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SMOOTHLY SYMMETRIZABLE SYSTEMS AND THE REDUCED DIMENSIONS II

By

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1. Introduction

Let L be a first order system

$$L(x, D) = \sum_{j=1}^{n} A_j(x) D_j$$

where $A_1 = I$ is the identity matrix of order m and $A_j(x)$ are $m \times m$ matrix valued smooth functions. In this note we continue the study [1] on the question when we can symmetrize $L(x, D)$ smoothly. In particular we discuss some connections between the symmetrizability of $L(x, D)$ at every frozen x and the smooth symmetrizability. Let $L(x, \xi)$ be the symbol of $L(x, D)$:

$$L(x, \xi) = \sum_{j=1}^{n} A_j(x) \xi_j = (\phi_j^i(x, \xi))_{i,j=1}^{m}$$

where $\phi_j^i(x, \xi)$ stands for the (i, j)-th entry of $L(x, \xi)$ which is linear form in ξ. Recall that

$$d(L(x, \cdot)) = \dim \operatorname{span}\{\phi_j^i(x, \cdot)\}$$

is called the reduced dimension of L at x. This is nothing but the dimension of the linear subspace of $M(m; \mathbf{R})$, the space of all real $m \times m$ matrices, spanned by $A_1(x), \ldots, A_n(x)$.

Our aim in this note is to prove

THEOREM 1.1. *Assume that $L(x, \xi)$ is symmetrizable at every x near $\bar{x}$, that is there exists a non singular matrix $S(x)$ which is possibly non smooth in x such that*

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$S(x)^{-1}L(x,\xi)S(x)$ is symmetric for every ξ and the reduced dimension of $L(\bar{x},\cdot) \geq m(m+1)/2 - [m/2]$ and $m \geq 3$. Then $L(x,\xi)$ is smoothly symmetrizable near $\bar{x}$, that is there is a smooth non singular matrix $T(x)$ defined near $\bar{x}$ such that

$$T(x)^{-1}L(x,\xi)T(x)$$

is symmetric for any ξ and any x near $\bar{x}$.

In the series of papers [2], [3], [4] and [5] the second author proved that if $L(D)$ is strongly hyperbolic and the reduced dimension of $L(\cdot) \geq m(m+1)/2 - 2$ then there exists a constant matrix S such that $S^{-1}L(\xi)S$ is symmetric for every ξ. Combining with the above theorem we conclude that the strong hyperbolicity of $L(x,D)$ at every frozen x implies the strong hyperbolicity of $L(x,D)$ if the reduced dimension of $L(x,\cdot) \geq m(m+1)/2 - 2$. This result, when the reduced dimension of $L(x,\cdot) \geq m(m+1)/2 - 1$, was proved in our previous paper [1].

2. A Lemma

Recall that $L(x,\xi) = (\phi_j^i(x,\xi))_{i,j=1}^m$ where i and j denotes i-th row and j-th column respectively.

LEMMA 2.1. *Assume that there exist two rows, say p-th and q-th rows such that $\phi_j^p(\bar{x},\cdot)$, $1 \leq j \leq m$, $\phi_i^q(\bar{x},\cdot)$, $1 \leq i \leq m$, $i \neq p$ are linearly independent and for every x we can find a positive definite $H(x)$ such that*

$$L(x,\xi)H(x) = H(x)^t L(x,\xi). \tag{2.1}$$

Then $H(x)/h_p^p(x)$ is smooth near $\bar{x}$ where we have denoted $H(x) = (h_j^i(x))$.

PROOF. Since $h_p^p(x) > 0$ then $H(x)/h_p^p(x)$ is again positive definite and verifies (2.1). We denote $H(x)/h_p^p(x)$ by $H(x)$ again. Let us consider the (p,j)-th entry of the equation (2.1):

$$\sum_{k=1}^m \phi_k^p(x,\xi)h_j^k(x) - \sum_{k=1}^m \phi_k^j(x,\xi)h_k^p(x) = 0. \tag{2.2}$$

Take $j = q$ then we get

$$\sum_{k=1}^m \phi_k^p(x,\xi)h_q^k(x) - \sum_{k=1,k\neq p}^m \phi_k^q(x,\xi)h_k^p(x) = \phi_p^q(x,\xi)$$

because $h_p^p(x) = 1$. To simplify notations let us write

$$\{\phi_k^p, 1 \le k \le m, \phi_j^q, 1 \le j \le m, j \ne p\} = \{\theta_j \mid 1 \le j \le 2m-1\}$$

$$\{h_q^k, 1 \le k \le m, h_j^p, 1 \le j \le m, j \ne p\} = \{y_j \mid 1 \le j \le 2m-1\}.$$

Since $\theta_i(\bar{x}, \cdot)$ are linearly independent, with

$$\theta_i(x, \xi) = \sum_{k=1}^{n} C_k^i(x)\xi_k$$

one can find $j_1 < \cdots < j_{2m-1}$ so that

$$\det(C_{j_k}^i(x))_{i,k=1}^{2m-1} \ne 0$$

which holds near $\bar{x}$. Then solving the equation

$$\sum_{i=1}^{m-1} C_{j_k}^i(x) y_i(x) = \text{smooth}, \quad k = 1, 2, \dots, 2m-1$$

we conclude that $y_i(x)$ are smooth near $\bar{x}$.

