TSUKUBA J. MATH.
Vol. 27 No. 2 (2003), 359–374

BOUND FOR THE WEIERSTRASS WEIGHTS OF POINTS ON A SMOOTH PLANE ALGEBRAIC CURVE

By

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Abstract. Let C be a smooth plane algebraic curve of degree $n \geq 3$. We give the upper bound for the weights of points on C and if C has an involution, i.e., an automorphism of order 2, then we give the lower bound for the weights of fixed points of the involution on C. Furthermore, we obtain all the possible Weierstrass gap sequences and weights of fixed points of the involution for the case $n = 5$ or 6.

1. Introduction

Let C be a smooth plane algebraic curve of genus g and let P be a point on C. Then we can choose a basis of holomorphic differentials $\varphi_1, \varphi_2, \dots, \varphi_g$ such that

$$0 = k_1 < k_2 < \cdots < k_g \leq 2g - 2,$$

where k_i is the order of the zero of φ_i at P. We define the (*Weierstrass*) *weight* of P by $w(P) = \sum_{j=1}^{g}(k_j + 1 - j)$. A point P on C is called a *Weierstrass point* if $w(P) > 0$. The sequence $k_1 + 1, k_2 + 1, \dots, k_g + 1$ is called the *Weierstrass gap sequence* at P. In particular, we call the sequence $k_1, k_2, \dots, k_g$ the *order sequence* at P. (See [2]). Furthermore, suppose that the degree of C is n and let T be the tangent line to C at P. If $(C \cdot T)_P = e \geq 3$, then P is called an $(e-2)$-*inflection point*, where $(C \cdot T)_P$ is the intersection number of C and T at P. In particular, an $(n-2)$-inflection point is called a *total inflection point*. (See [1]).

In Section 2, we give the upper bound for the weights of points on a smooth plane algebraic curve. Moreover, concerning the weights for fixed points of an involution, Towse [9] gave the lower bound for the weights of fixed points of an involution on the Fermat curve. We give the lower bound for the weights of fixed points of an involution on a smooth plane algebraic curve.

Received March 7, 2002.
Revised September 11, 2002.

It is known that all the possible order sequences of the points on a smooth plane algebraic curve of degree 4 is either $0, 1, 2$ or $0, 1, 4$. In Section 3, we obtain all the possible order sequences of the points which are fixed by an involution on a smooth plane algebraic curve of degree 5 or 6. We see that the five cases, $\{0,1,2,3,4,6\}$, $\{0,1,2,3,4,8\}$, $\{0,1,2,3,4,10\}$, $\{0,1,2,4,5,8\}$ or $\{0,1,2,5,6,10\}$ occur for a plane curve of degree 5 and the ten cases occur for a plane curve of degree 6.

In Section 4, we give examples of curves having order sequence appeared in Proposition 1, thus we can find curves having Weierstrass points on which equality does not hold in Propositions 1 or 2.

The author would like to thank Professor T. Kato for his invaluable comments. He would also like to thank the referee for his useful suggestions.

2. Results

First we give the upper bound for the weights of points on a smooth plane algebraic curve.

THEOREM 1. *Assume that C is a smooth plane curve of degree $n \geq 3$. Then*

$$w(P) \leq \frac{1}{24}(n-1)(n-2)(n-3)(n+4)$$

for all P on C, where equality holds if and only if P is a total inflection point.

PROOF. Assume that D is a divisor on C of degree k. Let $r = l(D) - 1$, where $l(D) = \dim\{f$ is a meromorphic function on $C \mid (f) + D \geq 0\}$. Noether [6] (cf. [4]) proved the following fact:

(i) If $k > n(n-3)$, then $r = k - (1/2)(n-1)(n-2)$.

(ii) If $k \leq n(n-3)$, then write $k = tn - s$ with $0 \leq t \leq n-3$ and $0 \leq s < n$. Then one has

$$\begin{cases} r \leq \frac{1}{2}(t-1)(t+2) & \text{if } s > t+1, \\ r \leq \frac{1}{2}t(t+3) - s & \text{if } s \leq t+1. \end{cases}$$

Choose a point P on C and let $D = kP$. Let

$$r(k) = \begin{cases} \frac{1}{2}(t-1)(t+2), & tn-(n-1) \leq k \leq tn-(t+1),\ 1 \leq t \leq n-3, \\ \frac{1}{2}t(t+3) + k - tn, & tn - t \leq k \leq tn,\ 1 \leq t \leq n-3, \\ k - \frac{1}{2}(n-1)(n-2), & k \geq n(n-3)+1, \end{cases}$$

and

$$N = \{k \in \mathbf{N} \mid (t-1)n + 1 \le k \le (t-1)n + n - t - 1), 1 \le t \le n-2\},$$

where $\mathbf{N}$ is the set of non-negative integers. Then $r(k)$ is monotone increasing and N satisfies $\#N = (1/2)(n-1)(n-2) = g$. In this case, the weight of N is given by

$$\sum_{t=1}^{n-2} \sum_{l=1}^{n-t-1} ((t-1)n + l) - \sum_{l=1}^{(n-1)(n-2)/2} l$$
$$= \frac{1}{24}(n-1)(n-2)(n-3)(n+4).$$

