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ASYMPTOTIC ESTIMATES FOR DENSITIES OF MULTI-DIMENSIONAL STABLE DISTRIBUTIONS

By

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1. Introduction and Results

Let $\mu(dx)$ be a stable distribution on $\mathbf{R}^d$ with exponent $0 < \alpha < 2$. Its log-characteristic function $\Psi(z) := \log \int_{\mathbf{R}^d} e^{ixz}\mu(dx)$ $(i = \sqrt{-1})$ is given by the following:

$$\Psi(z) = \begin{cases} -\int_{\mathbf{S}^{d-1}} |\langle z,\theta\rangle|^{\alpha}\left[1 - i(\operatorname{sgn}\langle z,\theta\rangle)\tan\frac{\pi\alpha}{2}\right]\lambda(d\theta) + i\langle z,b\rangle & (\alpha \neq 1), \\ -\int_{\mathbf{S}^{d-1}} |\langle z,\theta\rangle|\left[1 + i\frac{2}{\pi}(\operatorname{sgn}\langle z,\theta\rangle)\log|\langle z,\theta\rangle|\right]\lambda(d\theta) + i\langle z,b\rangle & (\alpha = 1), \end{cases}$$

where $\langle z,\theta\rangle = \sum_{j=1}^{d} z_j\theta_j$ for $z = (z_1,\dots,z_d)$, $\theta = (\theta_1,\dots,\theta_d)$, "sgn x" is the sign function, i.e., $\operatorname{sgn} x = 1$ $(x > 0)$, $= 0$ $(x = 0)$, $= -1$ $(x < 0)$, $\lambda(d\theta)$ is a finite measure on $\mathbf{S}^{d-1}$ and $b \in \mathbf{R}^d$. Moreover if μ is non-degenerate, then μ has a C^∞-density function $p(x)$ with respect to the Lebesgue measure dx, i.e., $\mu(dx) = p(x)\,dx$ and

$$p(x) = \frac{1}{(2\pi)^d}\int_{\mathbf{R}^d} \exp[-i\langle x,z\rangle + \Psi(z)]\,dz. \tag{1.1}$$

The non-degeneracy of μ means $\mathbf{Span\,Spt}\,\mu = \mathbf{R}^d$ and it is equivalent to $\mathbf{Span\,Spt}\,\lambda = \mathbf{R}^d$, where $\mathbf{Spt}\,\mu$ (resp. $\mathbf{Spt}\,\lambda$) is a support of μ (resp. λ) and for a set $S \subset \mathbf{R}^d$, $\mathbf{Span}\,S$ is a linear subspace of $\mathbf{R}^d$ spanned by S (cf. [3]).

In the present paper we would like to investigate the asymptotic behavior of $p(r\sigma)$ as $r \to \infty$ for each direction $\sigma \in \mathbf{S}^{d-1}$ under the following assumption.

ASSUMPTION 1. *Let $b = 0$. For some number $m \geq 0$,*

$$\mathbf{Spt}\,\lambda = \{\sigma^{(1)},\sigma^{(2)},\dots,\sigma^{(d+m)}\} \subset \mathbf{S}^{d-1} \quad \textit{and} \quad \mathbf{Span\,Spt}\,\lambda = \mathbf{R}^d,$$

that is, the support of λ is only finitely many points which linearly spans $\mathbf{R}^d$.

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Note that we always denote vectors as $\sigma^{(j)} = (\sigma_1^{(j)}, \dots, \sigma_d^{(j)})$.

In the one-dimensional case the asymptotic behavior of $p(y)$ as $y \to \pm\infty$ is well-known as follows. If λ has mass at $\{+1\}$, then $p(y) \sim C(\alpha) y^{-1-\alpha}$ as $y \to +\infty$, with some constant $C(\alpha) > 0$ which is determined by α and $\lambda(\{\pm 1\})$. Also if λ does not have mass at $\{-1\}$, then $p(y) = 0$ if and only if $y \le 0$ and $0 < \alpha < 1$. Moreover

$$\alpha = 1 \Rightarrow p(y) \sim \frac{1}{2\sqrt{ce}} \exp\left[\frac{\pi|y|}{4c} - \frac{2c}{\pi e} \exp\left(\frac{\pi|y|}{2c}\right)\right] \quad (y \to -\infty),$$

$$1 < \alpha < 2 \Rightarrow p(y) \sim C(\alpha)' |y|^{(2-\alpha)/(2\alpha-2)} \exp[-\gamma |y|^{\alpha/(\alpha-1)}] \quad (y \to -\infty),$$

where constants $C(\alpha)', c, \gamma > 0$ are determined by α and $\lambda(\{-1\})$ (cf. [2]). Note that for positive functions $f(r), g(r)$ of $r \ge 1$, $f(r) \sim g(r)$ $(r \to \infty)$ means $\lim_{r\to\infty} f(r)/g(r) = 1$.

In the two-dimensional case and in some special cases of three-dimension, we gave the asymptotic behavior of $p(r\sigma)$ in [1].

In this paper we give the asymptotic behavior of $p(r\sigma)$ in the general dimension $d \ge 1$. For each $n = 1, 2, \dots, d$, let

$$S(n) := \left\{ \sum_{s=1}^{n} a_s \sigma^{(j_s)}; a_s \ge 0, j_s = 1, 2, \dots, d+m \ (s = 1, 2, \dots, n) \right\} \cap \mathbf{S}^{d-1}$$

and

$$T(n) := S(n) \backslash S(n-1) \quad \text{with } S(0) := \emptyset.$$

That is, $\sigma \in T(n)$ means σ can be expressed by a linear sum of just n-number of independent vectors of $T(1) = S(1) = \{\sigma^{(1)}, \sigma^{(2)}, \dots, \sigma^{(d+m)}\}$ with positive coefficients and it can not be by less than n-number of independent vectors with positive coefficients (note that σ may be also expressed by more than n-number of independent vectors with positive coefficients).

Let $\mathbf{Int}\, S(d)$ denote the interior of $S(d)$ in $\mathbf{S}^{d-1}$ and for $r \ge 1$,

$$h_\alpha(r) := \begin{cases} \exp\left[\frac{\pi r}{4} - \frac{2}{\pi e} \exp\left(\frac{\pi r}{2}\right)\right] & (\alpha = 1), \\ r^{(2-\alpha)/(2\alpha-2)} \exp[-r^{\alpha/(\alpha-1)}] & (1 < \alpha < 2). \end{cases}$$

THEOREM 1. *Under Assumption* 1, *the following hold with some constants* $C(\alpha, \sigma) > 0$, $0 < C_1 \le C_2$, $\gamma_1 > \gamma_2 > 0$ *which are independent of* $r \ge 1$.

(i) *Let* $0 < \alpha < 1$. *If* $\sigma \in T(n) \cap \mathbf{Int}\, S(d)$ *for some* $n = 1, \dots, d$, *then* $p(r\sigma) \sim C(\alpha, \sigma) r^{-n(1+\alpha)}$ *as* $r \to \infty$. *If* $\sigma \notin \mathbf{Int}\, S(d)$, *then* $p(r\sigma) = 0$ *for all* $r \ge 0$.

(ii) *Let* $1 \le \alpha < 2$. *If* $\sigma \in T(n)$ *for some* $n = 1, \ldots, d$, *then* $p(r\sigma) \sim C(\alpha, \sigma) r^{-n(1+\alpha)}$ *as* $r \to \infty$. *If* $\sigma \notin S(d)$, *then* $C_1 h_\alpha(\gamma_1 r) \le p(r\sigma) \le C_2 h_\alpha(\gamma_2 r)$ *for all* $r \ge 1$.

It is possible to determine the constant $C(\alpha, \sigma)$ exactly. We shall give a more detailed result at the end of the next section (see Theorem 2). From the above result the following is immediately obtained.

Corollary 1. *If* $S(d) = \mathbf{S}^{d-1}$ *and* $\sigma \in T(n)$ *for some* $n = 1, \ldots, d$, *then* $p(r\sigma) \sim C(\alpha, \sigma) r^{-n(1+\alpha)}$ *as* $r \to \infty$.

2. Further Results

Let $e^{(j)}$ be the unit vector in x_j-axis direction $(j = 1, \ldots, d)$. Adding to Assumption 1, we may suppose $\{\sigma^{(1)}, \sigma^{(2)}, \ldots, \sigma^{(d)}\}$ linearly spans $\mathbf{R}^d$ and there is a $d \times d$-regular matrix Q such that $\sigma^{(j)} = Q e^{(j)}$, by changing the order of $\{\sigma^{(j)};\ j = 1, 2, \ldots, d + m\}$ if necessary, where we regard $\sigma^{(j)}, e^{(j)}$ as column vectors (Q is given by $Q = (\sigma^{(1)} \cdots \sigma^{(d)})$). Let

$$p_Q(x) := |\det Q| p(Qx), \quad \text{or equivalently,} \quad p(x) = |\det Q|^{-1} p_Q(Q^{-1}x).$$

If we denote

$$\Psi(z) = \Psi_\lambda(z) = \int_{\mathbf{S}^{d-1}} F(\langle z, \theta \rangle) \lambda(d\theta)$$

with a suitable function F, and let ${}^tQ$ be a transposed matrix of Q, then the log-characteristic function $\Psi_Q(z)$ of $p_Q(x)$ is given by $\Psi_\lambda({}^tQ^{-1}z) = \Psi_{\lambda_Q}(z)$, where $\lambda_Q(d\theta) = \lambda(Q\, d\theta)$ on $Q^{-1}(\mathbf{S}^{d-1})$. Thus $\mathbf{Spt}\, \lambda_Q$ contains $e^{(j)} = Q^{-1}\sigma^{(j)}$ $(j = 1, \ldots, d)$. In fact,

$$\begin{aligned}(2\pi)^d p_Q(x) &= |\det Q| \int_{\mathbf{R}^d} \exp[-i\langle Qx, z \rangle + \Psi_\lambda(z)]\, dz \\ &= |\det Q| \int_{\mathbf{R}^d} \exp[-i\langle x, {}^tQz \rangle + \Psi_\lambda(z)]\, dz \\ &= \int_{\mathbf{R}^d} \exp[-i\langle x, w \rangle + \Psi_\lambda({}^tQ^{-1}w)]\, dw.\end{aligned}$$

Moreover by $\langle {}^tQ^{-1}w, \theta \rangle = \langle w, Q^{-1}\theta \rangle$ we have

$$\Psi_\lambda({}^tQ^{-1}w) = \int_{\mathbf{S}^{d-1}} F(\langle w, Q^{-1}\theta \rangle) \lambda(d\theta) = \int_{Q^{-1}(\mathbf{S}^{d-1})} F(\langle w, \tilde{\theta} \rangle) \lambda(Q\, d\tilde{\theta}) = \Psi_{\lambda_Q}(w).$$

This implies $\Psi_Q = \Psi_{\lambda_Q}$. Therefore our results are invariant for regular linear transformations Q by changing $\mathbf{S}^{d-1}$ to $Q^{-1}(\mathbf{S}^{d-1})$.

For each $j = 1, 2, \ldots, d+m$ and $t \in \mathbf{R}$, let

$$\Psi_j(t) = \begin{cases} -\lambda(\{\sigma^{(j)}\})|t|^\alpha \left[1 - i(\operatorname{sgn} t) \tan \frac{\pi\alpha}{2}\right] & (\alpha \neq 1), \\ -\lambda(\{\sigma^{(j)}\})|t| \left[1 + i\frac{2}{\pi}(\operatorname{sgn} t) \log|t|\right] & (\alpha = 1). \end{cases}$$

and let $p_j(y)$ be the one-dimensional α-stable density corresponding to $\Psi_j(t)$. Then $p_j(y)$ is a C^∞ function satisfying the following: $p_j(y) \sim C_j(\alpha) y^{-1-\alpha}$ as $y \to +\infty$. $p_j(y) = 0$ if and only if $y \le 0$, $0 < \alpha < 1$. Moreover

$$\alpha = 1 \Rightarrow p_j(y) \sim \frac{1}{2\sqrt{c_j e}} \exp\left[\frac{\pi|y|}{4c_j} - \frac{2c_j}{\pi e} \exp\left(\frac{\pi|y|}{2c_j}\right)\right] \quad (y \to -\infty),$$

$$1 < \alpha < 2 \Rightarrow p_j(y) \sim C_j(\alpha)' |y|^{(2-\alpha)/(2\alpha-2)} \exp[-\gamma_j |y|^{\alpha/(\alpha-1)}] \quad (y \to -\infty).$$

Here constants $C_j(\alpha), C_j(\alpha)', c_j, \gamma_j > 0$ are determined by α and $\lambda(\{\sigma^{(j)}\})$.