We next study (2.2) with j ($\ne q$):

$$\sum_{k=1}^{m} \phi_k^p(x, \xi) h_j^k(x) = \sum_{k=1}^{m} \phi_k^j(x, \xi) h_k^p(x).$$

Since $h_k^p(x)$, $1 \le k \le m$ are smooth near $\bar{x}$, applying the same arguments as above we conclude that $h_j^1(x), \dots, h_j^m(x)$ are smooth near $\bar{x}$ because $\phi_k^p(\bar{x}, \cdot)$, $1 \le k \le m$ are linearly independent. This shows that $H(x)$ is smooth near $\bar{x}$ and hence the result. □

3. A Special Case

Let us denote $J = \{(i, j) \mid i > j\}$ and $\bar{J} = \{(i, j) \mid i \ge j\}$. We show

Proposition 3.1. *Let $m = 4$ and $d(L(\bar{x}, \cdot)) = 8$. Assume that $L(\bar{x}, \xi)$ is symmetric and for every x near $\bar{x}$ there is a positive definite $H(x)$ such that*

$$L(x, \xi)H(x) = H(x)^t L(x, \xi).$$

Then there is p such that $H(x)/h_p^p(x)$ is smooth near $\bar{x}$.

Proof. We first note that for any permutation matrix P, $P^{-1}L(x, \xi)P$ verifies the hypothesis with $H(x)$ replaced by $P^{-1}H(x)P$ and if the statement holds for $P^{-1}H(x)P$ then so does for $H(x)$. Let us denote by $E(i, j)$ the matrix obtained from the zero matrix by replacing the (i,j) entry by 1. Then for a permutation matrix P we define the index $(i,j)^P$ by

$$P^{-1}E(i,j)P = E((i,j)^P).$$

Let K be a subset of indices (i,j) then we denote

$$K_P = \{(i,j)^P \mid (i,j) \in K\}.$$

We devide the cases into three according to the dimension of E:

$$E = \operatorname{span}\{\phi_j^i(\bar{x},\cdot) \mid i > j\}.$$

Note that $4 \le \dim E \le 6$ by our assumption.

I) $\dim E = 6$. This shows that there are two μ, ν such that $\phi_\mu^\mu(\bar{x},\cdot)$ and $\phi_\nu^\nu(\bar{x},\cdot)$ are linear combinations of the other $\phi_j^i(\bar{x},\cdot)$, $(i,j) \in \bar{J}\backslash\{(\mu,\mu),(\nu,\nu)\}$ which are linearly independent. The two rows which contains neither ϕ_μ^μ nor ϕ_ν^ν verify the hypothesis of Lemma 2.1 and hence we have the assertion thanks to Lemma 2.1.

II) $\dim E = 4$. By the assumption there are $(p,q), (\tilde{p},\tilde{q}) \in J$ such that $\phi_q^p(\bar{x},\cdot)$ and $\phi_{\tilde{q}}^{\tilde{p}}(\bar{x},\cdot)$ are linear combinations of $\phi_j^i(\bar{x},\cdot)$, $(i,j) \in J\backslash\{(p,q),(\tilde{p},\tilde{q})\} = J\backslash K$ where we have set

$$K = \{(p,q),(\tilde{p},\tilde{q})\}.$$

Taking a suitable permutation matrix P we may assume that $(2,1) \in K_P$. We drop the suffix P in K_P. We still devide the cases into two:

II)$_a$ the other entry of K is on the third row

II)$_b$ the other entry of K is on the last row.

Assume II)$_a$. Then either $K = \{(2,1),(3,1)\}$ or $\{(2,1),(3,2)\}$. Recall that

$$L(x,\xi)H(x) = H(x)\,{}^t L(x,\xi). \tag{3.1}$$

Dividing $H(x)$ by $h_4^4(x)$ which is positive we may suppose that $h_4^4(x) = 1$ in (3.1). Let us put

$$\hat{H}(x) = {}^t(h_1^1(x), h_2^2(x), h_3^3(x), h_2^1(x), h_3^1(x), h_4^1(x), h_3^2(x), h_4^2(x), h_4^3(x)).$$

Equating the $(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)$-th entries in both sides of (3.1) in this order, we get

$$\hat{L}(x,\xi)\hat{H}(x) = \hat{F}(x,\xi) \tag{3.2}$$

where $\hat{L}(x,\xi)$ is a 6×9 matrix and

$$\hat{F}(x,\xi) = {}^t(0,0,-\phi_4^1(x,\xi),0,-\phi_4^2(x,\xi),-\phi_4^3(x,\xi)).$$

We choose $\xi^{(1)}$ so that

$$\phi_1^1(\bar{x},\xi^{(1)}) = 1, \quad \phi_j^i(\bar{x},\xi^{(1)}) = 0, \quad \forall(i,j) \notin K,\ (i,j) \neq (1,1),\ i \geq j.$$