We show that N gives the maximum weight. Let

$$\{\mu_1, \dots, \mu_g\} = \{1, \dots, 2g\} \backslash N$$
$$= \{k \in \mathbf{N} \mid tn - t \le k \le tn, 1 \le t \le n-3\} \cup \{(n-3)n + 2\},$$

where $\mu_i < \mu_j$ if $i < j$. In order to prove that N gives the maximum weight, it is enough to prove that for any non-gap sequence $\{m_1, \dots, m_g\}$, where $m_i < m_j$ if $i < j$,

$$\mu_j \le m_j, \quad j = 1, \dots, g.$$

From

$$r(tn - t) - r(tn - t - 1) = 1, \quad 1 \le t \le n-3,$$

we have

$$r(\mu_j) - r(\mu_j - 1) = 1, \quad j = 1, \dots, g.$$

Furthermore, from

$$r((t+1)n - (t+1)) - r(tn) = 1, \quad 1 \le t \le n-2,$$

and

$$r((n-3)n + 2) - r((n-3)n) = 1,$$

we have

$$r(\mu_j) - r(\mu_{j-1}) = 1, \quad j = 2, \dots, g.$$

Since $r(\mu_1) = 1$, it is easily seen that

$$r(\mu_j) = j, \quad j = 1, \dots, g.$$

It follows that

$$\min\{1 \leq k \leq 2g \mid r(k) = j\} = \mu_j, \quad j = 1, \dots, g.$$

From $r(\mu_j) = j$ and $l(m_j P) = j + 1$ for any point P on C, we have

$$r(\mu_j) + 1 = j + 1 = l(m_j P).$$

On the other hand,

$$l(m_j P) \leq r(m_j) + 1.$$

Hence we have

$$r(\mu_j) + 1 \leq r(m_j) + 1,$$

i.e.,

$$r(\mu_j) \leq r(m_j).$$

Since $r(k)$ is monotone increasing, if $r(\mu_j) < r(m_j)$, then

$$\mu_j < m_j,$$

and since $\mu_j = \min\{1 \leq k \leq 2g \mid r(k) = j\}$, if $r(\mu_j) = r(m_j)$, then

$$\mu_j \leq m_j.$$

Therefore we have

$$\mu_j \leq m_j, \quad j = 1, \dots, g.$$

M. Coppens and T. Kato [1] proved that P is a total inflection point on a smooth plane algebraic curve of degree n if and only if the set of non-gaps at P is $\{a(n-1) + bn \mid a, b \in \mathbf{N}\}$. Hence, we see that the weight of a total inflection point of a smooth plane algebraic curve of degree n is $(1/24)(n-1)(n-2)(n-3) \cdot (n+4)$. Q.E.D.

We next consider the lower bound for the weights of points which are fixed by an involution on a smooth plane algebraic curve.

THEOREM 2. *Let C be a smooth plane algebraic curve of degree $n \geq 3$ given by $F(X, Y, Z) = 0$. Let σ be the involution of the projective plane given by $\sigma(X : Y : Z) = (X : -Y : Z)$. Suppose that C is invariant by σ. Then, for each fixed point $P \in C$ of σ, we have*

$$w(P) \geq \begin{cases} (n-1)(n-3)/8, & n \text{ odd}, \\ (n-2)(n-4)/8, & n \text{ even}. \end{cases}$$

PROOF. The fixed points of σ in the projective plane are the points of the line $Y=0$ and the point $Q=(0:1:0)$. Since C is invariant by σ, we have

$$F(X,Y,Z)=\sum_{j=0}^{[n/2]}\left(\sum_{k=0}^{n-2j}c_{n-2j-k,2j,k}X^{n-2j-k}Z^k\right)Y^{2j}.$$

Put $G(X)=F(X,0,1)$. Then $G(X)$ is a polynomial of degree n. Furthermore, we see that $G(X)$ has no multiple factors. For, if $X=a$ is a multiple root of $G(X)=0$, then $G(a)=G'(a)=0$. From

$$G(X)=\sum_{k=0}^{n-1}c_{n-k,0,k}X^{n-k}+c_{00n},$$

we have

$$G'(X)=\sum_{k=0}^{n-1}c_{n-k,0,k}(n-k)X^{n-k-1}.$$

On the other hand, from

$$F(X,Y,Z)=\sum_{j=0}^{[n/2]}\left(\sum_{k=0}^{n-2j-1}c_{n-2j-k,2j,k}X^{n-2j-k}Z^k+c_{0,2j,n-2j}Z^{n-2j}\right)Y^{2j},$$

we have

$$\frac{\partial F(X,Y,Z)}{\partial X}=\sum_{j=0}^{[n/2]}\left(\sum_{k=0}^{n-2j-1}c_{n-2j-k,2j,k}(n-2j-k)X^{n-2j-k-1}Z^k\right)Y^{2j}.$$

Hence we obtain

$$\left.\frac{\partial F(X,Y,Z)}{\partial X}\right|_{(X,Y,Z)=(a,0,1)}=\sum_{k=0}^{n-1}c_{n-k,0,k}(n-k)a^{n-k-1}.$$

We also obtain

$$\left.\frac{\partial F(X,Y,Z)}{\partial Y}\right|_{(X,Y,Z)=(a,0,1)}=0.$$

Therefore, if $G'(a)=\sum_{k=1}^{n-1}c_{n-k,0,k}(n-k)a^{n-k-1}=0$, then

$$\frac{\partial F}{\partial X}(a,0,1)=\frac{\partial F}{\partial Y}(a,0,1)=0.$$

Hence the point $(a:0:1)$ on C is a singular point. This contradicts that C is smooth. Thus the line $Y=0$ intersects the curve C at n distinct points. The point

Q belongs to C if and only if n is odd. Putting all together, we obtain that there are n (resp. $n+1$) fixed points of σ on C if n is even (resp. n is odd). Now the result follows from [9, Theorem 5] if $n \geq 5$. (This result was first proved by Torres in [8] under some additional hypothesis.) If $n = 3$, then the dimension of the space of holomorphic differentials on C is 1. Hence $w(P) = 0$. If $n = 4$, then the order sequence at P is either $0, 1, 2$ or $0, 1, 4$. Hence $w(P) = 0$ or 2. Q.E.D.

3. Case of Degree 5 or 6

We can obtain all the possible Weierstrass gap sequences of the points which are fixed by an involution on a smooth plane algebraic curve of degree 5 or 6.

