\|\chi E_\rho f(y_0,:\rho)\|_q\,dy_0\leq\frac{C}{(-\mathrm{Im}\,\gamma)}\rho^{2q}$$

further since $(\mathrm{Im}\,\gamma)y_0+|y|^2\geq\delta'(>0)$ on $[P_\rho,\chi]E_\rho v_\rho^N\neq 0$, we have

$$\int_{-\infty}^0\|([P_\rho,\chi]E_\rho v_\rho^N)(y_0,\cdot)\|_q\,dy_0\leq Ce^{-\delta_0\rho}.$$

From the well-posedness, the inequality

$$\delta\rho^{-(m+n)/4-(s+1)q}\leq\frac{C}{(-\mathrm{Im}\,\gamma)}(\rho^{s+2+2q-N}+\rho^{s+2+2q-N'/2})+Ce^{-\delta_0\rho}$$

holds for any large ρ and any integers N and N', and this is the contradiction.

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