Note that we have

$$\phi_j^i(\bar{x},\xi^{(1)}) = 0, \quad \forall(i,j) \neq (1,1) \tag{3.3}$$

because for $(i,j) \in K$, $\phi_j^i(\bar{x},\cdot)$ is a linear combination of $\phi_j^i(\bar{x},\cdot)$, $i > j$, $(i,j) \notin K$ and $L(\bar{x},\cdot)$ is symmetric. We take the first three equations in (3.2) with $\xi = \xi^{(1)}$. We next choose $\xi^{(2)}$ so that

$$\phi_2^2(\bar{x},\xi^{(2)}) = 1, \quad \phi_j^i(\bar{x},\xi^{(2)}) = 0, \quad \forall(i,j) \notin K,\ i \geq j,\ (i,j) \neq (2,2)$$

and take 4-th and 5-th equations of (3.2) with $\xi = \xi^{(2)}$. Choose $\xi^{(3)}$ so that

$$\phi_3^3(\bar{x},\xi^{(3)}) = 1, \quad \phi_j^i(\bar{x},\xi^{(3)}) = 0, \quad \forall(i,j) \notin K,\ i \geq j,\ (i,j) \neq (3,3)$$

and take the 6-th equation of (3.2) with $\xi = \xi^{(3)}$. We choose $\xi^{(4)}, \xi^{(5)}, \xi^{(6)}$ so that

$$\phi_j^4(\bar{x},\xi^{(3+j)}) = 1, \quad \phi_\nu^\mu(\bar{x},\xi^{(3+j)}) = 0, \quad \forall(\mu,\nu) \notin K,\ \mu > \nu$$

where $j = 1,2,3$ and take 3-rd, 5-th and 6-th equations of (3.2) with $\xi = \xi^{(4)}$, $\xi^{(5)}, \xi^{(6)}$ respectively. Collecting these nine equations we get

$$M(x)\hat{H}(x) = G(x) \tag{3.4}$$

where

$$G(x) = -{}^t(0,0,\phi_4^1(x,\xi^{(1)}),0,\phi_4^2(x,\xi^{(2)}),\phi_4^3(x,\xi^{(3)}),\phi_4^1(x,\xi^{(4)}),\phi_4^2(x,\xi^{(5)}),\phi_4^3(x,\xi^{(6)}))$$

and $M(x)$ is a 9×9 matrix. It is easy to see that

$$M(\bar{x}) = \left(\begin{array}{ccc|ccccc} & & & 1 & & & 0 \\ & O & & 0 & 1 & & \\ & & & & & \ddots & \\ & & & 0 & & & 1 \\ \hline -1 & 0 & 0 & & & & \\ 0 & -1 & 0 & & * & & \\ 0 & 0 & -1 & & & & \end{array}\right).$$

Then $M(\bar{x})$ is non singular and hence near $\bar{x}$ there is a smooth inverse of $M(x)$ and hence

$$\hat{H}(x) = M(x)^{-1}G(x)$$

which proves the assertion.

We turn to the case II)$_b$. If the entry on the last row is $(4,j) \neq (4,3)$ then by $P^{-1}L(x,\xi)P$ with a suitable permutation matrix this case is reduced to the case II)$_a$. Thus we may assume that the reference entry of K is $(4,3)$. We choose the same $\xi^{(1)},\dots,\xi^{(5)}$ and the same eight equations of (3.2) with $\xi = \xi^{(1)},\dots,\xi^{(5)}$ as in the case II)$_a$. Choose $\xi^{(6)}$ so that

$$\phi_1^3(\bar{x},\xi^{(6)}) = 1, \quad \phi_j^i(\bar{x},\xi^{(6)}) = 0, \quad \forall (i,j) \notin K,\ (i,j) \neq (3,1),\ i > j$$

and take the 2-nd equation of (3.2) with $\xi = \xi^{(6)}$. Then $M(x)$ in (3.4) at $\bar{x}$ yields

$$M(\bar{x}) = \left(\begin{array}{ccc|cccc} & & & 1 & & & 0 \\ & O & & 0 & 1 & & \\ & & & & & \ddots & \\ & & & 0 & & & 1 \\ \hline -1 & 0 & 0 & & & & \\ 0 & -1 & 0 & & & * & \\ -1 & 0 & 1 & & & & \end{array}\right).$$

This is invertible and we get the desired assertion.

III) $\dim E = 5$. By the assumption there is (i_0, j_0), $i_0 > j_0$ such that $\phi_{j_0}^{i_0}(\bar{x},\cdot)$ is a linear combination of $\phi_j^i(\bar{x},\cdot)$, $(i,j) \neq (i_0,j_0)$, $i > j$ and there is s such that $\phi_s^s(\bar{x},\cdot)$ is a linear combination of $\phi_j^i(\bar{x},\cdot)$, $i \geq j$, $(i,j) \neq (s,s),(i_0,j_0)$. Let us set

$$K = \{(s,s),(i_0,j_0)\}.$$

Considering $P^{-1}L(x,\xi)P$ with a suitable permutation matrix we may assume that $(1,1) \in K$. Again taking $P^{-1}L(x,\xi)P$ we may suppose that either $K = \{(1,1),(2,1)\}$ or $K = \{(1,1),(3,2)\}$. Note that at least two of

$$(\phi_1^1 - \phi_2^2)(\bar{x},\cdot), \quad (\phi_1^1 - \phi_3^3)(\bar{x},\cdot), \quad (\phi_1^1 - \phi_4^4)(\bar{x},\cdot)$$