PROPOSITION 1. *Let* $(a : 0 : 1)$ *be a fixed point of the involution* $(X, Y, Z) \to (X, -Y, Z)$ *on a smooth plane algebraic curve of degree* 5 *defined by* $F(X, Y, Z) = \sum_{j=0}^{2}(\sum_{k=0}^{5-2j} c_{5-2j-k,2j,k}X^{5-2j-k}Z^k)Y^{2j} = 0$. *Put* $p_{2j}(x) = \sum_{k=0}^{5-2j} c_{5-2j-k,2j,k}x^{5-2j-k}$ $(j = 0, 1, 2)$ *and* $p_0(x) = (x-a)\tilde{p}_0(x)$. *Then the order sequence at* $(a : 0 : 1)$ *is*

(i) $0, 1, 2, 4, 5, 8$ *if* $a_2 = 0$,
(ii) $0, 1, 2, 3, 4, 6$ *if* $a_2 \neq 0$, $a_6 \neq 2a_4^2/a_2$,
(iii) $0, 1, 2, 3, 4, 8$ *if* $a_2 \neq 0$, $a_6 = 2a_4^2/a_2$, $a_8 \neq 5a_4^3/a_2^2$

or

(iv) $0, 1, 2, 3, 4, 10$ *if* $a_2 \neq 0$, $a_6 = 2a_4^2/a_2$, $a_8 = 5a_4^3/a_2^2$,

where

$$a_2 = -p_2(a)/\tilde{p}_0(a),$$

$$a_4 = -(a_2p_2'(a) + a_2^2\tilde{p}_0'(a) + p_4(a))/\tilde{p}_0(a),$$

$$a_6 = -(a_2^2p_2''(a)/2 + a_2^3\tilde{p}_0''(a)/2 + a_4p_2'(a) + 2a_2a_4\tilde{p}_0'(a) + a_2p_4'(a))/\tilde{p}_0(a),$$

$$a_8 = -(a_2^3p_2^{(3)}(a)/6 + a_2^4\tilde{p}_0^{(3)}(a)/6 + a_2a_4p_2''(a) + 3a_2^2a_4\tilde{p}_0''(a)/2 + a_4^2\tilde{p}_0'(a)$$
$$+ a_6p_2'(a) + 2a_2a_6\tilde{p}_0'(a) + a_4p_4'(a))/\tilde{p}_0(a).$$

Furthermore, the order sequence at $(0 : 1 : 0)$ *is*

(v) $0, 1, 2, 3, 4, 6$ *if* $b_3 \neq 0$

or

(vi) $0, 1, 2, 5, 6, 10$ *if* $b_3 = 0$, *where*

$$b_3 = \sum_{k=0}^{3}(-1)^k p_4^k(0)\{p_4'(0)\}^{-k} p_2^{(k)}(0)/k!.$$

Proof. Put $f(x,y) = F(x,y,1)$. Since $f_x(a,0) = p_0'(a) \neq 0$, there exists a function

$$x = x(y) = a + a_2 y^2 + a_4 y^4 + \cdots$$

which is analytic in some neighborhood of $y=0$ such that $f(x(y),y) \equiv 0$. Then we consider the order sequence of

$$\left\{\frac{x^i y^j}{f_y}\, dx \,|\, 0 \leq i+j \leq 2\right\}$$

at $(a,0)$. Since $\mathrm{ord}_{(a,0)}(dx/f_y) = 0$, it is enough to check the order sequence of $\{1, x, y, x^2, xy, y^2\}$ at $y=0$, where $x = a + a_2 y^2 + \cdots$. Since x is an even function of y, we consider the even functions part $\{1, x, x^2, y^2\}$ and the odd functions part $\{y, xy\}$, separately. From $x^2 = a^2 + 2aa_2 y^2 + (2aa_4 + a_2^2)y^4 + (2aa_6 + 2a_2a_4)y^6 + (2aa_8 + 2a_2a_6 + a_4^2)y^8 + (2aa_{10} + 2a_2a_8 + 2a_4a_6)y^{10} + \cdots$, the coefficient matrix of the terms, $y^0, y^2, y^4, \ldots, y^{10}$ of $\{1, x, x^2, y^2\}$ is

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ a & a_2 & a_4 & \cdots & a_{10} \\ a^2 & 2aa_2 & 2aa_4 + a_2^2 & \cdots & 2aa_{10} + 2a_2a_8 + 2a_4a_6 \\ 0 & 1 & 0 & \cdots & 0 \end{pmatrix}$$

and the coefficient matrix of the terms, y, y^3, y^5 of $\{y, xy\}$ is

$$\begin{pmatrix} 1 & 0 & 0 \\ a & a_2 & a_4 \end{pmatrix}.$$

After a suitable elementary deformation, we have

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_4 & a_6 & a_8 & a_{10} \\ 0 & 0 & a_2^2 & 2a_2a_4 & 2a_2a_6 + a_4^2 & 2a_2a_8 + 2a_4a_6 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & a_2 & a_4 \end{pmatrix}.$$

If $a_2 = 0$, then we have

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_4 & a_6 & a_8 & a_{10} \\ 0 & 0 & 0 & 0 & a_4^2 & 2a_4a_6 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & a_4 \end{pmatrix}.$$

If $a_2 = 0$, then $a_4 \neq 0$. Therefore, if $a_2 = 0$, then the order sequence at $(a, 0)$ is $0, 1, 2, 4, 5, 8$.