Let $p^{(d)}(x) := p_1(x_1) \cdots p_d(x_d)$ for $x = (x_1, \ldots, x_d)$. If $m = 0$, then $p_Q(x) = p^{(d)}(x)$. If $m \ge 1$, then by $\Psi_Q(z) = \sum_{j=1}^{d+m} \Psi_j(\langle z, Q^{-1}\sigma^{(j)}\rangle)$ we have

$$(2.1) \qquad p_Q(x) = \int_{-\infty}^{\infty} dy_1 \cdots \int_{-\infty}^{\infty} dy_m p^{(d)}(x - y_1 Q^{-1}\sigma^{(d+1)} - \cdots - y_m Q^{-1}\sigma^{(d+m)}) p_{d+1}(y_1) \cdots p_{d+m}(y_m).$$

In fact, in general, if $\tilde{p}(x)$ is a d-dimensional density with a log-characteristic function $\tilde{\Psi}(z) := \Psi_Q(z) - \Psi_j(\langle z, Q^{-1}\sigma^{(j)}\rangle)$, then

$$\begin{aligned} (2\pi)^d p_Q(x) &= \int_{\mathbf{R}^d} \exp[-i\langle x, z\rangle + \Psi_Q(z)]\, dz \\ &= \int_{\mathbf{R}^d} \exp[-i\langle x, z\rangle + \tilde{\Psi}(z)] \exp[\Psi_j(\langle z,, Q^{-1}\sigma^{(j)}\rangle)]\, dz \\ &= \int_{\mathbf{R}^d} \exp[-i\langle x, z\rangle + \tilde{\Psi}(z)] \left(\int_{-\infty}^{\infty} \exp[iy\langle z, Q^{-1}\sigma^{(j)}\rangle] p_j(y)\, dy\right) dz \\ &= \int_{-\infty}^{\infty} dy \int_{\mathbf{R}^d} \exp[-i\langle x - yQ^{-1}\sigma^{(j)}, z\rangle + \tilde{\Psi}(z)] p_j(y)\, dz \\ &= (2\pi)^d \int_{-\infty}^{\infty} \tilde{p}(x - yQ^{-1}\sigma^{(j)}) p_j(y)\, dy. \end{aligned}$$

Hence we have (2.1).

When $\sigma \in T(n)$, we define a family of indexes

$$J(n) := \{\{j_1, \dots, j_n\} \subset \{1, \dots, d+m\}; \sigma = a_1\sigma^{(j_1)} + \dots + a_n\sigma^{(j_n)},$$
$$a_s > 0 \ (s = 1, \dots, n), \{\sigma^{(j_1)}, \dots, \sigma^{(j_n)}\} \text{ are linearly independent}\}.$$

For each $\{j_1, \dots, j_n\} \in J(n)$, we always fix $\{\sigma^{(j_{n+1})}, \dots, \sigma^{(j_d)}\}$ such that $\{\sigma^{(j_1)}, \dots, \sigma^{(j_d)}\}$ is a basis of $\mathbf{R}^d$ and a $d \times d$-matrix $Q_{j_1,\dots,j_n}$ such that $Q_{j_1,\dots,j_n} e^{(i_s)} = \sigma^{(j_s)}$ $(s = 1, \dots, d)$, where $(i_1, \dots, i_d)$ is a permutation of $(1, \dots, d)$. Moreover if $n < d$, then let

$$\Psi^{\perp}_{j_1,\dots,j_n}(z_{i_{n+1}}, \dots, z_{i_d}) := \Psi_{Q_{j_1,\dots,j_n}}(w_1, \dots, w_d) \quad \text{with } w_i = z_{i_s} \ (i = i_s), \ w_i = 0 \ (i \neq i_s)$$

and $p^{\perp}_{j_1,\dots,j_n}(x_{i_{n+1}}, \dots, x_{i_d})$ be a $(d-n)$-dimensional stable density corresponding to $\Psi^{\perp}_{j_1,\dots,j_n}$. It is expressed by

$$\int_{-\infty}^{\infty} dy_1 p_{j_{d+1}}(y_1) \cdots \int_{-\infty}^{\infty} dy_m p_{j_{d+m}}(y_m)$$
$$\times p_{j_{n+1}}\left(x_{i_{n+1}} - \sum_{s=1}^{m} y_s \xi^{(j_{d+s})}_{i_{n+1}}\right) \cdots p_{j_d}\left(x_{i_d} - \sum_{s=1}^{m} y_s \xi^{(j_{d+s})}_{i_d}\right),$$

where $\{j_{d+1}, \dots, j_{d+m}\} := \{1, \dots, d+m\} \backslash \{j_1, \dots, j_d\}$ and $\xi^{(j_{d+s})} := R\sigma^{(j_{d+s})} \in \mathbf{R}^d$ with $R = Q^{-1}_{j_1,\dots,j_n}$. We also set $p^{\perp}_{j_1,\dots,j_d}(0, \dots, 0) := 1$. Now we state a more detailed result than Theorem 1 in case of $\sigma \in T(n)$.

THEOREM 2. *Let $\sigma \in T(n)$ (and $\sigma \in \mathbf{Int}\, S(d)$ if $0 < \alpha < 1$). It holds that*

$$p(r\sigma) \sim \sum_{\{j_1,\dots,j_n\} \in J(n)} |\det Q_{j_1,\dots,j_n}|^{-1} p_{j_1}(ra_1) \cdots p_{j_n}(ra_n) p^{\perp}_{j_1,\dots,j_n}(0, \dots, 0)$$

as $r \to \infty$, where each $p^{\perp}_{j_1,\dots,j_n}(0, \dots, 0)$ is positive and $(a_1, \dots, a_n)$ is determined by $\sigma = \sum_{s=1}^{n} a_s \sigma^{(j_s)}$ such that $a_s > 0$ $(s = 1, \dots, n)$.

3. Proofs of Theorems

Adding Assumption 1, we may also assume $(\sigma^{(1)}, \dots, \sigma^{(d)}) = (e^{(1)}, \dots, e^{(d)})$ and $m \geq 1$. For simplicity, let $\eta^{(j)} := \sigma^{(d+j)}$ $(j = 1, \dots, m)$. Then by (2.1) with $Q = E_d$ (the $d \times d$-unit matrix) we have

$$p(x) = \int_{-\infty}^{\infty} dy_1 \cdots \int_{-\infty}^{\infty} dy_m p^{(d)}(x - y_1\eta^{(1)} - \dots - y_m\eta^{(m)})$$
$$\times p_{d+1}(y_1) \cdots p_{d+m}(y_m),$$

where $p^{(d)}(x) = p_1(x_1) \cdots p_d(x_d)$ for $x = (x_1, \dots, x_d)$.

We first show the latter half of each result of (i) and (ii) in Theorem 1.

PROPOSITION 1. *Let* $S(d) \neq \mathbf{S}^{d-1}$.

(i) *If* $0 < \alpha < 1$ *and* $\sigma \notin \mathbf{Int}\, S(d)$, *then* $p(r\sigma) = 0$ *for* $r \geq 0$.

(ii) *If* $1 \leq \alpha < 2$ *and* $\sigma \notin S(d)$, *then* $C_1 h_\alpha(\gamma_1 r) \leq p(r\sigma) \leq C_2 h_\alpha(\gamma_2 r)$ *for all* $r \geq 1$, *where* $0 < C_1 \leq C_2 < \infty$, $\gamma_1 > \gamma_2 > 0$ *are independent of* $r \geq 1$.

PROOF. Since $e^{(1)}, \dots, e^{(d)} \in S(d) \neq \mathbf{S}^{d-1}$ and $\sigma \notin \mathbf{Int}\, S(d)$, there is a number $i_0 = 1, \dots, d$ such that $\sigma_{i_0} \leq 0$ and we may assume that $S(d) \subset \{\theta \in \mathbf{S}^{d-1}; \theta_{i_0} \geq 0\}$ by using a regular linear transformation if necessary. For simplicity, let $i_0 = 1$. Hence $\eta_1^{(j)} \geq 0$ $(j = 1, \dots, m)$. Moreover $\sigma \notin S(d)$ implies $\sigma_1 < 0$.

(i) Let $0 < \alpha < 1$. If $\sigma \notin \mathbf{Int}\, S(d)$, then $\sigma_1 \leq 0$. By $p_j(y) = 0$ $(y \leq 0)$ for every j,

$$p(r\sigma) = \int_0^\infty dy_1 \cdots \int_0^\infty dy_m p^{(d)}(r\sigma - y_1\eta^{(d+1)} - \cdots - y_m\eta^{(d+m)}) \\ \times p_{d+1}(y_1) \cdots p_{d+m}(y_m).$$

Thus $r\sigma_1 - y_1\eta_1^{(1)} - \cdots - y_m\eta_1^{(m)} \leq 0$ by $\eta_1^{(j)} \geq 0$ for every j. Therefore $p_1(r\sigma_1 - y_1\eta_1^{(1)} - \cdots - y_m\eta_1^{(m)}) = 0$ and hence $p(r\sigma) = 0$.

(ii) Let $1 \leq \alpha < 2$. If $\sigma \notin S(d)$, then $\sigma_1 < 0$. Let $\varepsilon > 0$ be a sufficiently small number such that $-\sigma_1 - \varepsilon(\eta_1^{(1)} + \cdots + \eta_1^{(m)}) > \varepsilon$. By the definition of $h_\alpha(r)$, there exist constants $C_0, \gamma_2 > 0$ such that $p_j(y) \leq C_0 h_\alpha(\gamma_2 r)$ whenever $y \leq -\varepsilon r$, $r \geq 1$ for every $j = 1, \dots, d+m$. We have

$$p(r\sigma) = \sum_{k=0}^{m} \sum_{\substack{\{j_1,\dots,j_k\} \\ \subset \{1,\dots,m\}}} \int_{-\infty}^{-\varepsilon r} dy_{j_1} p_{d+j_1}(y_{j_1}) \cdots \\ \int_{-\infty}^{-\varepsilon r} dy_{j_k} p_{d+j_k}(y_{j_k}) \int_{-\varepsilon r}^{\infty} dy_{j_{k+1}} p_{d+j_{k+1}}(y_{j_{k+1}}) \cdots \\ \int_{-\varepsilon r}^{\infty} dy_{j_m} p_{d+j_m}(y_{j_m}) p^{(d)}(r\sigma - y_1\eta^{(1)} - \cdots - y_m\eta^{(m)}),$$

where $\{j_{k+1}, \dots, j_m\} = \{1, \dots, m\} \backslash \{j_1, \dots, j_k\}$. In the right-hand side if $k = 0$, then the corresponding term satisfies

$$\int_{-\varepsilon r}^{\infty} dy_1 p_{d+1}(y_1) \cdots \int_{-\varepsilon r}^{\infty} dy_m p_{d+m}(y_m) p^{(d)}(r\sigma - y_1\eta^{(1)} - \cdots - y_m\eta^{(m)}) \leq C_0' h_\alpha(\gamma_2 r)$$

for some $C_0' > 0$. In fact, if $y_j \geq -\varepsilon r$ for every j, then

$$r\sigma_1 - y_1\eta_1^{(1)} - \cdots - y_m\eta_1^{(m)} \leq r(\sigma_1 + \varepsilon(\eta_1^{(1)} + \cdots + \eta_1^{(m)})) < -\varepsilon r.$$

Hence $p_1(r\sigma_1 - y_1\eta_1^{(1)} - \cdots - y_m\eta_1^{(m)}) \leq C_0 h_\alpha(\gamma_2 r)$, which implies the above inequality. If $k \geq 1$, then it is easy to see that

$$\int_{-\infty}^{\infty} p^{(d)}(r\sigma - y_1\eta^{(1)} - \cdots - y_m\eta^{(m)})\, dy_j$$

is bounded in $(y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_m)$. Therefore for some constants $C_0'' > 0$,

$$\begin{aligned}
&\int_{-\infty}^{-\varepsilon r} dy_{j_1} p_{d+j_1}(y_{j_1}) p^{(d)}(r\sigma - y_1\eta^{(1)} - \cdots - y_m\eta^{(m)}) \\
&\quad \leq C_0 h_\alpha(\gamma_2 r) \int_{-\infty}^{\infty} p^{(d)}(r\sigma - y_1\eta^{(1)} - \cdots - y_m\eta^{(m)})\, dy_{j_1} \\
&\quad \leq C_0'' h_\alpha(\gamma_2 r).
\end{aligned}$$

Thus we have $p(r\sigma) \leq C_2 h_\alpha(\gamma_2 r)$. Finally, for the lower estimate, since $p_j(y)$ is strictly positive and continuous, if $0 \leq y_j \leq 1$ for every j, then

$$p^{(d)}(r\sigma - y_1\eta^{(1)} - \cdots - y_m\eta^{(m)}) \geq C' h_\alpha(\gamma_1 r)$$

for all $r \geq 1$ with some constants $C' > 0$, $\gamma_1 > 0$. Therefore

$$\begin{aligned}
p(r\sigma) &\geq \int_0^1 dy_1 p_{d+1}(y_1) \cdots \int_0^1 dy_m p_{d+m}(y_m) p^{(d)}(r\sigma - y_1\eta^{(1)} - \cdots - y_m\eta^{(m)}) \\
&\geq C_1 h_\alpha(\gamma_1 r).
\end{aligned}$$

■

Next in order to show the first half of each (i), (ii) in Theorem 1, it suffices to show Theorem 2. We always assume $\sigma \in T(n)$ for some $n = 1, \ldots, d$ (and $\sigma \in$ **Int** $S(d)$ if $0 < \alpha < 1$). Then by using a regular linear transformation, we may also assume that $\sigma = \sigma_1 e^{(1)} + \cdots + \sigma_n e^{(n)}$ with $\sigma_1 > 0, \ldots, \sigma_n > 0$.