are linearly independent when $\phi_j^i(\bar{x},\cdot) = 0$, $i > j$, $(i,j) \notin K$ by the assumption. Let us assume that $(\phi_1^1 - \phi_3^3)(\bar{x},\cdot)$, $(\phi_1^1 - \phi_4^4)(\bar{x},\cdot)$, $\phi_j^i(\bar{x},\cdot)$, $i > j$, $(i,j) \notin K$ are linearly independent. We choose $\xi^{(8)},\xi^{(9)}$ so that

$$(\phi_1^1 - \phi_3^3)(\bar{x}, \xi^{(8)}) = 1, \quad \phi_j^i(\bar{x}, \xi^{(8)}) = 0, \quad \forall (i,j) \notin K,\ i > j$$
$$(\phi_1^1 - \phi_4^4)(\bar{x}, \xi^{(9)}) = 1, \quad \phi_j^i(\bar{x}, \xi^{(9)}) = 0, \quad \forall (i,j) \notin K,\ i > j$$

and take the second and third equations of (3.2) with $\xi = \xi^{(8)}, \xi^{(9)}$. Choose the same $\xi^{(2)}, \xi^{(3)}, \xi^{(4)}, \xi^{(5)}, \xi^{(6)}$ and the same equations as before, that is 4-th, 5-th of (3.2) with $\xi = \xi^{(2)}$, 6-th of (3.2) with $\xi = \xi^{(3)}$, 3-rd, 5-th, 6-th of (3.2) with $\xi = \xi^{(4)}, \xi^{(5)}, \xi^{(6)}$ respectively. Finally we choose $\xi^{(7)}$ so that

$$\phi_2^4(\bar{x}, \xi^{(7)}) = 1, \quad \phi_j^i(\bar{x}, \xi^{(7)}) = 0, \quad \forall (i,j) \notin K,\ i > j,\ (i,j) \neq (4,2)$$

and take the third equation of (3.2) with $\xi = \xi^{(7)}$. Then we get the equation

$$M(x)\hat{H}(x) = G(x) \tag{3.5}$$

where $G(x)$ is

$$-{}^t(0, \phi_4^1(x, \xi^{(9)}), 0, \phi_4^2(x, \xi^{(2)}), \phi_4^3(x, \xi^{(3)}), \phi_4^1(x, \xi^{(4)}), \phi_4^2(x, \xi^{(5)}), \phi_4^3(x, \xi^{(6)}), \phi_4^1(x, \xi^{(7)})).$$

It is easy to see that

$$M(\bar{x}) = \left(\begin{array}{cccc:ccccc} & & & & 1 & & & & 0 \\ & O & & & 0 & 1 & & & \\ & & & & & & \ddots & & \\ & & & & 0 & & & & 1 \\ \hdashline -1 & 0 & 0 & 0 & & & & & \\ 0 & -1 & 0 & 0 & & & * & & \\ 0 & 0 & -1 & 0 & & & & & \\ 0 & 0 & 0 & -1 & & & & & \end{array}\right)$$

which is non singular. Thus we get the desired assertion. The remaining case can be proved by the same arguments. □

4. Proof of Theorem

We first show the next lemma.

LEMMA 4.1. *Let $m \geq 3$. Assume that $L(\bar{x}, \xi)$ is symmetric $m \times m$ matrix with*

$$d(L(\bar{x}, \cdot)) \geq \frac{m(m+1)}{2} - \left[\frac{m}{2}\right].$$

and for every x near $\bar{x}$ there is a positive definite $H(x)$ such that

$$L(x,\xi)H(x) = H(x){}^tL(x,\xi). \tag{4.1}$$

Then there is a $1 \le p \le m$ such that $H(x)/h_p^p(x)$ is smooth near $\bar{x}$.

PROOF. We prove this lemma by induction on the size of the matrix $L(x,\xi)$. When $m = 3$ or $m = 4$ with $d(\bar{x},\cdot) \ge 9$, the assertion was proved in our previous paper [1] (see the proof of Theorem 1.1 in [1]) and the case $m = 4$ with $d(\bar{x},\cdot) = 8$ is just Proposition 3.1. Suppose that the assertion holds for $L(x,\xi)$ of size at most $m-1$ with $m \ge 5$. Let

$$\left[\frac{m}{2}\right] = k$$

so that $m = 2k$ or $m = 2k+1$. We devide the cases into two.

Case I:

$$\dim \operatorname{span}\{\phi_j^i(\bar{x},\cdot) \mid i > j\} = \frac{m(m+1)}{2} - m - k,$$

and

Case II:

$$\dim \operatorname{span}\{\phi_j^i(\bar{x},\cdot) \mid i > j\} \ge \frac{m(m+1)}{2} - m - k + 1.$$

We first treat Case I. We denote by K the set of indices (i,j), $i > j$ such that $\phi_j^i(\bar{x},\cdot)$, $(i,j) \in K$ are linear combinations of the other $m(m+1)/2 - m - k$ entries $\phi_j^i(\bar{x},\cdot)$, $i > j$ which are linearly independent. By the assumption, $\phi_j^i(\bar{x},\cdot)$, $i \ge j$, $(i,j) \notin K$ are linearly independent. Considering $P^{-1}L(x,\xi)P$ with a suitable permutation matrix P, we may assume that $(2,1) \in K_P$. As before we drop the suffix P in K_P. We further devide Case I into two cases: we first assume that K contains no (i,j) with $i \ge 3$, $j = 1,2$.