Assume $a_2 \neq 0$. Then we have

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_6 - \frac{2a_4^2}{a_2} & a_8 - \frac{2a_4a_6}{a_2} - \frac{a_4^3}{a_2^2} & a_{10} - \frac{2a_4a_8}{a_2} - \frac{2a_4^2a_6}{a_2^2} \\ 0 & 0 & a_2^2 & 2a_2a_4 & 2a_2a_6 + a_4^2 & 2a_2a_8 + 2a_4a_6 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & a_2 & a_4 \end{pmatrix}.$$

Hence, if $a_6 \neq 2a_4^2/a_2$, then the order sequence at $(a, 0)$ is $0, 1, 2, 3, 4, 6$.

Assume $a_6 = 2a_4^2/a_2$. Then we have

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_8 - \frac{5a_4^3}{a_2^2} & a_{10} - \frac{2a_4a_8}{a_2} - \frac{4a_4^4}{a_2^3} \\ 0 & 0 & a_2^2 & 2a_2a_4 & 5a_4^2 & 2a_2a_8 + \frac{4a_4^3}{a_2} \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence, if $a_8 \neq 5a_4^3/a_2^2$, then the order sequence at $(a, 0)$ is $0, 1, 2, 3, 4, 8$.

Assume $a_8 = 5a_4^3/a_2^2$. Then we have

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{10} - \frac{14a_4^4}{a_2^3} \\ 0 & 0 & a_2^2 & 2a_2a_4 & 5a_4^2 & \frac{14a_4^3}{a_2} \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence the order sequence at $(a, 0)$ is $0, 1, 2, 3, 4, 10$. Note that the maximum value of the order is 10.

Next we study the representations of a_2, a_4, a_6 and a_8. Put $\tilde{p}_0(x) = d_4x^4 + \cdots + d_1x + d_0$. Note that $\tilde{p}_0(a) \neq 0$. Consider the coefficient of y^2 of $f(x(y), y)$. Then only $p_2(x(y))y^2$ and $p_0(x(y))$ have nonzero coefficient of y^2. Hence we have

$$\begin{aligned} 0 &= f(x(y), y) \\ &= ((c_{320}a^3 + c_{221}a^2 + c_{122}a + c_{023}) + a_2(d_4a^4 + \cdots + d_1a + d_0))y^2 + \cdots. \end{aligned}$$

It follows that

$$a_2 = -\frac{p_2(a)}{\tilde{p}_0(a)}.$$

Similarly, we have

$$\begin{aligned} a_4 &= -(a_2p_2'(a) + a_2^2\tilde{p}_0'(a) + p_4(a))/\tilde{p}_0(a), \\ a_6 &= -(a_2^2p_2''(a)/2 + a_2^3\tilde{p}_0''(a)/2 + a_4p_2'(a) + 2a_2a_4\tilde{p}_0'(a) + a_2p_4'(a))/\tilde{p}_0(a), \\ a_8 &= -(a_2^3p_2^{(3)}(a)/6 + a_2^4\tilde{p}_0^{(3)}(a)/6 + a_2a_4p_2''(a) + 3a_2^2a_4\tilde{p}_0''(a)/2 + a_4^2\tilde{p}_0'(a) \\ &\qquad + a_6p_2'(a) + 2a_2a_6\tilde{p}_0'(a) + a_4p_4'(a))/\tilde{p}_0(a). \end{aligned}$$

Finally, we check the order sequence at $(0:1:0)$. Put $g(u, v) = F(u, 1, v)$. Since we may assume that $g_u(0, 0) \neq 0$, there exists a function

$$u = u(v) = b_1v + b_3v^3 + \cdots$$

which is analytic in some neighborhood of $v = 0$ such that $g(u(v), v) \equiv 0$. Then we consider the order sequence of

$$\left\{\frac{u^iv^j}{g_v}\,du \,|\, 0 \leq i + j \leq 2\right\}$$

at $(0, 0)$. Since $\mathrm{ord}_{(0,0)}(du/g_v) = 0$, it is enough to check the order sequence of $\{1, v, u, v^2, uv, u^2\}$ at $v = 0$, where $u = b_1v + b_3v^3 + \cdots$. Since u is an odd function of v, we consider the even functions part $\{1, v^2, uv, u^2\}$ and the odd functions part $\{v, u\}$, separately. From $u^2 = b_1^2v^2 + 2b_1b_3v^4 + (2b_1b_5 + b_3^2)v^6 + 2(b_1b_7 + b_3b_5)v^8 + (2b_1b_9 + 2b_3b_7 + b_5^2)v^{10} + \cdots$, the coefficient matrix of the terms, $v^0, v^2, v^4, \ldots, v^{10}$ of $\{1, v^2, uv, u^2\}$ is

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & b_1 & b_3 & \cdots & b_9 \\ 0 & b_1^2 & 2b_1b_3 & \cdots & 2b_1b_9 + 2b_3b_7 + b_5^2 \end{pmatrix}$$

and the coefficient matrix of the terms, v, v^3, v^5 of $\{v, u\}$ is

$$\begin{pmatrix} 1 & 0 & 0 \\ b_1 & b_3 & b_5 \end{pmatrix}.$$

After a suitable elementary deformation, we have

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_3 & b_5 & b_7 & b_9 \\ 0 & 0 & 0 & b_3^2 & 2b_3b_5 & 2b_3b_7 + b_5^2 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & b_3 & b_5 \end{pmatrix}.$$

Hence, if $b_3 \neq 0$, then the order sequence at $(0,0)$ is $0,1,2,3,4,6$.

Assume $b_3 = 0$. Then we have

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_5 & b_7 & b_9 \\ 0 & 0 & 0 & 0 & 0 & b_5^2 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & b_5 \end{pmatrix}.$$

Hence, if $b_3 = 0$, then the order sequence at $(0,0)$ is $0,1,2,5,6,10$.