Proof of Theorem 2. Let $\varepsilon > 0$ be a sufficiently small number such that

$$c_0 := \min_{j=1,\ldots,n} \{\sigma_j - \varepsilon(|\eta_j^{(1)}| + \cdots + |\eta_j^{(m)}|)\} > 0$$

and $\varepsilon_0 := \varepsilon\, dm \max\{|\eta_j^{(s)}|; j = 1, \ldots, d, s = 1, \ldots, m\}$. We have

$$p(r\sigma)=\sum_{k=0}^{m}\sum_{\substack{\{j_1,\dots,j_k\}\\ \subset\{1,\dots,m\}}}\int_{|y_{j_1}|\ge\varepsilon r}dy_{j_1}p_{d+j_1}(y_{j_1})\cdots$$
$$\int_{|y_{j_k}|\ge\varepsilon r}dy_{j_k}p_{d+j_k}(y_{j_k})\int_{|y_{j_{k+1}}|<\varepsilon r}dy_{j_{k+1}}p_{d+j_{k+1}}(y_{j_{k+1}})\cdots$$
$$\int_{|y_{j_m}|<\varepsilon r}dy_{j_m}p_{d+j_m}(y_{j_m})p^{(d)}(r\sigma-y_1\eta^{(1)}-\cdots-y_m\eta^{(m)}),$$

where $\{j_{k+1},\dots,j_m\}=\{1,\dots,m\}\backslash\{j_1,\dots,j_k\}$.

In the following for positive functions $f_\varepsilon(r),f(r)$ of $r\ge 1$ $(\varepsilon>0)$, let

$$f_\varepsilon(r)\sim f(r)\ \text{as}\ r\to\infty,\ \varepsilon\downarrow 0\quad\text{denote}\quad \lim_{\varepsilon\downarrow 0}\lim_{r\to\infty}f_\varepsilon(r)/f(r)=1.$$

For instance, if $\sigma_j>0$, then $p_j(r\sigma_j\pm\varepsilon)\sim p_j(r\sigma_j)$ as $r\to\infty$, $\varepsilon\downarrow 0$ by $p_j(r)\sim C_j(\alpha)r^{-1-\alpha}$ as $r\to\infty$.

In the case $k=0$, the corresponding term satisfies

$$\int_{|y_1|<\varepsilon r}dy_1p_{d+1}(y_1)\cdots\int_{|y_m|<\varepsilon r}dy_mp_{d+m}(y_m)p^{(d)}(r\sigma-y_1\eta^{(1)}-\cdots-y_m\eta^{(m)})$$
$$\sim p_1(r\sigma_1)\cdots p_n(r\sigma_n)p^{\perp}_{1,\dots,n}(0,\dots,0)$$

as $r\to\infty$, $\varepsilon\downarrow 0$, where $p^{\perp}_{1,\dots,n}(0,\dots,0)$ is given by

(3.1)
$$\int_{-\infty}^{\infty}dy_1p_{d+1}(y_1)\cdots\int_{-\infty}^{\infty}dy_mp_{d+m}(y_m)p_{n+1}\left(-\sum_{s=1}^{m}y_s\eta^{(s)}_{n+1}\right)\cdots p_d\left(-\sum_{s=1}^{m}y_s\eta^{(s)}_d\right)$$

if $n<d$, and $p^{\perp}_{1,\dots,d}(0,\dots,0)=1$ if $n=d$. In fact, let $\tilde\sigma:=(\sigma_1,\dots,\sigma_n)$ and $\tilde\eta^{(s)}:=(\eta^{(s)}_1,\dots,\eta^{(s)}_n)$ $(s=1,\dots,m)$. For each $j=1,\dots,n$, by $p_j(r\sigma_j\pm\varepsilon)\sim p_j(r\sigma_j)$ as $r\to\infty$, $\varepsilon\downarrow 0$, and

$$r\sigma_j-\sum_{s=1}^{m}y_s\eta^{(s)}_j\begin{cases}<r(\sigma_j+\varepsilon(|\eta^{(1)}_j|+\cdots+|\eta^{(m)}_j|)),\\ >r(\sigma_j-\varepsilon(|\eta^{(1)}_j|+\cdots+|\eta^{(m)}_j|))\ge rc_0,\end{cases}$$

we have $p^{(n)}(r\tilde\sigma-y_1\tilde\eta^{(1)}-\cdots-y_m\tilde\eta^{(m)})\sim p^{(n)}(r\tilde\sigma)$ as $r\to\infty$ and $\varepsilon\downarrow 0$. Hence by

$$\begin{aligned}p^{(d)}&(r\sigma-y_1\eta^{(1)}-\cdots-y_m\eta^{(m)})\\ &=p^{(n)}(r\tilde\sigma-y_1\tilde\eta^{(1)}-\cdots-y_m\tilde\eta^{(m)})\\ &\quad\times p_{n+1}(-y_1\eta^{(1)}_{n+1}-\cdots-y_m\eta^{(m)}_{n+1})\cdots p_d(-y_1\eta^{(1)}_d-\cdots-y_m\eta^{(m)}_d),\end{aligned}$$

the above asymptotic is obtained if $p^{\perp}_{1,\dots,n}(0,\dots,0) > 0$. We show that if $n < d$, then

$$\int_{-\infty}^{\infty} dy_1 p_{d+1}(y_1) \cdots \int_{-\infty}^{\infty} dy_m p_{d+m}(y_m) p_{n+1}\left(-\sum_{s=1}^{m} y_s \eta_{n+1}^{(s)}\right) \cdots p_d\left(-\sum_{s=1}^{m} y_s \eta_d^{(s)}\right) > 0$$

(note that it is obvious $p^{\perp}_{1,\dots,n}(0,\dots,0)$ is given by the above formula). When $1 \le \alpha < 2$, $p_j(y)$ is strictly positive and continuous. Hence it is evident. When $0 < \alpha < 1$, we also assumed $\sigma \in \mathbf{Int}\, S(d)$. By $p_j(y) = 0$ for $y \le 0$, $p^{\perp}_{1,\dots,n}(0,\dots,0)$ is equal to

$$\int_{0}^{\infty} dy_1 p_{d+1}(y_1) \cdots \int_{0}^{\infty} dy_m p_{d+m}(y_m) p_{n+1}\left(-\sum_{s=1}^{m} y_s \eta_{n+1}^{(s)}\right) \cdots p_d\left(-\sum_{s=1}^{m} y_s \eta_d^{(s)}\right).$$

The following lemma ensure $p^{\perp}_{1,\dots,n}(0,\dots,0) > 0$ by $p_j(y) > 0$ for $y > 0$ and the continuity of $p_j(y)$.

LEMMA 1. *Let* $1 \le n \le d-1$ *and* $\sigma = \sigma_1 e^{(1)} + \cdots + \sigma_n e^{(n)}$ *with* $\sigma_1 > 0, \dots, \sigma_n > 0$. *If* $\sigma \in \mathbf{Int}\, S(d)$, *then there exists a vector* $(y_1, \dots, y_m)$ *such that* $y_1 > 0, \dots, y_m > 0$ *and* $y_1 \eta_k^{(1)} + \cdots + y_m \eta_k^{(m)} < 0$ *for all* $k = n+1, \dots, d$.

PROOF. For $x \in \mathbf{R}^d$, we denote $\hat{x} := (x_{n+1}, \dots, x_d)$, and $\hat{x} \in \mathbf{Int}(\mathbf{R}_-^{d-n})$ if $x_k < 0$ for every $k = n+1, \dots, d$. We have to show that

$$y_1 \hat{\eta}^{(1)} + \cdots + y_m \hat{\eta}^{(m)} \in \mathbf{Int}(\mathbf{R}_-^{d-n}) \quad \text{for some } y_1 > 0, \dots, y_m > 0.$$

Let $\hat{S}_0 = \mathbf{Con}\{\hat{\sigma}^{(n+1)}, \dots, \hat{\sigma}^{(d+m)}\} \subset \mathbf{R}^{d-n}$ be the convex cone subtended by $\{\hat{\sigma}^{(n+1)}, \dots, \hat{\sigma}^{(d+m)}\} = \{\hat{e}^{(n+1)}, \dots, \hat{e}^{(d)}, \hat{\eta}^{(1)}, \dots, \hat{\eta}^{(m)}\}$. Noting that $\sigma \in \mathbf{R}^n \times \{0\}^{d-n}$, if $\hat{S}_0$ is contained in a half space of $\mathbf{R}^{d-n}$, then $\sigma \in \partial S(d)$. Hence $\sigma \in \mathbf{Int}\, S(d)$ implies $\hat{S}_0 = \mathbf{R}^{d-n}$. Therefore there exists a basis $\{\hat{\sigma}^{(i_1)}, \dots, \hat{\sigma}^{(i_{d-n})}\} \subset \{\hat{\sigma}^{(n+1)}, \dots, \hat{\sigma}^{(d+m)}\}$ of $\mathbf{R}^{d-n}$ such that the cone $\hat{S} = \mathbf{Con}\{\hat{\sigma}^{(i_1)}, \dots, \hat{\sigma}^{(i_{d-n})}\} \subset \mathbf{R}^{d-n}$ satisfies $\mathbf{Int}\, \hat{S} \cap \mathbf{Int}(\mathbf{R}_-^{d-n}) \ne \varnothing$. Thus we fix a point $\hat{x} \in \mathbf{Int}\, \hat{S} \cap \mathbf{Int}(\mathbf{R}_-^{d-n})$ such that $\hat{x} \ne \eta^{(j)}$ $(j = 1, \dots, m)$. Then $\hat{x} = a_1 \hat{\sigma}^{(i_1)} + \cdots + a_{d-n} \hat{\sigma}^{(i_{d-n})}$ with positive numbers $a_i > 0$. Now we can consider the following two cases.

[First case] $\{\hat{\sigma}^{(i_1)}, \dots, \hat{\sigma}^{(i_{d-n})}\}$ does not contain any $\hat{e}^{(k)}$ $(k = n+1, \dots, d)$, i.e.,

$$\{\hat{\sigma}^{(i_1)}, \dots, \hat{\sigma}^{(i_{d-n})}\} = \{\hat{\eta}^{(j_1)}, \dots, \hat{\eta}^{(j_{d-n})}\}.$$

Thus $\hat{x} = a_1 \hat{\eta}^{(j_1)} + \cdots + a_{d-n} \hat{\eta}^{(j_{d-n})}$ with $a_i > 0$. We would like to add other $\hat{\eta}^{(j)}$ $(\ne \hat{\eta}^{(j_i)}, i = 1, \dots, d-n)$ with positive coefficients. In this case for some $\{i_1, \dots, i_q\} \subset \{1, 2, \dots, d-n\}$ $(0 \le q \le d-n)$, $\hat{\eta}^{(j)}$ can be expressed by $\hat{\eta}^{(j)} =$

$\sum_{s=1}^q b_s\hat{\eta}^{(j_{i_s})} - \sum_{i\notin\{i_s\}} c_i\hat{\eta}^{(j_i)}$ with $b_s \geq 0$, $c_i \geq 0$. Note that if $q=0$, then $\hat{\eta}^{(j)} = -(c_1\hat{\eta}^{(j_1)} + \cdots + c_{d-n}\hat{\eta}^{(j_{d-n})})$. Hence

$$\hat{x} = \hat{\eta}^{(j)} + (a_1 + c_1)\hat{\eta}^{(j_1)} + \cdots + (a_{d-n} + c_{d-n})\hat{\eta}^{(j_{d-n})}.$$

On the other hand, if $q \geq 1$, then $\sum_{s=1}^q b_s\hat{\eta}^{(j_{i_s})} = \hat{\eta}^{(j)} + \sum_{i\notin\{i_s\}} c_i\hat{\eta}^{(j_i)}$. Thus for a sufficiently small $\varepsilon > 0$ such that $a_{j_s} - \varepsilon b_s > 0$ $(s = 1, \dots, q)$, we have

$$\begin{aligned}\hat{x} &= \sum_{s=1}^q (a_{j_s} - \varepsilon b_s)\hat{\eta}^{(j_{i_s})} + \varepsilon\sum_{s=1}^q b_q\hat{\eta}^{(j_{i_s})} + \sum_{i\notin\{i_s\}} a_i\hat{\eta}^{(j_i)} \\ &= \sum_{s=1}^q (a_{j_s} - \varepsilon b_s)\hat{\eta}^{(j_{i_s})} + \varepsilon\hat{\eta}^{(j)} + \sum_{i\notin\{i_s\}} (a_i + \varepsilon c_i)\hat{\eta}^{(j_i)}.\end{aligned}$$

Therefore $\hat{x}$ can be expressed by $y_1\hat{\eta}^{(1)} + \cdots + y_m\hat{\eta}^{(m)} \in \mathbf{Int}(\mathbf{R}_-^{d-n})$ with $y_j > 0$.