Write

$$L(x,\xi) = \begin{pmatrix} L_{11}(x,\xi) & L_{12}(x,\xi) \\ L_{21}(x,\xi) & L_{22}(x,\xi) \end{pmatrix} \tag{4.2}$$

where $L_{22}(x,\xi)$ is the $(m-2)\times(m-2)$ submatrix consisting of the last $(m-2)$ rows and the last $(m-2)$ columns of $L(x,\xi)$. Let

$$H(x) = \begin{pmatrix} H_{11}(x) & H_{12}(x) \\ H_{21}(x) & H_{22}(x) \end{pmatrix}$$

where the blocking corresponds to that of (4.2). Then (4.1) is written as

$$L_{21}H_{12} + L_{22}H_{22} = H_{21}{}^tL_{21} + H_{22}{}^tL_{22} \tag{4.3}$$

$$L_{21}H_{11} + L_{22}H_{21} = H_{21}{}^tL_{11} + H_{22}{}^tL_{12}. \tag{4.4}$$

Since $\phi_j^i(\bar{x}, \cdot)$, $i \geq 3$, $j = 1, 2$ are linearly independent, near $\bar{x}$ one can solve $L_{21}(x, \xi) = 0$ so that $\xi_b = (\xi_{i_1}, \ldots, \xi_{i_N})$, $N = 2(m-2)$ are linear combinations of the other $\xi_a = (\xi_{j_1}, \ldots, \xi_{j_M})$ with coefficients which are smooth functions of x where $\xi = (\xi_a, \xi_b)$ is some partition of the variables ξ. Substituting these ξ_b into $L(x, \xi)$ the equation (4.3) becomes

$$L_{22}(x, \xi_a)H_{22}(x) = H_{22}(x)^tL_{22}(x, \xi_a). \tag{4.5}$$

Note that

$$\begin{aligned} d(L_{22}(\bar{x}, \cdot)) &\geq \frac{(m-2)(m-1)}{2} - (k-1) \\ &\geq \frac{(m-2)(m-1)}{2} - \left[\frac{m-2}{2}\right] \end{aligned}$$

and $H_{22}(x)$ is positive definite. By the induction hypothesis there is $h_i^i(x)$, $3 \leq i \leq m$ such that $H_{22}(x)/h_i^i(x)$ is smooth near $\bar{x}$. Then denoting $H(x)/h_i^i(x)$ by $\tilde{H}(x)$ we have (4.3) and (4.4) for $\tilde{H}(x)$ where $\tilde{H}_{22}(x)$ is smooth. Solve

$$\phi_j^i(x, \xi) = 0, \quad \forall (i, j) \notin K, \, i > j$$

which gives $\xi_b = f(x, \xi_a)$, with a partition of the ξ variables $\xi = (\xi_a, \xi_b)$ as above, where $f(x, \xi_a)$ is linear in ξ_a with smooth coefficients in x. Substituting this relation into (4.4) we get

$$L_{22}(x, \xi_a)\tilde{H}_{21}(x) - \tilde{H}_{21}(x)^tL_{11}(x, \xi_a) = (g_j^i(x, \xi_a)) \tag{4.6}$$

where $g_j^i(x)$ are smooth. Note that

$$L_{22}(\bar{x}, \xi_a)\tilde{H}_{21} - \tilde{H}_{21}{}^tL_{11}(\bar{x}, \xi_a) = 0$$

implies that

$$[\phi_j^j(\bar{x}, \xi_a) - \phi_k^k(\bar{x}, \xi_a)]\tilde{h}_k^j = 0, \quad k = 1, 2, \, j \geq 3$$

because $\phi_j^i(\bar{x}, \xi_a) = 0$ if $i \neq j$ and hence $\tilde{H}_{21} = 0$. This proves that the coefficient matrix of the linear equation (4.6) is non singular at $\bar{x}$. Thus (4.6) is smoothly invertible and we conclude that $\tilde{H}_{21}(x)$ is smooth near $\bar{x}$. We finally study $\tilde{H}_{11}(x)$. Considering $(1,2)$-th, $(3,2)$-th and $(3,1)$-th entries of (4.1) we get

$$\begin{pmatrix} -\phi_1^2 & \phi_2^1 & \phi_1^1 - \phi_2^2 \\ 0 & \phi_2^3 & \phi_1^3 \\ \phi_1^3 & 0 & \phi_2^3 \end{pmatrix} \begin{pmatrix} \tilde{h}_1^1 \\ \tilde{h}_2^1 \\ \tilde{h}_2^2 \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} \tag{4.7}$$

where g_j are known to be smooth near $\bar{x}$. Take $\bar{\xi}$ so that $\phi_1^3(\bar{x}, \bar{\xi}) = \phi_2^3(\bar{x}, \bar{\xi}) \neq 0$ and $\phi_2^2(\bar{x}, \bar{\xi}) - \phi_1^1(\bar{x}, \bar{\xi}) \neq 0$ and consider the equation (4.7) with $\xi = \bar{\xi}$. Then one sees that the determinant of the coefficient matrix at $\bar{x}$ is

$$[\phi_2^2(\bar{x}, \bar{\xi}) - \phi_1^1(\bar{x}, \bar{\xi})]\phi_1^3(\bar{x}, \bar{\xi})^2 \neq 0$$

so that we can conclude that $\tilde{h}_1^1(x), \tilde{h}_2^1(x)$ and $\tilde{h}_2^2(x)$ are smooth near $\bar{x}$. This proves the assertion.