Using the same method as getting the representation of $a_2, \dots, a_8$, we can obtain

$$b_3 = \frac{1}{p_4'(0)^4} \sum_{k=0}^{3} (-1)^{k+1} p_4^k(0) \{p_4'(0)\}^{3-k} p_2^{(k)}(0)/k!.$$

Note that $p_4'(0) = c_{140} \neq 0$. Since our concern is whether b_3 is equal to 0 or not, we may take

$$b_3 = \sum_{k=0}^{3} (-1)^k p_4^k(0) \{p_4'(0)\}^{-k} p_2^{(k)}(0)/k!. \qquad \text{Q.E.D.}$$

Using a similar method to the proof of Proposition 1, we can obtain the following:

PROPOSITION 2. *Let* $(a:0:1)$ *be a fixed point of the involution* $(X,Y,Z)\to(X,-Y,Z)$ *on a smooth plane algebraic curve of degree* 6 *given by* $F(X,Y,Z)=0$. *Then the order sequence at* $(a:0:1)$ *is*

(i) $0,1,2,3,4,5,6,8,9,12,$
(ii) $0,1,2,3,6,7,8,12,13,18,$
(iii) $0,1,2,3,4,5,6,8,9,10,$
(iv) $0,1,2,3,4,5,6,10,11,12,$
(v) $0,1,2,3,4,5,6,12,13,14,$
(vi) $0,1,2,3,4,5,6,7,8,10,$
(vii) $0,1,2,3,4,5,6,7,8,12,$
(viii) $0,1,2,3,4,5,6,7,8,14,$
(ix) $0,1,2,3,4,5,6,7,8,16$

or

(x) $0,1,2,3,4,5,6,7,8,18.$

REMARK. Put $F(X,Y,Z)=\sum_{j=0}^{3}(\sum_{k=0}^{6-2j}c_{6-2j-k,2j,k}X^{6-2j-k}Z^k)Y^{2j}=0$, $p_{2j}(x)=\sum_{k=0}^{6-2j}c_{6-2j-k,2j,k}x^{6-2j-k}$ $(j=0,1,2,3)$ and $p_0(x)=(x-a)\tilde{p}_0(x)$. Then each order sequence appeared in Proposition 2 occurs under the following condition:

(i) $a_2=0,\ a_4\neq 0,$
(ii) $a_2=0,\ a_4=0,$
(iii) $a_2\neq 0,\ a_6=2a_4^2/a_2,\ a_8\neq 5a_4^3/a_2^2,$
(iv) $a_2\neq 0,\ a_6=2a_4^2/a_2,\ a_8=5a_4^3/a_2^2,\ a_{10}\neq 14a_4^4/a_2^3,$
(v) $a_2\neq 0,\ a_6=2a_4^2/a_2,\ a_8=5a_4^3/a_2^2,\ a_{10}=14a_4^4/a_2^3,$
(vi) $a_2\neq 0,\ a_6\neq 2a_4^2/a_2,\ a_{10}\neq A_{10},$
(vii) $a_2\neq 0,\ a_6\neq 2a_4^2/a_2,\ a_{10}=A_{10},\ a_{12}\neq A_{12},$
(viii) $a_2\neq 0,\ a_6\neq 2a_4^2/a_2,\ a_{10}=A_{10},\ a_{12}=A_{12},\ a_{14}\neq A_{14},$
(ix) $a_2\neq 0,\ a_6\neq 2a_4^2/a_2,\ a_{10}=A_{10},\ a_{12}=A_{12},\ a_{14}=A_{14},\ a_{16}\neq A_{16},$
(x) $a_2\neq 0,\ a_6\neq 2a_4^2/a_2,\ a_{10}=A_{10},\ a_{12}=A_{12},\ a_{14}=A_{14},\ a_{16}=A_{16},$

where

$$a_2=-p_2(a)/\tilde{p}_0(a),$$

$$a_4=-(a_2p_2'(a)+a_2^2\tilde{p}_0'(a)+p_4(a))/\tilde{p}_0(a),$$

$$a_6=-(a_2^2p_2''(a)/2+a_2^3\tilde{p}_0''(a)/2+a_4p_2'(a)+2a_2a_4\tilde{p}_0'(a)+a_2p_4'(a)+p_6(a))/\tilde{p}_0(a),$$

$$a_8=-(a_2^3p_2^{(3)}(a)/6+a_2^4\tilde{p}_0^{(3)}(a)/6+a_2a_4p_2''(a)+3a_2^2a_4\tilde{p}_0''(a)/2+a_4^2\tilde{p}_0'(a)$$
$$+a_6p_2'(a)+2a_2a_6\tilde{p}_0'(a)+a_4p_4'(a)+a_2^2p_4''(a)/2)/\tilde{p}_0(a),$$