[Second case] $\{\hat{\sigma}^{(i_1)}, \dots, \hat{\sigma}^{(i_{d-n})}\}$ contains some $\hat{e}^{(k)}$ $(k = n+1, \dots, d)$, that is,

$$\{\hat{\sigma}^{(i_1)}, \dots, \hat{\sigma}^{(i_{d-n})}\} = \{\hat{e}^{(j_1)}, \dots, \hat{e}^{(j_q)}, \hat{\eta}^{(j_{q+1})}, \dots, \hat{\eta}^{(j_{d-n})}\}.$$

Then $\hat{x} = a_1\hat{e}^{(j_1)} + \cdots + a_q\hat{e}^{(j_q)} + b_1\hat{\eta}^{(j_{q+1})} + \cdots + b_{d-n-q}\hat{\eta}^{(j_{d-n})}$ with $a_s > 0$, $b_t > 0$. In this case by the same way as above, we have

$$\hat{x} = c_1\hat{e}^{(j_1)} + \cdots + c_q\hat{e}^{(j_q)} + y_1\hat{\eta}^{(1)} + \cdots + y_m\hat{\eta}^{(m)} \quad \text{with } c_s > 0,\ y_j > 0.$$

This implies $y_1\hat{\eta}^{(1)} + \cdots + y_m\hat{\eta}^{(m)} = \hat{x} - (c_1\hat{e}^{(j_1)} + \cdots + c_q\hat{e}^{(j_q)}) \in \mathbf{Int}(\mathbf{R}_-^{d-n})$. ■

REMARK 1. By this lemma, it can be also shown that $p_{j_1,\dots,j_n}^\perp(0,\dots,0) > 0$ in Theorem 2. In fact, for each $s = 1, \dots, d$, $R\sigma^{(j_s)} = e^{(i_s)}$ holds by $Q_{j_1,\dots,j_n}e^{(i_s)} = \sigma^{(j_s)}$ $(R = Q_{j_1,\dots,j_n}^{-1})$. Hence $\sigma = \sum_{s=1}^n a_s\sigma^{(j_s)}$ implies $R\sigma = \sum_{s=1}^n a_sR\sigma^{(j_s)} = \sum_{s=1}^n a_se^{(i_s)}$. Therefore $p_{j_1,\dots,j_n}^\perp(0,\dots,0)$ is given by the same formula as in (3.1) with $\{R\sigma^{(j_s)}\}_{s=d+1}^{d+m}$ instead of $\{\eta^{(s)}\}_{s=1}^m$.

In the case $k \geq 1$, it is possible to show the following Claim 1. If $k \leq n$ and $\{\eta^{(j_1)}, \dots, \eta^{(j_k)}\}$ are linearly independent, then let

$$J_{j_1,\dots,j_k} := \left\{ \begin{aligned} &\{i_{k+1}, \dots, i_n\} \subset \{1, \dots, d\}; \\ &\sigma = \sum_{s=1}^k a_s\eta^{(j_s)} + \sum_{s=k+1}^n b_se^{(i_s)} \text{ with } a_s > 0, b_s > 0, \\ &\text{where } \{\eta^{(j_1)}, \dots, \eta^{(j_k)}, e^{(i_{k+1})}, \dots, e^{(i_n)}\} \text{ are linearly independent} \end{aligned} \right\}.$$

Note that $J(n)$ can be expressed by the following disjoint unioin:

$$J(n) = \{\{1,\dots,n\}\} \cup \bigcup_{k=1}^{n} \bigcup_{\substack{\{j_1,\dots,j_k\} \\ \subset \{1,\dots,m\}}} \{\{d+j_1,\dots,d+j_k,i_{k+1},\dots,i_n\}; \{i_{k+1},\dots,i_n\} \in J_{j_1,\dots,j_k}\}.$$

(Claim 1) If $k \le n$ and $J_{j_1,\dots,j_k} \neq \varnothing$, then

$$\int_{|y_{j_1}|\ge \varepsilon r} dy_{j_1} p_{d+j_1}(y_{j_1}) \cdots \int_{|y_{j_k}|\ge \varepsilon r} dy_{j_k} p_{d+j_k}(y_{j_k}) \int_{|y_{j_{k+1}}|<\varepsilon r} dy_{j_{k+1}} p_{d+j_{k+1}}(y_{j_{k+1}}) \cdots$$

$$\int_{|y_{j_m}|<\varepsilon r} dy_{j_m} p_{d+j_m}(y_{j_m}) p^{(d)}(r\sigma - y_1\eta^{(1)} - \cdots - y_m\eta^{(m)})$$

$$\sim \sum_{\substack{\{i_{k+1},\dots,i_n\} \\ \in J_{j_1,\dots,j_k}}} C_{i_{k+1},\dots,i_n} p_{d+j_1}(ra_1) \cdots p_{d+j_k}(ra_k) p_{i_{k+1}}(rb_{k+1}) \cdots p_{i_n}(rb_n)$$

as $r \to \infty$, $\varepsilon \downarrow 0$. Otherwise the above term is $o(r^{-n(1+\alpha)})$ as $r \to \infty$ for any small $\varepsilon > 0$. Here $C_{i_{k+1},\dots,i_d} = 1$ $(n = d)$ and if $n < d$, then

$$C_{i_{k+1},\dots,i_n} = \int_{-\infty}^{\infty} dy_{j_{k+1}} p_{d+j_{k+1}}(y_{j_{k+1}}) \cdots \int_{-\infty}^{\infty} dy_{j_m} p_{d+j_m}(y_{j_m})$$

$$\int_{-\infty}^{\infty} dy_{j_1} \cdots \int_{-\infty}^{\infty} dy_{j_k} \prod_{\substack{i=1,\dots,d; \\ i \neq i_{k+1},\dots,i_n}} p_i\left(-\sum_{s=1}^{m} y_s \eta_i^{(s)}\right).$$

Note that $C_{i_{k+1},\dots,i_n}$ is positive. In fact, denote $\{i_1,\dots,i_k,i_{n+1},\dots,i_d\} := \{1,\dots,d\}\backslash\{i_{k+1},\dots,i_n\}$ and let $Q = Q_{d+j_1,\dots,d+j_k,i_{k+1},\dots,i_n}$ be a $d \times d$-matrix such that $Qe^{(i_s)} = \eta^{(j_s)} = \sigma^{(d+j_s)}$ $(s = 1,\dots,k)$ and $Qe^{(i_s)} = e^{(i_s)}$ $(s = k+1,\dots,d)$. By change of variables $(y_{j_1},\dots,y_{j_k})$ to $(\tilde{y}_1,\dots,\tilde{y}_k)$ such that

$$-\tilde{y}_s := \sum_{j=1}^{m} y_j \eta_{i_s}^{(j)} = \sum_{t=1}^{k} y_{j_t} \eta_{i_s}^{(j_t)} + \sum_{t=k+1}^{m} y_{j_t} \eta_{i_s}^{(j_t)} \quad (s = 1,\dots,k),$$

we have the following.

LEMMA 2. *If $n < d$, then*

$$C_{i_{k+1},\dots,i_n} = |\det Q|^{-1} p^{\perp}_{d+j_1,\dots,d+j_k,i_{k+1},\dots,i_n}(0,\dots,0) \quad (>0).$$

PROOF. The positivity of $p^{\perp}_{d+j_1,\dots,d+j_k,i_{k+1},\dots,i_n}(0,\dots,0)$ was mentioned in Remark 1. For the equation, it is enough to show the case $(j_1,\dots,j_m) = (1,\dots,m)$. By the definition, $p^{\perp}_{d+1,\dots,d+k,i_{k+1},\dots,i_n}(0,\dots,0)$ is given by

$$\int_{-\infty}^{\infty} dy_{k+1} p_{d+k+1}(y_{k+1}) \cdots \int_{-\infty}^{\infty} dy_m p_{d+m}(y_m) \int_{-\infty}^{\infty} d\tilde{y}_1 p_{i_1}(\tilde{y}_1) \cdots \int_{-\infty}^{\infty} d\tilde{y}_k p_{i_k}(\tilde{y}_k)$$
$$p_{i_{n+1}}\left(-\sum_{s=1}^{k} \tilde{y}_s (Re^{(i_s)})_{i_{n+1}} - \sum_{t=k+1}^{m} y_t (R\eta^{(t)})_{i_{n+1}}\right)$$
$$\cdots p_{i_d}\left(-\sum_{s=1}^{k} \tilde{y}_s (Re^{(i_s)})_{i_d} - \sum_{t=k+1}^{m} y_t (R\eta^{(t)})_{i_d}\right)$$

For simplicity, we consider the case $(i_1,\dots,i_d) = (1,\dots,d)$. Denote $Q = (Q_{s,t})_{1\le s,t\le d}$ and $R = (R_{s,t})_{1\le s,t\le d}$. Then $Q_{s,t} = \eta_s^{(t)}$ $(t \le k)$ and $Q_{s,t} = \delta_{s,t}$ $(t \ge k+1)$, where $\delta_{s,t} = 1$ $(s = t)$, $= 0$ $(s \ne t)$. Let $Q_k := (Q_{s,t})_{1\le s,t\le k} = (\eta_s^{(t)})_{1\le s,t\le k}$ and $E_j = (\delta_{s,t})_{1\le s,t\le j}$. By $R = Q^{-1}$, we have

$$Q = \begin{pmatrix} Q_k & O \\ * & E_{d-k} \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} Q_k^{-1} & O \\ * & E_{d-k} \end{pmatrix}.$$

Let $u = n+1,\dots,d$. For $t = 1,\dots,k$,

$$\sum_{s=1}^{k} R_{u,s}\eta_s^{(t)} = \sum_{s=1}^{d} R_{u,s}\eta_s^{(t)} - \sum_{s=k+1}^{d} R_{u,s}\eta_s^{(t)} = \delta_{u,t} - \eta_u^{(t)} = -\eta_u^{(t)}.$$

For $s = 1,\dots,k$ and $t = k+1,\dots,m$,

$$(Re^{(i_s)})_{i_u} = (Re^{(s)})_u = R_{u,s} \quad \text{and} \quad (R\eta^{(t)})_{i_u} = (R\eta^{(t)})_u = \sum_{s=1}^{k} R_{u,s}\eta_s^{(t)} + \eta_u^{(t)}.$$

Therefore by change of variables $(\tilde{y}_1,\dots,\tilde{y}_k)$ to $(y_1,\dots,y_k)$ such that

$$-\tilde{y}_s = \sum_{j=1}^{m} y_j \eta_s^{(j)} = \sum_{t=1}^{k} y_t \eta_s^{(t)} + \sum_{t=k+1}^{m} y_t \eta_s^{(t)} \quad (s = 1,\dots,k),$$

we have $d\tilde{y}_1 \cdots d\tilde{y}_k = |\det Q_k|\, dy_1 \cdots dy_k$ and for $u \ge n+1$,

$$-\sum_{s=1}^{k}\tilde{y}_s(Re^{(i_s)})_{i_u}-\sum_{t=k+1}^{m}y_t(R\eta^{(t)})_{i_u}$$

$$=\sum_{s=1}^{k}\left(\sum_{t=1}^{k}y_t\eta_s^{(t)}+\sum_{t=k+1}^{m}y_t\eta_s^{(t)}\right)R_{u,s}-\sum_{t=k+1}^{m}y_t\left(\sum_{s=1}^{k}R_{u,s}\eta_s^{(t)}+\eta_u^{(t)}\right)$$

$$=\sum_{t=1}^{k}y_t\left(\sum_{s=1}^{k}\eta_s^{(t)}R_{u,s}\right)-\sum_{t=k+1}^{m}y_t\eta_u^{(t)}=-\sum_{t=1}^{m}y_j\eta_u^{(t)}.$$

Hence $p^{\perp}_{d+1,\dots,d+k,k+1,\dots,n}(0,\dots,0)=|\det Q|C_{k+1,\dots,n}$ with

$$C_{k+1,\dots,n}=\int_{-\infty}^{\infty}dy_{k+1}p_{d+k+1}(y_{k+1})\cdots\int_{-\infty}^{\infty}dy_m p_{d+m}(y_m)$$

$$\int_{-\infty}^{\infty}dy_1\cdots\int_{-\infty}^{\infty}dy_k\prod_{\substack{i=1,\dots,d;\\ i\neq k+1,\dots,n}}p_i\left(-\sum_{j=1}^{m}y_s\eta_i^{(j)}\right).\qquad\blacksquare$$

We show Claim 1. If $y_j\le-\varepsilon r$, then $p_{d+j}(y_j)$ has an exponential decay and

$$\int_{y_j\le-\varepsilon r}p^{(d)}(r\sigma-y_1\eta^{(1)}-\cdots-y_m\eta^{(m)})\,dy_j\le\int_{-\infty}^{\infty}p^{(d)}(r\sigma-y_1\eta^{(1)}-\cdots-y_m\eta^{(m)})\,dy_j$$

is bounded in $(y_1,\dots,y_{j-1},y_{j+1},\dots,y_m)$. Thus Claim 1 is reduced to the following. Let $v:=-y_{j_{k+1}}\eta^{(j_{k+1})}-\cdots-y_{j_m}\eta^{(j_m)}$, then $|v|\le\varepsilon_0 r$ by $|y_{j_{k+1}}|\le\varepsilon r,\dots,$ $|y_{j_m}|\le\varepsilon r$ (recall $\varepsilon_0=\varepsilon\, dm\max\{|\eta_j^{(s)}|;j=1,\dots,d,s=1,\dots,m\}$).