We turn to the second case that K contains (i,j) with $i \geq 3$, $1 \leq j \leq 2$. Let us consider the set

$$\check{K} = \{(i,j) \mid (i,j) \in K \text{ or } (j,i) \in K\}.$$

Assume that K contains more than two such entries then it is clear that

$$\#(\check{K} \cap \{\text{the first } 2 \text{ rows}\}) \geq 4$$

and this implies that

$$\#(\check{K} \cap \{\text{the last } m-2 \text{ rows}\}) \leq 2k - 4 \leq m - 4.$$

Hence, among the last $m-2$ rows, we can choose two rows which verify the hypothesis of Lemma 2.1. Then one can apply Lemma 2.1 to conclude the assertion. Thus we may assume that K contains only one such (i,j).

Considering $P^{-1}L(x,\xi)P$ with a suitable permutation matrix P we may assume that either $K \supset \{(2,1),(3,1)\}$ or $K \supset \{(2,1),(3,2)\}$. We show that there is a p-th row with $p \geq 4$ such that

$$\check{K} \cap \{p\text{-th row}\} = \emptyset.$$

If not we would have

$$\#(\check{K}) \geq 4 + (m-3) = m + 1 \geq 2k + 1$$

since $\check{K}$ has at least 4 entries in the first three rows. This is a contradiction because $\#(\check{K}) \leq 2k$. Again considering $P^{-1}L(x,\xi)P$ we may assume that $\check{K} \cap \{4\text{-th row}\} = \emptyset$. Denote

$$L = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}$$

where L_{22} is the $(m-3) \times (m-3)$ submatrix consisting of the last $(m-3)$ rows and columns of $L(x, \xi)$. We may assume that K contains no (i, j) with $i \geq 4$, $1 \leq j \leq 3$. If not we have at least 5 entries of $\check{K}$ on the first three rows and hence

$$\#(\check{K} \cap \{\text{the last } m-3 \text{ rows}\}) \leq 2k - 5 \leq m - 5.$$

Thus one can choose two rows among the last $m-3$ rows which verify the hypothesis of Lemma 2.1. Applying Lemma 2.1 we get the desired assertion.

Solving $L_{21}(x, \xi) = 0$ we apply the same arguments as above. Note that

$$\begin{aligned} d(L_{22}(\bar{x}, \cdot)) &\geq \frac{(m-3)(m-2)}{2} - (k-2) \\ &\geq \frac{(m-3)(m-2)}{2} - \left\lceil \frac{m-3}{2} \right\rceil \end{aligned}$$

since K contains 2 entries in lower diagonal part of $L_{11}(\bar{x}, \cdot)$. If $m \geq 6$ then from the induction hypothesis we conclude that there is $i \geq 4$ such that $H_{22}/h_i^i(x)$ is smooth near $\bar{x}$. If $m = 5$ and hence $k = 2$ then the existence of such i follows from Theorem 1.1 in [1] or rather its proof. Denote $H(x)/h_i^i(x)$ by the same $H(x)$. It remains to show that $H_{11}(x)$ and $H_{21}(x)$ are smooth near $\bar{x}$. Recall the equation

$$L_{21}H_{11} + L_{22}H_{21} = H_{21}{}^tL_{11} + H_{22}{}^tL_{12}. \tag{4.8}$$

Solving again $\phi_j^i(x, \xi) = 0$, $\forall(i, j) \notin K$, $i > j$, the equation (4.8) becomes

$$L_{22}(x, \xi_a)H_{21}(x) - H_{21}(x){}^tL_{11}(x, \xi_a) = (g_j^i(x, \xi_a))$$

where the right-hand side is known to be smooth in x near $\bar{x}$ and $\xi = (\xi_a, \xi_b)$ is some partition of the variables ξ. Note that this equation turns out at $x = \bar{x}$

$$\begin{pmatrix} (\phi_j^j - \phi_1^1)(\bar{x}, \xi_a) & 0 & 0 \\ 0 & (\phi_j^j - \phi_2^2)(\bar{x}, \xi_a) & 0 \\ 0 & 0 & (\phi_j^j - \phi_3^3)(\bar{x}, \xi_a) \end{pmatrix} \begin{pmatrix} h_1^j \\ h_2^j \\ h_3^j \end{pmatrix} = \text{smooth} \tag{4.9}$$

because $\phi_1^2(\bar{x}, \xi_a) = 0$, $\phi_1^3(\bar{x}, \xi_a) = 0$, $\phi_2^3(\bar{x}, \xi_a) = 0$ and $L(\bar{x}, \cdot)$ is symmeric where $j \geq 4$. We choose $\bar{\xi}_a$ so that

$$(\phi_j^j - \phi_k^k)(\bar{x}, \bar{\xi}_a) \neq 0, \quad k = 1, 2, 3, \; j \geq 4$$

and study (4.8) with $\xi_a = \bar{\xi}_a$ fixed. Then (4.9) shows that the coefficient matrix of the equation at $x = \bar{x}$ is non singular and hence we conclude that $H_{21}(x)$ is smooth near $\bar{x}$. We turn to the equation for $H_{11}(x)$. These can be written as