$$
\begin{aligned}
a_{10} = &-(a_2^4 p_2^{(4)}(a)/24 + a_2^5 \tilde{p}_0^{(4)}(a)/24 + a_2^2 a_4 p_2^{(3)}(a)/2 + 2a_2^3 a_4 \tilde{p}_0^{(3)}(a)/3 \\
&+ a_4^2 p_2''(a)/2 + 3a_2 a_4^2 \tilde{p}_0''(a)/2 + a_2 a_6 p_2''(a) + 3a_2^2 a_6 \tilde{p}_0''(a)/2 + 2a_4 a_6 \tilde{p}_0'(a) \\
&+ a_8 p_2'(a) + 2a_2 a_8 \tilde{p}_0'(a) + a_6 p_4'(a) + a_2 a_4 p_4''(a))/\tilde{p}_0(a), \\
a_{12} = &-(a_2^6 \tilde{p}_0^{(5)}(a)/120 + a_2^3 a_4 p_2^{(4)}(a)/6 + 5a_2^4 a_4 \tilde{p}_0^{(4)}(a)/24 + a_2 a_4^2 p_2^{(3)}(a)/2 \\
&+ a_2^2 a_4^2 \tilde{p}_0^{(3)}(a) + a_4^3 \tilde{p}_0''(a)/2 + a_2^2 a_6 p_2^{(3)}(a)/2 + 2a_2^3 a_6 \tilde{p}_0^{(3)}(a)/3 + a_4 a_6 p_2''(a) \\
&+ 3a_2 a_4 a_6 \tilde{p}_0''(a) + a_6^2 \tilde{p}_0'(a) + a_2 a_8 p_2''(a) + 3a_2^2 a_8 \tilde{p}_0''(a)/2 + 2a_4 a_8 \tilde{p}_0'(a) \\
&+ a_{10} p_2'(a) + 2a_2 a_{10} \tilde{p}_0'(a) + a_8 p_4'(a) + a_4^2 p_4''(a)/2 + a_2 a_6 p_4''(a))/\tilde{p}_0(a), \\
a_{14} = &-(a_2^5 a_4 \tilde{p}_0^{(5)}(a)/20 + a_2^2 a_4^2 p_2^{(4)}(a)/4 + 5a_2^3 a_4^2 \tilde{p}_0^{(4)}(a)/12 + a_4^3 p_2^{(3)}(a)/6 \\
&+ 2a_2 a_4^3 \tilde{p}_0^{(3)}(a)/3 + a_2^3 a_6 p_2^{(4)}(a)/6 + 5a_2^4 a_6 \tilde{p}_0^{(4)}(a)/24 + a_2 a_4 a_6 p_2^{(3)}(a) \\
&+ 2a_2^2 a_4 a_6 \tilde{p}_0^{(3)}(a) + 3a_4^2 a_6 \tilde{p}_0''(a)/2 + a_6^2 p_2''(a)/2 + 3a_2 a_6^2 \tilde{p}_0''(a)/2 \\
&+ a_2^2 a_8 p_2^{(3)}(a)/2 + 2a_2^3 a_8 \tilde{p}_0^{(3)}(a)/3 + a_4 a_8 p_2''(a) + 3a_2 a_4 a_8 \tilde{p}_0''(a) + 2a_6 a_8 \tilde{p}_0'(a) \\
&+ a_2 a_{10} p_2''(a) + 3a_2^2 a_{10} \tilde{p}_0''(a)/2 + 2a_4 a_{10} \tilde{p}_0'(a) + a_{12} p_2'(a) + 2a_2 a_{12} \tilde{p}_0'(a) \\
&+ a_{10} p_4'(a) + a_4 a_6 p_4''(a) + a_2 a_8 p_4''(a))/\tilde{p}_0(a), \\
a_{16} = &-(a_2^4 a_4^2 \tilde{p}_0^{(5)}(a)/8 + a_2 a_4^3 p_2^{(4)}(a)/6 + 5a_2^2 a_4^3 \tilde{p}_0^{(4)}(a)/12 + a_4^4 \tilde{p}_0^{(3)}(a)/6 \\
&+ a_2^5 a_6 \tilde{p}_0^{(5)}(a)/20 + a_2^2 a_4 a_6 p_2^{(4)}(a)/2 + 5a_2^3 a_4 a_6 \tilde{p}_0^{(4)}(a)/6 + a_4^2 a_6 p_2^{(3)}(a)/2 \\
&+ 2a_2 a_4^2 a_6 \tilde{p}_0^{(3)}(a) + a_2 a_6^2 p_2^{(3)}(a)/2 + a_2^2 a_6^2 \tilde{p}_0^{(3)}(a) + 3a_4 a_6^2 \tilde{p}_0''(a)/2 \\
&+ a_2^3 a_8 p_2^{(4)}(a)/6 + 5a_2^4 a_8 \tilde{p}_0^{(4)}(a)/24 + a_2 a_4 a_8 p_2^{(3)}(a) + 2a_2^2 a_4 a_8 \tilde{p}_0^{(3)}(a) \\
&+ 3a_4^2 a_8 \tilde{p}_0''(a)/2 + a_6 a_8 p_2''(a) + 3a_2 a_6 a_8 \tilde{p}_0''(a) + a_8^2 \tilde{p}_0'(a) + a_2^2 a_{10} p_2^{(3)}(a)/2 \\
&+ 2a_2^3 a_{10} \tilde{p}_0^{(3)}(a)/3 + a_4 a_{10} p_2''(a) + 3a_2 a_4 a_{10} \tilde{p}_0''(a) + 2a_6 a_{10} \tilde{p}_0'(a) + a_2 a_{12} p_2''(a) \\
&+ 3a_2^2 a_{12} \tilde{p}_0''(a)/2 + 2a_4 a_{12} \tilde{p}_0'(a) + a_{14} p_2'(a) + 2a_2 a_{14} \tilde{p}_0'(a) + a_{12} p_4'(a) \\
&+ a_6^2 p_4''(a)/2 + a_4 a_8 p_4''(a) + a_2 a_{10} p_4''(a))/\tilde{p}_0(a), \\
A_{10} = &-(3a_4^6 - 6a_2 a_4^4 a_6 + 2a_2^2 a_4^2 a_6^2 - 3a_2^3 a_6^3 + 2a_2^2 a_4^3 a_8 \\
&+ 4a_2^3 a_4 a_6 a_8 - a_2^4 a_8^2)/(a_2^3(-2a_4^2 + a_2 a_6)), \\
A_{12} = &-(a_4^9 - 24a_2 a_4^7 a_6 + 45a_2^2 a_4^5 a_6^2 - 15a_2^3 a_4^3 a_6^3 + 7a_2^4 a_4 a_6^4 + a_2^2 a_8(9a_4^6 - 32a_2 a_4^4 a_6 \\
&+ 9a_2^2 a_4^2 a_6^2 - 7a_2^3 a_6^3) + a_2^4 a_4 a_8^2(-a_4^2 + 8a_2 a_6) - a_2^6 a_8^3)/(a_2^4(-2a_4^2 + a_2 a_6)^2),
\end{aligned}
$$