(Claim 2) If $1\le k\le n$ and $J_{j_1,\dots,j_k}\neq\emptyset$, then

$$\text{(3.2)}\quad\int_{y_{j_1}\ge\varepsilon r}dy_{j_1}p_{d+j_1}(y_{j_1})\cdots\int_{y_{j_k}\ge\varepsilon r}dy_{j_k}p_{d+j_k}(y_{j_k})p^{(d)}(r\sigma-y_{j_1}\eta^{(j_1)}$$

$$-\cdots-y_{j_k}\eta^{(j_k)}+v)$$

$$\sim\sum_{\substack{\{i_{k+1},\dots,i_n\}\\ \in J_{j_1,\dots,j_k}}}C_{i_{k+1},\dots,i_n}(v)p_{d+j_1}(ra_1)\cdots p_{d+j_k}(ra_k)p_{i_{k+1}}(rb_{k+1})\cdots p_{i_n}(rb_n)$$

as $r\to\infty$, $\varepsilon\downarrow 0$, bounded and pointwise in $|v|\le\varepsilon_0 r$. Otherwise, i.e., if $k>n$ or $J_{j_1,\dots,j_k}=\emptyset$, then it is $o(r^{-n(1+\alpha)})$ as $r\to\infty$ for any small $\varepsilon>0$. Here

$$C_{i_{k+1},\dots,i_n}(v):=\int_{-\infty}^{\infty}dy_{j_1}\cdots\int_{-\infty}^{\infty}dy_{j_k}\prod_{\substack{i=1,\dots,d;\\ i\neq i_{k+1},\dots,i_n}}p_i\left(-\sum_{s=1}^{k}y_{j_s}\eta_i^{(j_s)}+v\right).$$

In the above, for positive functions $f(r,\varepsilon,v), g(r)$ ($r \geq 1$, sufficiently small $\varepsilon > 0$ and $v \in \mathbf{R}^d$),

$$f(r,\varepsilon,v) \sim g(r) \quad \text{as } r \to \infty,\ \varepsilon \downarrow 0, \text{ bounded and pointwise in } |v| \leq \varepsilon_0 r$$

means that

$$f(r,\varepsilon,v)1_{\{|v| \leq \varepsilon_0 r\}}/g(r) \quad \text{is bounded in } (r,\varepsilon,v) \text{ and}$$

$$\lim_{\varepsilon \downarrow 0} \lim_{r \to \infty} f(r,\varepsilon,v)1_{\{|v| \leq \varepsilon_0 r\}}/g(r) = 1 \quad \text{for every } v \in \mathbf{R}^d.$$

For simplicity, we consider the case $(j_1,\dots,j_k) = (1,\dots,k)$, that is, $(\eta^{(j_1)},\dots,\eta^{(j_k)}) = (\eta^{(1)},\dots,\eta^{(k)})$ and $(y_{j_1},\dots,y_{j_k}) = (y_1,\dots,y_k)$. Let

$$B := \mathbf{Con}\{\eta^{(1)},\dots,\eta^{(k)}\} = \left\{\sum_{s=1}^{k} a_s\eta^{(s)}; a_s \geq 0, s = 1,2,\dots,k\right\}$$

and $k_0 := \dim B$ ($\leq k$). Fix a basis $\{\eta^{(j_1)},\dots,\eta^{(j_{k_0})}\} \subset \{\eta^{(1)},\dots,\eta^{(k)}\}$ of **Span** B. We may set $\{\eta^{(j_1)},\dots,\eta^{(j_{k_0})}\} = \{\eta^{(1)},\dots,\eta^{(k_0)}\}$.

In the following we always use the same notation $C > 0$ as constants which are independent of $r \geq 1$. They may be different in each line.

Let $k_0 > n$. By using change of variables it is easy to see that

$$\int_{\mathbf{R}} dy_1 \cdots \int_{\mathbf{R}} dy_{k_0} p^{(d)}(r\sigma - y_1\eta^{(1)} - \cdots - y_k\eta^{(k)} + v) \leq C,$$

where C is independent of $r \geq 1$ and $(y_{k_0+1},\dots,y_k)$. Hence we have, by $p_{d+1}(y_1)\cdots p_{d+k_0}(y_{k_0}) \leq Cr^{-k_0(1+\alpha)}$,

$$\begin{aligned}
&\int_{y_1 \geq \varepsilon r} dy_1 p_{d+1}(y_1) \cdots \int_{y_k \geq \varepsilon r} dy_k p_{d+k}(y_k) p^{(d)}\left(r\sigma - \sum_{s=1}^{k} y_s\eta^{(s)} + v\right) \\
&\leq Cr^{-k_0(1+\alpha)} \int_{\mathbf{R}} p_{d+k_0+1}(y_{k_0+1})\, dy_{k_0+1} \cdots \int_{\mathbf{R}} p_{d+k}(y_k)\, dy_k \int_{y_1 \geq \varepsilon r} dy_1 \\
&\quad \cdots \int_{y_{k_0} \geq \varepsilon r} dy_{k_0} p^{(d)}\left(r\sigma - \sum_{s=1}^{k} y_s\eta^{(s)} + v\right) \\
&\leq Cr^{-k_0(1+\alpha)} = o(r^{-n(1+\alpha)}) \quad \text{as } r \to \infty \text{ by } k_0 > n.
\end{aligned}$$

Next let $k_0 \leq n$. We first show the above term is $O(r^{-n(1+\alpha)})$ ($k = k_0$) or $o(r^{-n(1+\alpha)})$ ($k > k_0$) as $r \to \infty$ for any small $\varepsilon > 0$. If $k_0 = n$, then it is evident. Let $k_0 < n$. We need the following lemma. For each $r \geq 1$, let

$$H_\varepsilon(r) := \left\{ x = r\sigma - \sum_{s=1}^{k} y_s \eta^{(s)};\ y_s \geq \varepsilon r \ \ (s = 1, \ldots, k) \right\}.$$

Moreover let

$$I_{k_0} := \left\{ \{i_1, \ldots, i_{k_0}\} \subset \{1, \ldots, d\}; \det \begin{pmatrix} \eta_{i_1}^{(1)} & \eta_{i_1}^{(2)} & \cdots & \eta_{i_1}^{(k_0)} \\ \eta_{i_2}^{(1)} & \eta_{i_2}^{(2)} & \cdots & \eta_{i_2}^{(k_0)} \\ \vdots & \vdots & \ddots & \vdots \\ \eta_{i_{k_0}}^{(1)} & \eta_{i_{k_0}}^{(2)} & \cdots & \eta_{i_{k_0}}^{(k_0)} \end{pmatrix} \neq 0 \right\}$$

and denote $\{i_1, \ldots, i_{k_0}\}^c := \{1, \ldots, d\} \backslash \{i_1, \ldots, i_{k_0}\}$.

LEMMA 3. *Let $k_0 < n$. There exists $\delta > 0$ such that for all $r \geq 1$,*

$$H_\varepsilon(r) \subset \left(\bigcup_{i=1}^{d} C_i^\delta(r) \right) \cup \left(\bigcup_{\{i_1,\ldots,i_{k_0}\} \in I_{k_0}} \ \bigcup_{\substack{\{i_{k_0+1},\ldots,i_n\} \\ \subset \{i_1,\ldots,i_{k_0}\}^c}} D_{i_{k_0+1},\ldots,i_n}^\delta(r) \right),$$

where $\delta > 0$ is independent of $r \geq 1$ and

$$C_i^\delta(r) := \{x \in \mathbf{R}^d; x_i \leq -\delta r\},$$

$$D_{i_{k_0+1},\ldots,i_n}^\delta(r) := \{x \in \mathbf{R}^d; x_{i_{k_0+1}} \geq \delta r, \ldots, x_{i_n} \geq \delta r\}.$$

We shall give the proof in the next section. By this lemma we have

$$\int_{y_1 \geq \varepsilon r} dy_1 p_{d+1}(y_1) \cdots \int_{y_k \geq \varepsilon r} dy_k p_{d+k}(y_k) p^{(d)} \left(r\sigma - \sum_{s=1}^{k} y_s \eta^{(s)} + v \right)$$

$$\leq \int \cdots \int_{\mathbf{R}^k} dy_1 \cdots dy_k \left(\sum_{i=1}^{d} 1_{C_i^\delta(r) \cap H_\varepsilon(r)} \left(r\sigma - \sum_{s=1}^{k} y_s \eta^{(s)} \right) \right.$$

$$\left. + \sum_{\{i_1,\ldots,i_{k_0}\} \subset I_{k_0}} \ \sum_{\substack{\{i_{k_0+1},\ldots,i_n\} \\ \subset \{i_1,\ldots,i_{k_0}\}^c}} 1_{D_{i_{k_0+1},\ldots,i_n}^\delta(r) \cap H_\varepsilon(r)} \left(r\sigma - \sum_{s=1}^{k} y_s \eta^{(s)} \right) \right)$$

$$\times p^{(d)} \left(r\sigma - \sum_{s=1}^{k} y_s \eta^{(s)} + v \right) p_{d+1}(y_1) \cdots p_{d+k}(y_k).$$

Here we may assume $\delta > \varepsilon_0 > 0$ by taking a sufficiently small $\varepsilon > 0$ from the beginning. If $r\sigma - \sum_{s=1}^{k} y_s \eta^{(s)} \in C_i^\delta(r)$, then by $r\sigma_i - \sum_{s=1}^{k} y_s \eta_i^{(s)} \geq -\delta r$ and $|v| \leq \varepsilon_0 r$ we have

$$p^{(d)}\left(r\sigma-\sum_{s=1}^{k}y_s\eta^{(s)}+v\right)\le Cp_i(-\delta' r)\|p_1\|\cdots\|p_{i-1}\|\,\|p_{i+1}\|\cdots\|p_d\|,$$

where $\delta'=\delta-\varepsilon_0>0$ and $\|\cdot\|=\|\cdot\|_\infty$ denotes the supremum norm. Hence the corresponding term has an exponential decay. Next if

$$x=(x_1,\dots,x_d):=r\sigma-\sum_{s=1}^{k}y_s\eta^{(s)}\in D^{\delta}_{i_{k_0+1},\dots,i_n}(r)\cap H_\varepsilon(r)$$

for some $\{i_1,\dots,i_{k_0}\}\in I_{k_0}$ and $\{i_{k_0+1},\dots,i_n\}\subset\{i_1,\dots,i_{k_0}\}^c$, then by using change of variables,

$$\int_{\mathbf{R}}\cdots\int_{\mathbf{R}}dy_1\cdots dy_{k_0}p_{i_1}(x_{i_1})\cdots p_{i_{k_0}}(x_{k_0})\le C$$

and by $y_s\ge\varepsilon r$, we have $p_{d+1}(y_1)\cdots p_{d+k_0}(y_{k_0})\le Cr^{-k_0(1+\alpha)}$. Furthermore by $p^{(d)}=p_{i_1}\cdots p_{i_{k_0}}\cdot p_{i_{k_0+1}}\cdots p_{i_n}\cdot p_{i_{n+1}}\cdots p_{i_d}$ and $p_{i_{k_0+1}}(x_{k_0+1})\cdots p_{i_n}(x_{i_n})\le Cr^{-(n-k_0)(1+\alpha)}$, it holds

$$\begin{aligned}&\int\cdots\int_{\mathbf{R}^{k_0}}dy_1\cdots dy_{k_0}1_{D^{\delta}_{i_{k_0+1},\dots,i_n}(r)\cap H_\varepsilon(r)}\left(r\sigma-\sum_{s=1}^{k}y_s\eta^{(s)}\right)p^{(d)}\left(r\sigma-\sum_{s=1}^{k}y_s\eta^{(s)}+v\right)\\
&\le Cr^{-(n-k_0)(1+\alpha)}\int_{y_1\ge\varepsilon r}\cdots\int_{y_{k_0}\ge\varepsilon r}dy_1\cdots dy_{k_0}p_{i_1}(x_{i_1})\cdots p_{i_{k_0}}(x_{i_{k_0}})\\
&\le Cr^{-(n-k_0)(1+\alpha)}.\end{aligned}$$

If $k>k_0$, then

$$\int_{\varepsilon r}^{\infty}dy_{k_0+1}\cdots\int_{\varepsilon r}^{\infty}dy_kp_{d+k_0+1}(y_{k_0+1})\cdots p_{d+k}(y_k)=O(r^{-(k-k_0)\alpha})\to 0$$

as $r\to\infty$. Hence for $k\ge k_0$,

$$\begin{aligned}&\int\cdots\int_{\mathbf{R}^{k}}dy_1\cdots dy_k1_{D^{\delta}_{i_{k_0+1},\dots,i_n}(r)\cap H_\varepsilon(r)}\left(r\sigma-\sum_{s=1}^{k}y_s\eta^{(s)}\right)\\
&\quad\times p^{(d)}\left(r\sigma-\sum_{s=1}^{k}y_s\eta^{(s)}+v\right)p_{d+1}(y_1)\cdots p_{d+k}(y_k)\\
&\begin{cases}\le Cr^{-n(1+\alpha)} & (k=k_0)\\ =o(r^{-n(1+\alpha)}) & (k>k_0).\end{cases}\end{aligned}$$

Thus we also have $p(r\sigma)\le Cr^{-n(1+\alpha)}$ for all $r\ge 1$.