$$(4.10)\quad \begin{pmatrix} -\phi_1^2 & \phi_2^1 & 0 & \phi_1^1-\phi_2^2 & -\phi_3^2 & \phi_3^1 \\ -\phi_1^3 & 0 & \phi_3^1 & -\phi_2^3 & \phi_1^1-\phi_3^3 & \phi_2^1 \\ 0 & -\phi_2^3 & \phi_3^2 & -\phi_1^3 & \phi_1^2 & \phi_2^2-\phi_3^3 \\ \phi_1^4 & 0 & 0 & \phi_2^4 & \phi_3^4 & 0 \\ 0 & \phi_2^4 & 0 & \phi_1^4 & 0 & \phi_3^4 \\ 0 & 0 & \phi_3^4 & 0 & \phi_1^4 & \phi_2^4 \end{pmatrix} \begin{pmatrix} h_1^1 \\ h_2^2 \\ h_3^3 \\ h_2^1 \\ h_3^1 \\ h_3^2 \end{pmatrix} = \text{smooth}.$$

Here we have equated the $(1,2),(1,3),(2,3),(1,4),(2,4),(3,4)$-th entries in both sides of (4.8) in this order. Choose $\bar{\xi}$ so that $\phi_k^4(\bar{x},\bar{\xi}) = 1$, $k = 1,2,3$ and

$$\phi_j^i(\bar{x},\bar{\xi}) = 0, \quad (i,j) \notin K, \quad (i,j) \neq (4,k), \quad k = 1,2,3,\ i > j$$

and $(\phi_1^1 - \phi_2^2)(\bar{x},\bar{\xi})$, $(\phi_1^1 - \phi_3^3)(\bar{x},\bar{\xi})$, $(\phi_2^2 - \phi_3^3)(\bar{x},\bar{\xi})$ are large enough. Let us study (4.10) with $\xi = \bar{\xi}$. Then it is clear that the coefficient matrix of the equation thus obtained is non singular at $x = \bar{x}$ and hence we conclude that $H_{11}(x)$ is smooth near $\bar{x}$.

We now study Case II. We show that we may assume that

$$(4.11)\qquad \dim \operatorname{span}\{\phi_j^i(\bar{x},\cdot) \mid i > j\} = \frac{m(m+1)}{2} - m - k + 1.$$

Otherwise setting $\dim \operatorname{span}\{\phi_j^i(\bar{x},\cdot) \mid i > j\} = m(m+1)/2 - m - \ell$, we have $\ell \leq k-2$. Then one has $k - \ell \geq 2$ entries on the diagonal which are linear combinations of the other $m(m+1)/2 - m - \ell$ entries. Hence

$$\#(\check{K}) \leq 2\ell + (k-\ell) = k + \ell \leq 2k - 2 \leq m - 2.$$

Thus one can find two rows which verify the assumptions of Lemma 2.1. From Lemma 2.1 we conclude the assertion. Assume (4.11). There is a subset $K_1 \subset J$ with $\#(K_1) = k-1$ such that $\phi_j^i(\bar{x},\cdot)$, $(i,j) \in K_1$ are linear combinations of $\phi_j^i(\bar{x},\cdot)$, $(i,j) \in J\backslash K_1$ and there is s such that $\phi_s^s(\bar{x},\cdot)$ is a linear combination of

$$\phi_j^i(\bar{x},\cdot), \quad (i,j) \notin K = K_1 \cup \{(s,s)\}, \quad i \geq j.$$

Considering $P^{-1}L(x,\xi)P$ with a suitable permutation matrix P we may assume $(1,1) \in K$. Assume that K contains no $(i,1)$ with $i \geq 2$. Write

$$L = \begin{pmatrix} \phi_1^1 & L_{12} \\ L_{21} & L_{22} \end{pmatrix}, \quad H = \begin{pmatrix} h_1^1 & H_{12} \\ h_1^2 & H_{22} \end{pmatrix}$$

where L_{22} is the $(m-1) \times (m-1)$ matrix consisting of the last $(m-1)$ rows and columns of L. We repeat the same argument as in the proof of Case I choosing ξ so that $L_{21}(x, \xi) = 0$. Since

$$d(L_{22}(\bar{x}, \cdot)) \geq \frac{(m-1)m}{2} - (k-1) \geq \frac{(m-1)m}{2} - \left[\frac{m-1}{2}\right]$$

we conclude from the induction hypothesis that there is i such that $H_{22}(x)/h_i^i(x)$ is smooth near $\bar{x}$. Denote $H(x)/h_i^i(x)$ by the same $H(x)$ then $H(x)$ still verifies (4.1). Let us consider (i,k)-th entry of $LH = H^tL$ with $i, k \geq 2$:

$$\phi_1^i h_k^1 + \sum_{j=2}^{m} \phi_j^i h_k^j = h_1^i \phi_1^k + \sum_{j=2}^{m} h_j^i \phi_j^k. \tag{4.12}$$

Since $\phi_1^i(\bar{x}, \cdot)$ and $\phi_1^k(\bar{x}, \cdot)$ are linearly independent if $i \neq k$, $i, k \geq 2$ and $h_j^i(x)$ are smooth for $i, j \geq 2$ it follows that $H_{12}(x)$ is smooth near $\bar{x}$. We next take $(i,1)$-th entry of $LH = H^tL$ with some $i \geq 2$:

$$\phi_1^i h_1^1 + \sum_{j=2}^{m} \phi_j^i h_1^j = \sum_{j=1}^{m} h_j^i \phi_j^1. \tag{4.13}$$

Since $\phi_1^i(\bar{x}, \cdot) \neq 0$ it follows from (4.13) that $h_1^1(x)$ is smooth near $\bar{x}$.