$$A_{14} = (17a_4^{12} - 60a_2a_4^{10}a_6 + 18a_2^2a_4^8a_6^2 + 52a_2^3a_4^6a_6^3 + 15a_2^4a_4^4a_6^4 - 24a_2^5a_4^2a_6^5 + 12a_2^6a_6^6 - 4a_2^2a_4a_8(a_4^2 - 3a_2a_6)(7a_4^6 - 18a_2a_4^4a_6 + 9a_2^2a_4^2a_6^2 - 4a_2^3a_6^3) + 6a_2^4a_8^2(-7a_4^6 + 10a_2a_4^4a_6 - a_2^2a_4^2a_6^2 + 2a_2^3a_6^3) + 4a_2^6a_4a_8^3(a_4^2 - 3a_2a_6) + a_2^8a_8^4)/(a_2^5(-2a_4^2 + a_2a_6)^3),$$

$$A_{16} = -(-95a_4^{15} + 628a_2a_4^{13}a_6 - 1481a_2^2a_4^{11}a_6^2 + 1652a_2^3a_4^9a_6^3 - 1159a_2^4a_4^7a_6^4 + 683a_2^5a_4^5a_6^5 - 255a_2^6a_4^3a_6^6 + 60a_2^7a_4a_6^7 - 5a_2^2a_8(37a_4^{12} - 76a_2a_4^{10}a_6 - 39a_2^2a_4^8a_6^2 + 98a_2^3a_4^6a_6^3 - 11a_2^4a_4^4a_6^4 - 9a_2^5a_4^2a_6^5 + 9a_2^6a_6^6) + 5a_2^4a_4a_8^2(14a_4^8 - 132a_2a_4^6a_6 + 177a_2^2a_4^4a_6^2 - 80a_2^3a_4^2a_6^3 + 27a_2^4a_6^4) - a_2^6a_8^3(-74a_4^6 + 76a_2a_4^4a_6 + 7a_2^2a_4^2a_6^2 + 18a_2^3a_6^3) + a_2^8a_4a_8^4(-7a_4^2 + 16a_2a_6) - a_2^{10}a_8^5)/(a_2^6(-2a_4^2 + a_2a_6)^4).$$

4. Examples

Let F_n be the Fermat curve defined by $X^n + Y^n + Z^n = 0$. As mentioned in [7, p. 72], Leopoldt pointed out that the $3n^2$ points

$$(1 : \alpha : \sqrt[n]{2}\beta), (1 : \sqrt[n]{2}\beta : \alpha), (1 : \beta/\sqrt[n]{2} : \alpha\beta/\sqrt[n]{2}), \quad \alpha^n = 1, \beta^n = -1,$$

are Weierstrass points on F_n. We call them the *Leopoldt Weierstrass points* on F_n. In [5], T. Kato and S. Kikuchi proved that equality in [9, Corollary 8] holds for the Leopoldt Weierstrass points for $n \le 14$. We use Propositions 1 and 2 to give an alternative proof for $n = 5$ and 6. Furthermore, in [3], Hasse proved that the $3n$ trivial points

$$(1 : 0 : \beta), (0 : 1 : \beta), (1 : \beta : 0), \quad \beta^n = -1,$$

are Weierstrass points on F_n and its weight is

$$\frac{1}{24}(n-1)(n-2)(n-3)(n+4).$$

We see that our results agree with Hasse's.

Example 1. For $n = 5$ or 6, each Leopoldt Weierstrass point has weight 1. Furthermore, each trivial Weierstrass point has weight 9 for $n = 5$ or weight 25 for $n = 6$.

Proof. Let P, Q be Leopoldt Weierstrass points or trivial Weierstrass points on F_n. Then there exists an automorphism T of F_n such that $T(P) = Q$.

Hence it is enough to consider the special points $(1:1:(-2)^{1/n})$ and $(1:-1:0)$ if n is odd. Let $F(X,Y,Z)=0$ be the equation of the Fermat curve transformed by $\tau:(X,Y,Z)\to(2^{1/n-1}(X+Y)-(-1)^{-1/n}Z,2^{1/n-1}(X-Y),(-1)^{-1/n}Z)$. We have $\tau(1:1:(-2)^{1/n})=(0:0:1)$ and $\tau(1:-1:0)=(0:1:0)$. Putting $f(x,y)=F(x,y,1)$, we have

$$f(x,y)=p_{2[n/2]}(x)y^{2[n/2]}+\cdots+p_2(x)y^2+p_0(x)$$

where $p_0(x)=(x+1)^n-1=x\sum_{k=0}^{n-1}\binom{n}{k}x^{n-k-1}$ and $p_k(x)=\binom{n}{k}(x+1)^{n-k}$ $(k=2,\dots,2[n/2])$. Put $\tilde{p}_0(x)=\sum_{k=0}^{n-1}\binom{n}{k}x^{n-k-1}$.

For $n=5$, we obtain $p_4(x)=5(x+1)$, $p_2(x)=10(x+1)^3$ and $\tilde{p}_0(x)=x^4+5x^3+10x^2+10x+5$. Then we have $p_4(0)=5$, $p_4'(0)=5$, $p_2(0)=10$, $p_2'(0)=30$, $p_2''(0)=60$, $p_2^{(3)}(0)=60$, $\tilde{p}_0(0)=5$, $\tilde{p}_0'(0)=10$ and $\tilde{p}_0''(0)=20$. Hence we have $a_2=-2$, $a_4=3$, $a_6=0$ and $b_3=0$, which satisfy the conditions (ii) and (vi) in Proposition 1. It follows that the order sequences at $(0:0:1)$ and $(0:1:0)$ are $0,1,2,3,4,6$ and $0,1,2,5,6,10$ respectively. Hence, the weights of $(0:0:1)$ and $(0:1:0)$ are 1 and 9 respectively.