We show the asymptotic behavior (3.1). From the above estimate, it is enough to consider the case $1 \le k = k_0 \le n$ and

$$x = (x_1, \dots, x_d) := r\sigma - \sum_{s=1}^{k} y_s \eta^{(s)} \in H_\varepsilon(r) \cap \{x_1 > -\delta r, \dots, x_d > -\delta r\}.$$

First consider the main term. Let $\{i_{k+1}, \dots, i_n\} \in J_{1,\dots,k}$ $(\neq \varnothing)$, that is, σ can be expressed by $\sigma = \sum_{s=1}^{k} a_s \eta^{(s)} + \sum_{s=k+1}^{n} b_s e^{(i_s)}$ with $a_s > 0$, $b_s > 0$ and linearly independent vectors $\{\eta^{(1)}, \dots, \eta^{(k)}, e^{(i_{k+1})}, \dots, e^{(i_n)}\}$.

$$r\sigma - \sum_{s=1}^{k} y_s \eta^{(s)} = \sum_{s=1}^{k} (ra_s - y_s)\eta^{(s)} + \sum_{s=k+1}^{n} rb_s e^{(i_s)}.$$

We divide the integral area $E_r := \{(y_1, \dots, y_k); y_s \ge \varepsilon r \ (s = 1, \dots, k)\}$ to $E_r = F_r \cup G_r$ such that

$$F_r := \bigcup_{\{i_{k+1},\dots,i_n\} \in J_{1,\dots,k}} F_{i_{k+1},\dots,i_n}(r) \quad \text{and} \quad G_r := \bigcap_{\{i_{k+1},\dots,i_n\} \in J_{1,\dots,k}} G_{i_{k+1},\dots,i_n}(r),$$

where, noting that $\{a_1, \dots, a_k\}$ is determined by $\{i_{k+1}, \dots, i_n\}$,

$$F_{i_{k+1},\dots,i_n}(r) := \{(y_1, \dots, y_k) \in E_r; |ra_s - y_s| < \varepsilon r \text{ for all } s = 1, \dots, k\},$$

$$G_{i_{k+1},\dots,i_n}(r) := \{(y_1, \dots, y_k) \in E_r; |ra_s - y_s| \ge \varepsilon r \text{ for some } s = 1, \dots, k\}.$$

If $\varepsilon > 0$ is sufficiently small, then $\{F_{i_{k+1},\dots,i_n}(r)\}$ are disjoint. If $J_{1,\dots,k} = \varnothing$, then $F_r = \varnothing$ and $G_r = E_r$. By change of variables $\tilde{y}_s = ra_s - y_s$, $F_{i_{k+1},\dots,i_n}(r)$ is changed to

$$\tilde{F}_{i_{k+1},\dots,i_n}(r) := \{(\tilde{y}_1, \dots, \tilde{y}_k); |\tilde{y}_s| < \varepsilon r \text{ for all } s = 1, \dots, k\}$$

and we have

$$\begin{aligned}
&\int_{y_1 \ge \varepsilon r} dy_1 p_{d+1}(y_1) \cdots \int_{y_k \ge \varepsilon r} dy_k p_{d+k}(y_k) 1_{F_{i_{k+1},\dots,i_n}(r)}(y_1, \dots, y_k) \\
&\qquad \times p^{(d)}(r\sigma - y_1\eta^{(1)} - \cdots - y_k\eta^{(k)} + v) \\
&\quad = \int_{\tilde{F}_{i_{k+1},\dots,i_n}(r)} d\tilde{y}_1 \cdots d\tilde{y}_k p_{d+1}(ra_1 + \tilde{y}_1) \cdots p_{d+k}(ra_k + \tilde{y}_k) \\
&\qquad \times p^{(d)}\left(\sum_{s=1}^{k} \tilde{y}_s \eta^{(s)} + \sum_{s=k+1}^{n} rb_s e^{(i_s)} + v\right) \\
&\quad \sim C(v) p_{d+1}(ra_1) \cdots p_{d+k}(ra_k) p_{i_{k+1}}(rb_{k+1}) \cdots p_{i_n}(rb_n)
\end{aligned}$$

as $r \to \infty$, $\varepsilon \downarrow 0$, bounded and pointwise in $v \le \varepsilon_0 r$, where

$$C(v) = \int_{-\infty}^{\infty} dy_1 \cdots \int_{-\infty}^{\infty} dy_k \prod_{\substack{i=1,\dots,d;\\ i \ne i_{k+1},\dots,i_n}} p_i\left(-\sum_{s=1}^{k} y_s \eta_i^{(s)} + v\right).$$

Next on G_r, in order to show the corresponding terms are $o(r^{-n(1+\alpha)})$, we need the following result which is more detail than Lemma 3. For each $\{i_1,\dots,i_k\} \in I_k$, denote $\{i_{k+1},\dots,i_d\} := \{i_1,\dots,i_k\}^c$, i.e., $\{\eta^{(1)},\dots,\eta^{(k)},e^{(i_{k+1})},\dots,e^{(i_d)}\}$ is a basis of $\mathbf{R}^d$. Let

$$I_{k,n} := \{\{i_1,\dots,i_k\} \in I_k;\ \text{there exists } \{i_{k+1},\dots,i_n\} \in J_{1,\dots,k} \text{ such that}$$
$$\{i_{k+1},\dots,i_n\} \subset \{i_1,\dots,i_k\}^c\}$$

and $I_{k,n}^c := I_k \backslash I_{k,n}$. Note that $\{i_1,\dots,i_k\} \in I_{k,n}$ means that σ can be expressed by

$$\sigma = \sum_{s=1}^{k} a_s \eta^{(s)} + \sum_{s=k+1}^{d} b_s e^{(i_s)} \quad \text{with } a_s > 0,\ b_s \ge 0,$$

where just $(n-k)$-number of $\{b_s\}$ are positive and $\{\eta^{(1)},\dots,\eta^{(k)},e^{(i_{k+1})},\dots,e^{(i_d)}\}$ is a basis of $\mathbf{R}^d$.

LEMMA 4. *Let $1 \le k = k_0 \le n$. There exists $\delta > 0$ such that for all $r \ge 1$,*

$$H_\varepsilon(r) \cap \{x_1 > -\delta r,\dots,x_d > -\delta r\} \subset A^\delta_{I_{k,n}}(r) \cup A^\delta_{I^c_{k,n}}(r),$$

where $\delta > 0$ is independent of $r \ge 1$, and

$$A^\delta_{I_{k,n}}(r) := \bigcup_{\{i_1,\dots,i_k\} \in I_{k,n}} \ \bigcup_{\substack{\{i_{k+1},\dots,i_n\}\\ \subset \{i_1,\dots,i_k\}^c}} D^\delta_{i_{k+1},\dots,i_n}(r),$$

$$A^\delta_{I^c_{k,n}}(r) := \bigcup_{\{i_1,\dots,i_k\} \in I^c_{k,n}} \left(\bigcup_{s=1}^{k} \bigcup_{\substack{\{i_{k+1},\dots,i_n\}\\ \subset \{i_1,\dots,i_k\}^c}} D^\delta_{i_s,i_{k+1},\dots,i_n}(r) \cup \bigcup_{\substack{\{i_{k+1},\dots,i_{n+1}\}\\ \subset \{i_1,\dots,i_k\}^c}} D^\delta_{i_{k+1},\dots,i_{n+1}}(r) \right).$$

We give the proof in the next section. We may also assume $\delta > \varepsilon_0 > 0$ by taking a sufficiently small $\varepsilon > 0$ from the beginning. Denote $x = (x_1,\dots,x_d) := r\sigma - \sum_{s=1}^{k} y_s \eta^{(s)}$. We can consider the following two cases.

(Case 1) $x = r\sigma - y_1\eta^{(1)} - \cdots - y_k\eta^{(k)} \in A^\delta_{I_{k,n}}(r)$.

There exist $\{i_1,\dots,i_k\} \in I_{k,n}$ and $\{i_{k+1},\dots,i_n\} \subset \{i_1,\dots,i_k\}^c$ such that $x \in D^\delta_{i_{k+1},\dots,i_n}(r)$. Thus by $|v| \le \varepsilon_0 r$ and $\delta > \varepsilon_0 > 0$, we have

$$p_{i_{k+1}}(x_{i_{k+1}} + v_{i_{k+1}}) \cdots p_{i_d}(x_{i_d} + v_{i_d}) \le C r^{-(n-k)(1+\alpha)} \tag{3.3}$$

for all $r \geq 1$ with some $C > 0$. Moreover by $\{i_1, \ldots, i_k\} \in I_{k,n}$,

$$x = r\sigma - \sum_{s=1}^{k} y_s \eta^{(s)} = \sum_{s=1}^{k} (ra_s - y_s)\eta^{(s)} + \sum_{s=k+1}^{d} rb_s e^{(i_s)} \quad \text{with } a_s > 0,\ b_s \geq 0,$$

where just $(n-k)$-number of $\{b_s\}$ are positive. By change of variables $\tilde{y}_s = ra_s - y_s$, let G_r be changed to $\tilde{G}_r$, then $\tilde{G}_r \subset \{|y_s| \geq \varepsilon r \text{ for some } s \geq k+1\}$. Hence we have

$$\begin{aligned}
&\int_{y_1 \geq \varepsilon r} dy_1 p_{d+1}(y_1) \cdots \int_{y_k \geq \varepsilon r} dy_k p_{d+k}(y_k) 1_{G_r}(y_1, \ldots, y_k) \\
&\qquad \times p^{(d)}(r\sigma - y_1\eta^{(1)} - \cdots - y_k\eta^{(k)} + v) 1_{D^{\delta}_{i_{k+1},\ldots,i_n}(r) \cap H_\varepsilon(r)}(x) \\
&\quad \leq Cr^{-k(1+\alpha)} \int_{G_r} dy_1 \cdots dy_k \\
&\qquad \times p^{(d)}\left(\sum_{s=1}^{k}(ra_s - y_s)\eta^{(s)} + \sum_{s=k+1}^{n} rb_s e^{(i_s)} + v\right) 1_{D^{\delta}_{i_{k+1},\ldots,i_n}(r) \cap H_\varepsilon(r)}(x). \\
&\quad \leq Cr^{-n(1+\alpha)} \int_{\tilde{G}_r} d\tilde{y}_1 \cdots d\tilde{y}_k p_{i_1}\left(\sum_{s=1}^{k} \tilde{y}_s \eta^{(s)}_{i_1} + v_{i_1}\right) \cdots p_{i_k}\left(\sum_{s=1}^{k} \tilde{y}_s \eta^{(s)}_{i_k} + v_{i_k}\right) \\
&\quad = o(r^{-n(1+\alpha)})
\end{aligned}$$

as $r \to \infty$ for any small $\varepsilon > 0$ (by $\tilde{G}_r \downarrow \emptyset$).

(Case 2) $x = r\sigma - y_1\eta^{(1)} - \cdots - y_k\eta^{(k)} \in A^{\delta}_{I^c_{k,n}}(r)$.