We now assume that K contains a $(i, 1)$ with $i \geq 2$. Considering $P^{-1}L(x, \xi)P$ we may assume that $(2, 1) \in K$. Then there is a p-th row with $p \geq 3$ such that

$$\check{K} \cap \{p\text{-th row}\} = \varnothing.$$

In fact otherwise we have

$$\#(\check{K}) \geq 3 + m - 2 \geq 2k + 1$$

which contradicts $\#(\check{K}) \leq 2k$. Then considering $P^{-1}L(x, \xi)P$ again we may assume that the third row contains no entry of $\check{K}$. Let us write

$$L = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}, \quad H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}$$

where L_{22} is the $(m-3) \times (m-3)$ submatrix consisting of the last $(m-3)$ rows and columns of $L(x, \xi)$. We may assume that K contains no entry (i, j) with $i \geq 4$, $j = 1, 2, 3$. If not we have

$$\#(\check{K} \cap \{\text{the last } m-2 \text{ rows}\}) \leq 2k - 4 \leq m - 4.$$

Then one can choose two rows among the last $m-2$ rows which verify the hypothesis of Lemma 2.1 and hence the result. Repeating the same argument as in Case I we conclude that there is $i \geq 4$ such that $H_{22}/h_i^i(x)$ is smooth near $\bar{x}$. Again we denote $H(x)/h_i^i(x)$ by $H(x)$. Solving $\phi_j^i(x,\xi) = 0$, $\forall (i,j) \notin K$, $i > j$, $(i,j) \neq (3,1)$ and substituting the relation thus obtained into (4.4) one gets

$$L_{22}(x,\xi_a)H_{21}(x) - H_{21}(x)^t L_{11}(x,\xi_a) = G(x,\xi_a) \tag{4.14}$$

where the right-hand side is smooth in x. Fix ξ_a and study the linear equation (4.14) with unknowns H_{21} at $x = \bar{x}$. Then it is easy to see that the coefficient matrix at $x = \bar{x}$ is the direct sum of

$$\begin{pmatrix} (\phi_j^j - \phi_1^1)(\bar{x},\xi_a) & -\phi_2^1(\bar{x},\xi_a) & -\phi_3^1(\bar{x},\xi_a) \\ -\phi_1^2(\bar{x},\xi_a) & (\phi_j^j - \phi_2^2)(\bar{x},\xi_a) & 0 \\ -\phi_1^3(\bar{x},\xi_a) & 0 & (\phi_j^j - \phi_3^3)(\bar{x},\xi_a) \end{pmatrix} \tag{4.15}$$

for $j = 4,\dots,m$. Since we can choose ξ_a so that

$$\phi_1^3(\bar{x},\xi_a) \neq 0, \quad (\phi_j^j - \phi_2^2)(\bar{x},\xi_a) \neq 0, \quad (\phi_j^j - \phi_3^3)(\bar{x},\xi_a) = 0, \quad j = 4,\dots,m$$

the coefficient matrix is non singular and we conclude that $H_{12}(x)$ is smooth near $\bar{x}$. Finally we study $H_{11}(x)$. Recall that $H_{11}(x)$ satisfies the equation (4.10). In (4.10) we choose $\bar{\xi}$ so that

$$\phi_1^4(\bar{x},\bar{\xi}) \neq 0, \quad \phi_3^4(\bar{x},\bar{\xi}) = \phi_2^4(\bar{x},\bar{\xi}) = 0, \quad \phi_1^3(\bar{x},\bar{\xi}) = 1, \quad \phi_2^3(\bar{x},\bar{\xi}) = 1$$

and

$$1 - \phi_2^1(\bar{x},\bar{\xi})^2 + \phi_2^1(\bar{x},\bar{\xi})[\phi_3^3(\bar{x},\bar{\xi}) - \phi_2^2(\bar{x},\bar{\xi})] \neq 0.$$

This is possible because $\phi_2^1(\bar{x},\cdot)$ does not depend on $\phi_i^i(\bar{x},\cdot)$. This shows that the coefficient matrix of the equation (4.10) is non singular at $(\bar{x},\bar{\xi})$ and hence $H_{11}(x)$ is smooth near $\bar{x}$. □

PROOF OF THEOREM 1.1. By the assumption for any x there is a $S(x)$ such that

$$S(x)^{-1}L(x,\xi)S(x)$$

is symmetric for every ξ. Taking $S(\bar{x})^{-1}L(x,\xi)S(\bar{x})$ instead of $L(x,\xi)$ we may assume that $L(\bar{x},\xi)$ is symmetric. Let us set

$$H(x) = S(x)^t S(x)$$

which is of course positive definite and satisfies $L(x,\xi)H(x) = H(x)^t L(x,\xi)$. Since the reduced dimension is invariant one can apply Lemma 4.1 to conclude that $\tilde{H}(x) = H(x)/h_p^p(x)$ is smooth near $\bar{x}$ with some p. Then $T(x) = \tilde{H}(x)^{1/2}$ is a desired one. $\square$

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