For $n=6$, we obtain $p_6(x)\equiv 1$, $p_4(x)=15(x+1)^2$, $p_2(x)=15(x+1)^4$ and $\tilde{p}_0(x)=x^5+6x^4+15x^3+20x^2+15x+6$. Then we have $p_4(0)=15$, $p_4'(0)=30$, $p_4''(0)=30$, $p_2(0)=15$, $p_2'(0)=60$, $p_2''(0)=180$, $p_2^{(3)}(0)=360$, $p_2^{(4)}(0)=360$, $\tilde{p}_0(0)=6$, $\tilde{p}_0'(0)=15$, $\tilde{p}_0''(0)=40$, $\tilde{p}_0^{(3)}(0)=90$ and $\tilde{p}_0^{(4)}(0)=144$. Hence we have $a_2=-5/2$, $a_4=55/8$, $a_6=-583/48$, $a_8=-20735/384$ and $a_{10}=137005/256$, which satisfy the condition (vi) in Proposition 2. It follows that the order sequence at $(0:0:1)$ is $0,1,2,3,4,5,6,7,8,10$, thus this point has weight 1.

It is easy to see that each point $(X:Y:Z)=(\beta:0:1)$, $(\beta^6=-1)$, on the Fermat curve, $X^6+Y^6+Z^6=0$, satisfies the condition (ii) in Proposition 2, whence these points have weight 25. Q.E.D.

We show that all the cases occur in Proposition 1. An example of the cases (ii) and (vi) in Proposition 1 is shown in Example 1.

EXAMPLE 2. The weights of the points $(X:Y:Z)=(0:0:1)$ and $(0:1:0)$ on the curve

$$(-2X-Z)Y^4+Z^3Y^2+X(X^4-3X^3Z+Z^4)=0$$

are 5 and 1, respectively. (This curve is an example satisfying the cases (iv) and (v) in Proposition 1).

PROOF. First of all, we prove that the curve

$$\tilde{C}:(-2x-1)y^4+y^2+p_0(x)=0, \tag{1}$$

where $p_0(x)=x(x^4-3x^3+1)$ is a smooth curve. Multiplying the equation of (1) by $(-2x-1)^3$, we have

$$(-2x-1)^4y^4+(-2x-1)^3y^2+(-2x-1)^3p_0(x)=0.$$

Putting $z=(-2x-1)y$, we have

$$z^4+(-2x-1)z^2+(-2x-1)^3p_0(x)=0.$$

Now we consider the automorphism $\sigma:(x,z)\to(x,-z)$. The order of σ is 2 and the fixed points of σ are $(x,z)=(-1/2,0),(a_i,0)$ $(i=1,\dots,5)$, where $a_1,\dots,a_5$ are the solutions of $p_0(x)=0$. Furthermore, these fixed points are also the branch points of the covering

$$\pi:\tilde{C}\to\tilde{C}/\langle\sigma\rangle.$$

Hence, by the Riemann-Hurwitz formula, we have

$$2g-2=2(2g'-2)+6,$$

i.e.,

$$g=2g'+2,$$

where g and g' are the genera of $\tilde{C}$ and $\tilde{C}/\langle\sigma\rangle$, respectively. The following three conditions are equivalent:

1. $\tilde{C}$ is smooth;
2. $g=6$;
3. $g'=2$.

Define the mapping σ as $u=x$, $v=z^2$. Then the equation of $\tilde{C}/\langle\sigma\rangle$ is given by

$$v^2+(-2u-1)v+(-2u-1)^3p_0(u)=0.$$

Putting $w=v/(-2u-1)+1/2$, we have

$$w^2=\frac{1}{4}+(2u+1)p_0(u).$$

The degree of $p(u)=1/4+(2u+1)p_0(u)$ is 6 and the discriminant of $p(u)$ is equal to $-1788325/128\neq0$. Hence $p(u)$ has six simple roots. Therefore $g'=2$, i.e., $\tilde{C}$ is smooth.

Put $p_4(x)=-2x-1$, $p_2(x)\equiv1$ and $\tilde{p}_0(x)=x^4-3x^3+1$. Then we have $p_4(0)=-1$, $p_4'(0)=-2$, $p_2(0)=1$, $p_2'(0)=0$, $p_2''(0)=0$, $p_2^{(3)}(0)=0$, $\tilde{p}_0(0)=1$, $\tilde{p}_0'(0)=0$, $\tilde{p}_0''(0)=0$ and $\tilde{p}_0^{(3)}(0)=-18$. Hence we have $a_2=-1$, $a_4=1$, $a_6=-2$, $a_8=5$ and $b_3=1$, which satisfy the conditions (iv) and (v) in Proposition 1. It follows that the order sequences at $(0:0:1)$ and $(0:1:0)$ are $0,1,2,3,4,10$ and $0,1,2,3,4,6$, respectively. Hence, the weights of $(0:0:1)$ and $(0:1:0)$ are 5 and 1, respectively. Q.E.D.

EXAMPLE 3. The weight of the point $(X:Y:Z)=(0:0:1)$ on the curve

$$(X+2Z)Y^4+XZ^2Y^2+X(X^4+4X^3Z-7X^2Z^2+24Z^4)=0$$

is 5. (This curve is an example satisfying the case (i) in Proposition 1).

EXAMPLE 4. The weight of the point $(X:Y:Z)=(0:0:1)$ on the curve

$$(-2X-Z)Y^4+Z^3Y^2+X(X^4+5X^3Z+Z^4)=0$$

is 3. (This curve is an example satisfying the case (iii) in Proposition 1).

Using a similar method to the proof of Example 2, we see that the curves given in Example 3 or Example 4 are smooth.

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