Fix $\{i_1, \ldots, i_k\} \in I^c_{k,n}$. If $x \in D^{\delta}_{i_s, i_{k+1},\ldots,i_n}(r)$ for some $s = 1, \ldots, k$ and $\{i_{k+1}, \ldots, i_n\} \subset \{i_1, \ldots, i_k\}^c$, then (3.3) also holds, and by change of variables $(y_1, \ldots, y_k)$ to $(x_{i_1}, \ldots, x_{i_k})$ we have

$$\begin{aligned}
&\int_{y_1 \geq \varepsilon r} dy_1 p_{d+1}(y_1) \cdots \int_{y_k \geq \varepsilon r} dy_k p_{d+k}(y_k) \\
&\qquad \times p^{(d)}(r\sigma - y_1\eta^{(1)} - \cdots - y_k\eta^{(k)} + v) 1_{D^{\delta}_{i_s, i_{k+1},\ldots,i_n}(r) \cap H_\varepsilon(r)}(x) \\
&\quad \leq Cr^{-n(1+\alpha)} \int_{-\infty}^{\infty} dx_{i_1} \cdots \int_{-\infty}^{\infty} dx_{i_k} p_{i_1}(x_{i_1} + v_{i_1}) \cdots p_{i_k}(x_{i_1} + v_{i_k}) 1_{\{x_{i_s} \geq \delta r\}}(x_{i_s}) \\
&\quad \leq Cr^{-n(1+\alpha)} \int_{\delta r}^{\infty} p_{i_s}(x_{i_s} + v_{i_s})\, dx_{i_s} \\
&\quad = Cr^{-n(1+\alpha)} r^{-\alpha} = o(r^{-n(1+\alpha)})
\end{aligned}$$

as $r \to \infty$ for any small $\varepsilon > 0$. If $x \in D^{\delta}_{i_{k+1},\dots,i_{n+1}}(r)$ $(n < d)$ for some $\{i_{k+1},\dots,i_{n+1}\} \subset \{i_1,\dots,i_k\}^c$, then it immediately holds that

$$\int_{y_1 \geq \varepsilon r} dy_1 p_{d+1}(y_1) \cdots \int_{y_k \geq \varepsilon r} dy_k p_{d+k}(y_k)$$

$$\times\, p^{(d)}(r\sigma - y_1\eta^{(1)} - \cdots - y_k\eta^{(k)} + v) 1_{D^{\delta}_{i_{k+1},\dots,i_{n+1}}(r) \cap H_\varepsilon(r)}(x).$$

$$\leq Cr^{-(n+1)(1+\alpha)} = o(r^{-n(1+\alpha)})$$

as $r \to \infty$ for any small $\varepsilon > 0$. ■

4. Proofs of Key Lemmas

We give the proofs of Lemma 3 and Lemma 4. First we give a fundamental result. The following result may be intuitively obvious at least for $d \leq 3$.

LEMMA 5. *If $x = \sum_{s=1}^{k} a_s \eta^{(s)}$ with $a_s > 0$ $(s = 1,\dots,k)$, then there exist a basis $\{\eta^{(i_1)},\dots,\eta^{(i_{k_0})}\} \subset \{\eta^{(1)},\dots,\eta^{(k)}\}$ of* **Span** *B and $c_s \geq 0$ $(s = 1,\dots,k_0)$ such that $x = \sum_{s=1}^{k_0} c_s \eta^{(i_s)}$.*

PROOF. We use the induction on k_0 and $k \geq k_0$. First if $k_0 = 1$, then $k = 1$ (i.e., $\eta^{(1)}$ only) or $k = 2$ (i.e., $\eta^{(1)} = -\eta^{(2)}$) and our claim clearly holds. Next let $\ell_0 \geq 2$. We assume that the result holds in case of $k_0 \leq \ell_0 - 1$ and $k \geq k_0$. We have to show the case $k_0 = \ell_0$ and $k \geq k_0$. If $k = k_0$, then the result is evident. Let $\ell \geq k_0$. We again assume that the result holds for $k_0 \leq k \leq \ell$. Let $x = \sum_{s=1}^{\ell+1} a_s \eta^{(s)}$ with $a_s > 0$ $(s = 1,\dots,\ell+1)$. It suffices to show that it can be expressed by $x = \sum_{s=1}^{k_0} c_s \eta^{(j_s)}$ with $c_s \geq 0$ $(s = 1,\dots,k_0)$, where $\{\eta^{(j_1)},\dots,\eta^{(j_{k_0})}\}$ need not be a basis of **Span** B (because by the assumption of the induction, it can be retaken as a basis). We have

$$x = \sum_{s=1}^{k} a_s \eta^{(s)} + a_{\ell+1}\eta^{(\ell+1)} = \sum_{s=1}^{k_0} c_s \eta^{(i_s)} + a_{\ell+1}\eta^{(\ell+1)} \quad \text{with } c_s \geq 0,$$

where $\{\eta^{(i_1)},\dots,\eta^{(i_{k_0})}\}$ is a basis of **Span** B. If some $c_s = 0$, then the claim holds. Let $c_s > 0$ for all $s = 1,\dots,k_0$. For simplicity, set $\hat{\eta}^{(i_s)} := c_s \eta^{(i_s)}$ and $\hat{\eta}^{(\ell+1)} := a_{\ell+1}\eta^{(\ell+1)}$. Then $\{\hat{\eta}^{(i_1)},\dots,\hat{\eta}^{(i_{k_0})}\}$ is also a basis of **Span** B. Hence

$$\hat{\eta}^{(\ell+1)} = -\sum_{s=1}^{t} b_s \hat{\eta}^{(i_s)} + \sum_{s=t+1}^{k_0} b_s \hat{\eta}^{(i_s)} \quad (b_s \geq 0, 0 \leq t \leq k_0).$$

It is enough to consider the case $t \geq 1$ and we may assume $b_1 \geq b_2 \geq \cdots \geq b_t \geq 0$ by changing the order of $s = 1, \dots, t$, if necessary. Thus

$$x = \sum_{s=1}^{t}(1 - b_s)\hat{\eta}^{(i_s)} + \sum_{s=t+1}^{k_0}(1 + b_s)\hat{\eta}^{(i_s)}.$$

When $b_1 \leq 1$, the claim follows. When $b_1 > 1$,

$$\hat{\eta}^{(i_1)} = -\frac{1}{b_1}\hat{\eta}^{(\ell+1)} - \sum_{s=2}^{t}\frac{b_s}{b_1}\hat{\eta}^{(i_s)} + \sum_{s=t+1}^{k_0}\frac{b_s}{b_1}\hat{\eta}^{(i_s)}.$$

Set $\hat{b}_1 := 1/b_1$ and $\hat{b}_s := b_s/b_1$ $(s = 2, \dots, k_0)$. Then $\hat{b}_s < 1$ $(s = 1, 2, \dots, t)$ and

$$x = (1 - \hat{b}_1)\hat{\eta}^{(\ell+1)} + \sum_{s=2}^{t}(1 - \hat{b}_s)\hat{\eta}^{(i_s)} + \sum_{s=t+1}^{k_0}(1 + \hat{b}_s)\hat{\eta}^{(i_s)}.$$

Therefore the claim holds for $k = \ell + 1$. ■

PROOF OF LEMMA 3. It is enough to show the case $r = 1$ by considering $(x/r, y_s/r)$ instead of (x, y_s). Moreover let $H := \sigma - B$, $C_i^\delta := C_i^\delta(1)$ and $D^\delta_{i_{k_0+1},\dots,i_n} := D^\delta_{i_{k_0+1},\dots,i_n}(1)$. By $H_\varepsilon(1) \subset H$, it suffices to show that for some $\delta > 0$,

$$H \subset \left(\bigcup_{i=1}^{d} C_i^\delta\right) \cup \left(\bigcup_{\{i_1,\dots,i_{k_0}\} \in I_{k_0}} \bigcup_{\substack{\{i_{k_0+1},\dots,i_n\} \\ \subset \{i_1,\dots,i_{k_0}\}^c}} D^\delta_{i_{k_0+1},\dots,i_n}\right). \tag{4.1}$$

[The First Claim] $(\bigcup_{i=1}^{d} C_i^\delta)^c = \{x \in \mathbf{R}^d; x_1 > -\delta, \dots, x_d > -\delta\}$ and

$$\left(\bigcup_{\substack{\{i_{k_0+1},\dots,i_n\} \\ \subset \{i_1,\dots,i_{k_0}\}^c}} D^\delta_{i_{k_0+1},\dots,i_n}\right)^c = \bigcup_{\substack{\{j_1,\dots,j_{d-n+1}\} \\ \subset \{i_1,\dots,i_{k_0}\}^c}} \{x \in \mathbf{R}^d; x_{j_1} < \delta, \dots, x_{j_{d-n+1}} < \delta\} \tag{4.2}$$

In fact, let $\{i_{k_0+1}, \dots, i_d\} := \{1, \dots, d\} \backslash \{i_1, \dots, i_{k_0}\}$. If x is in the left hand side, then x is not such that "at least $(n - k_0)$-number of $\{x_{i_{k_0+1}}, \dots, x_{i_d}\}$ satisfies $x_{i_s} \geq \delta$". That is (noting that the rest number is at most $(d - k_0) - (n - k_0) = d - n$), x is not such that "at most $(d - n)$-number of $\{x_{i_{k_0+1}}, \dots, x_{i_d}\}$ satisfies $x_{i_s} < \delta$". Hence x is such that "at least $(d - n + 1)$-number of $\{x_{i_{k_0+1}}, \dots, x_{i_d}\}$ satisfies $x_{i_s} < \delta$". This implies x is in the right-hand side. The reverse is also true. Thus we have (4.2).

[The Second Claim] It holds that

$$(4.3)\quad (H\cap \mathbf{R}_+^d)\cap \bigcap_{\{i_1,\dots,i_{k_0}\}\in I_{k_0}} \bigcup_{\substack{\{i_n,\dots,i_d\}\\ \subset\{i_1,\dots,i_{k_0}\}^c}} \{x\in \mathbf{R}^d; x_{i_n}=\cdots=x_{i_d}=0\}=\emptyset.$$

In fact, let $x\in H\cap \mathbf{R}_+^d$. If we assume that for any $\{i_1,\dots,i_{k_0}\}\in I_{k_0}$, there exists $\{i_{k_0+1},\dots,i_{n-1}\}\subset\{i_1,\dots,i_{k_0}\}^c$ such that

$$x\in \mathbf{Con}\{e^{(i_1)},\dots,e^{(i_{n-1})}\}=\mathbf{R}_+^d\cap\{x\in\mathbf{R}^d; x_{i_n}=\cdots=x_{i_d}=0\},$$

where $\{i_n,\dots,i_d\}:=\{1,\dots,d\}\backslash\{i_1,\dots,i_{n-1}\}$, then by $H=\sigma-B$, there exist $\beta=\sum_{s=1}^{k_0}a_s\eta^{(s)}\in B$ $(a_s\geq 0)$ such that $x=\sigma-\beta=\sum_{s=1}^{n-1}b_se^{(i_s)}$ $(b_s\geq 0)$. That is,

$$\sigma=\sum_{s=1}^{k_0}a_s\eta^{(s)}+\sum_{s=1}^{n-1}b_se^{(i_s)}\quad\text{with } a_s\geq 0,\ b_s\geq 0.$$

Fix $\{i_1,\dots,i_{k_0}\}\in I_{k_0}$ (which is equivalent to that $\{\eta^{(1)},\dots,\eta^{(k_0)},e^{(i_{k_0+1})},\dots,e^{(i_d)}\}$ is a basis of $\mathbf{R}^d$ by the definition of I_{k_0}). Let $I=\{s=1,\dots,k_0; e^{(i_s)}\notin \mathbf{Span}\,B\}$, $J:=\{1,\dots,k_0\}\backslash I$ and $\ell=\#I$. We may denote $I=\{1,\dots,\ell\}$, $J=\{\ell+1,\dots,k_0\}$ by changing the order. We show that $\ell\geq 1$ is essentially reduced to $\ell=0$ and this case has a contradiction.

First let $\ell=0$, i.e., $I=\emptyset$. Then $J=\{1,\dots,k_0\}$ and $e^{(i_s)}\in\mathbf{Span}\,B$ for all $s\in J$. By applying Lemma 5 with $B_J:=\mathbf{Con}\{B,e^{(i_s)};s\in J\}\subset\mathbf{Span}\,B$ instead of B, we have $\sigma\in T(n_0)$ for some $n_0\leq n-1$. In fact, by the above expression of σ and $\{\eta^{(1)},\dots,\eta^{(k_0)},e^{(i_{k_0+1})},\dots,e^{(i_{n-1})}\}$ are linearly independent, σ can be expressed by a linear sum of at most $(n-1)$-number of these vectors with positive coefficients. This contradicts with $\sigma\in T(n)$.

Next let $\ell\geq 1$. Then

$$\sigma=\tilde{\beta}+\sum_{s=1}^{\ell}b_se^{(i_s)}+\sum_{s=k_0+1}^{n-1}b_se^{(i_s)}\quad\text{with } \tilde{\beta}:=\sum_{s=1}^{k_0}a_s\eta^{(s)}+\sum_{s=\ell+1}^{k_0}b_se^{(i_s)}\in\mathbf{Span}\,B.$$

By Lemma 5, $\tilde{\beta}$ can be expressed by a linear sum of at most k_0-number of linearly independent vectors of $\{\eta^{(1)},\dots,\eta^{(k_0)},e^{(i_s)};s\in J\}$ with positive coefficients. Hence by $\sigma\in T(n)$, at least one $a_s>0$ $(s=1,\dots,\ell)$, we may let $s=1$. Since $e^{(i_1)}$ can be expressed by $e^{(i_1)}=\sum_{s=1}^{k_0}c_s\eta^{(s)}+\sum_{s=k_0+1}^{d}c_se^{(i_s)}$ $(c_s\in\mathbf{R})$, and by $e^{(i_1)}\notin\mathbf{Span}\,B$, we have $c_s\neq 0$ for some $s\geq n$, e.g., let $s=n$. Then $\{\eta^{(1)},\dots,\eta^{(k_0)}$, $e^{(i_{k_0+1})},\dots,e^{(i_{n-1})},e^{(i_1)},e^{(i_{n+1})},\dots,e^{(i_d)}\}$ is also a basis of $\mathbf{R}^d$, i.e., $\{i_n,i_2,\dots,i_{k_0}\}\in I_{k_0}$. Hence by the above assumption there exists $\{j_{k_0+1},\dots,j_{n-1}\}\subset\{i_n,i_2,\dots,i_{k_0}\}^c$ such that $x\in\mathbf{Con}\{e^{(i_n)},e^{(i_2)},\dots,e^{(i_{k_0})},e^{(j_{k_0+1})},\dots,e^{(j_{n-1})}\}$, i.e.,

$$x = b_n' e^{(i_n)} + \sum_{s=2}^{k_0} b_s' e^{(i_s)} + \sum_{s=k_0+1}^{n-1} b_s' e^{(j_s)} \quad \text{with } b_s' \geq 0.$$

Thus by $x = \sum_{s=1}^{n-1} b_s e^{(i_s)}$, we have $b_n' = 0$ and $b_s' = b_s$ $(s = 2, \ldots, k_0)$. Moreover for $s = k_0 + 1, \ldots, n-1$, if $b_s' > 0$, then j_s is a member of $\{i_s; s = k_0 + 1, \ldots, n-1\}$ and $b_s' = b_s > 0$. Thus we may assume $b_s' e^{(j_s)} = b_s e^{(i_s)}$ for all $s = k_0 + 1, \ldots, n-1$. Hence

$$\sigma = \tilde{\beta} + \sum_{s=2}^{\ell} b_s e^{(i_s)} + \sum_{s=k_0+1}^{n-1} b_s e^{(i_s)}.$$

This is the case $\ell - 1$ for $\{i_n, i_2, \ldots, i_{k_0}\} \in I_{k_0}$. Hence the case $\ell \geq 1$ is reduced to $\ell = 0$ and we have a contradict.

[**The Last Claim**] (4.3) implies (4.1) for some $\delta > 0$. In fact, if we first assume for every $\delta > 0$,

$$(H \cap \mathbf{R}_+^d) \cap \bigcap_{\{i_1,\ldots,i_{k_0}\} \in I_{k_0}} \bigcup_{\substack{\{i_n,\ldots,i_d\} \\ \subset \{i_1,\ldots,i_{k_0}\}^c}} \{x \in \mathbf{R}^d; x_{i_n} < \delta, \ldots, x_{i_d} < \delta\} \neq \varnothing.$$

That is, for each $\ell \geq 1$ (let $\delta = 1/\ell$), there exists $x^{(\ell)} \in H \cap \mathbf{R}_+^d$ such that

$$x^{(\ell)} \in \bigcap_{\{i_1,\ldots,i_{k_0}\} \in I_{k_0}} \bigcup_{\substack{\{i_n,\ldots,i_d\} \\ \subset \{i_1,\ldots,i_{k_0}\}^c}} \{x \in \mathbf{R}^d; x_{i_n} < 1/\ell, \ldots, x_{i_d} < 1/\ell\}.$$

This means there exists at least one $\{i_1, \ldots, i_{k_0}\} \in I_{k_0}$, and also exist $\{i_n, \ldots, i_d\} \subset \{i_1, \ldots, i_{k_0}\}^c$ and a subsequence $\{\ell_j\}$ such that for some $\beta^{(\ell_j)} \in B$,

$$\sigma - \beta^{(\ell_j)} = x^{(\ell_j)} \in \{x \in \mathbf{R}_+^d; 0 \leq x_{i_n} < 1/\ell_j, \ldots, 0 \leq x_{i_d} < 1/\ell_j\}.$$

Thus $\beta^{(\ell_j)}$ satisfies $\beta_i^{(\ell_j)} \leq \sigma_i$ $(i = 1, \ldots, d)$ and

$$\lim_{j \to \infty} \beta_{i_s}^{(\ell_j)} = \sigma_{i_s} \quad (s \neq i_n, \ldots, i_d).$$

Since B is a closed convex cone, we may assume $|\beta^{(\ell_j)}| \leq 1$ $(k \geq 1)$. Hence it is possible to take a further subsequence $\{\tilde{\ell}_j\} \subset \{\ell_j\}$ such that a limit point $\beta := \lim_{j \to \infty} \beta^{(\tilde{\ell}_j)}$ exists. Therefore $\beta \in B$, and $x := \sigma - \beta \in H \cap \mathbf{R}^d$ satisfies $x_{i_n} = \cdots = x_{i_d} = 0$. This is inconsistent with (4.3). Hence for some $\delta > 0$, we have

$$(H \cap \mathbf{R}_+^d) \cap \bigcap_{\{i_1,\ldots,i_{k_0}\} \in I_{k_0}} \bigcup_{\substack{\{i_n,\ldots,i_d\} \\ \subset \{i_1,\ldots,i_{k_0}\}^c}} \{x \in \mathbf{R}^d; x_{i_n} < \delta, \ldots, x_{i_d} < \delta\} = \varnothing.$$

Furthermore by the same way we have

$$H \cap \{x_1 > -\delta, \dots, x_d > -\delta\}$$
$$\cap \bigcap_{\{i_1,\dots,i_{k_0}\} \in I_{k_0}} \bigcup_{\substack{\{i_n,\dots,i_d\} \\ \subset \{i_1,\dots,i_{k_0}\}^c}} \{x \in \mathbf{R}^d; x_{i_n} < \delta, \dots, x_{i_d} < \delta\} = \emptyset.$$

Therefore by (4.2) we have (4.1). ∎

Proof of Lemma 4. Let $1 \le k = k_0 \le n$. This lemma can be proved by the same way as above. It is enough to consider the case $r = 1$. Let $H := \sigma - B$, $D^\delta_{i_s, i_{k+1}, \dots, i_n} := D^\delta_{i_s, i_{k+1}, \dots, i_n}(1)$ and $D^\delta_{i_{k+1}, \dots, i_{n+1}} := D^\delta_{i_{k+1}, \dots, i_{n+1}}(1)$. It suffices to show that for some $\delta > 0$,

$$H \cap \{x_1 > -\delta, \dots, x_d > -\delta\} \subset A^\delta_{I_{k,n}} \cup A^\delta_{I^c_{k,n}}, \tag{4.4}$$

where

$$A^\delta_{I_{k,n}} := \bigcup_{\{i_1,\dots,i_k\} \in I_{k,n}} \bigcup_{\substack{\{i_{k+1},\dots,i_n\} \\ \subset \{i_1,\dots,i_k\}^c}} D^\delta_{i_{k+1},\dots,i_n},$$

$$A^\delta_{I^c_{k,n}} := \bigcup_{\{i_1,\dots,i_k\} \in I^c_{k,n}} \left(\left(\bigcup_{s=1}^{k} \bigcup_{\substack{\{i_{k+1},\dots,i_n\} \\ \subset \{i_1,\dots,i_k\}^c}} D^\delta_{i_s, i_{k+1},\dots,i_n} \right) \cup \left(\bigcup_{\substack{\{i_{k+1},\dots,i_{n+1}\} \\ \subset \{i_1,\dots,i_k\}^c}} D^\delta_{i_{k+1},\dots,i_{n+1}} \right) \right).$$

Note that by the first claim of the previous proof, for a fixed $\{i_1, \dots, i_k\} \in I^c_{k,n}$, we have

$$\left(\bigcup_{s=1}^{k} \bigcup_{\substack{\{i_{k+1},\dots,i_n\} \\ \subset \{i_1,\dots,i_k\}^c}} D^\delta_{i_s, i_{k+1},\dots,i_n} \right)^c = \bigcap_{s=1}^{k} \left(\{x_{i_s} \ge \delta\} \cap \bigcup_{\substack{\{i_{k+1},\dots,i_n\} \\ \subset \{i_1,\dots,i_k\}^c}} D^\delta_{i_{k+1},\dots,i_n} \right)^c$$
$$= \{x_{i_1} < \delta, \dots, x_{i_k} < \delta\} \cup \left(\bigcup_{\substack{\{i_n,\dots,i_d\} \\ \subset \{i_1,\dots,i_k\}^c}} \{x_{i_n} < \delta, \dots, x_{i_d} < \delta\} \right)$$

and

$$\left(\bigcup_{\substack{\{i_{k+1},\dots,i_{n+1}\} \\ \subset \{i_1,\dots,i_k\}^c}} D^\delta_{i_{k+1},\dots,i_{n+1}} \right)^c = \bigcup_{\{i_{n+1},\dots,i_d\} \subset \{i_1,\dots,i_k\}^c} \{x_{i_{n+1}} < \delta, \dots, x_{i_d} < \delta\}.$$

Hence by

$$\bigcup_{\{i_n,\ldots,i_d\}\subset\{i_1,\ldots,i_k\}^c} \{x_{i_n}<\delta,\ldots,x_{i_d}<\delta\} \subset \bigcup_{\{i_{n+1},\ldots,i_d\}\subset\{i_1,\ldots,i_k\}^c} \{x_{i_{n+1}}<\delta,\ldots,x_{i_d}<\delta\},$$

we have (noting that if $B\subset C$, then $(A\cup B)\cap C=(A\cap C)\cup B$)

$$\left(\left(\bigcup_{s=1}^{k} \bigcup_{\substack{\{i_{k+1},\ldots,i_n\}\\ \subset\{i_1,\ldots,i_k\}^c}} D^{\delta}_{i_s,i_{k+1},\ldots,i_n}\right)\cup\left(\bigcup_{\substack{\{i_{k+1},\ldots,i_{n+1}\}\\ \subset\{i_1,\ldots,i_k\}^c}} D^{\delta}_{i_{k+1},\ldots,i_{n+1}}\right)\right)^c$$

$$=\left(\{x_{i_1}<\delta,\ldots,x_{i_k}<\delta\}\cap \bigcup_{\substack{\{i_{n+1},\ldots,i_d\}\\ \subset\{i_1,\ldots,i_k\}^c}} \{x_{i_{n+1}}<\delta,\ldots,x_{i_d}<\delta\}\right)$$

$$\cup\left(\bigcup_{\substack{\{i_n,\ldots,i_d\}\\ \subset\{i_1,\ldots,i_k\}^c}} \{x_{i_n}<\delta,\ldots,x_{i_d}<\delta\}\right).$$

In order to show (4.4), by the same way as in the last claim of the previous proof, it is enough to show that

$$(H\cap\mathbf{R}^d_+)\cap(A_{I_{k,n}})^c\cap(B_{I^c_{k,n}}\cup C_{I^c_{k,n}})=\varnothing,$$

where

$$(A_{I_{k,n}})^c := \bigcap_{\{i_1,\ldots,i_k\}\in I_{k,n}} \bigcup_{\substack{\{i_n,\ldots,i_d\}\\ \subset\{i_1,\ldots,i_k\}^c}} \{x_{i_n}=\cdots=x_{i_d}=0\},$$

$$B_{I^c_{k,n}} := \bigcap_{\{i_1,\ldots,i_k\}\in I^c_{k,n}} \left(\{x_{i_1}=\cdots=x_{i_k}=0\}\cap \bigcup_{\substack{\{i_{n+1},\ldots,i_d\}\\ \subset\{i_1,\ldots,i_k\}^c}} \{x_{i_{n+1}}=\cdots=x_{i_d}=0\}\right),$$

$$C_{I^c_{k,n}} := \bigcap_{\{i_1,\ldots,i_k\}\in I^c_{k,n}} \bigcup_{\{i_n,\ldots,i_d\}\subset\{i_1,\ldots,i_k\}^c} \{x_{i_n}=\cdots=x_{i_d}=0\}.$$

Note that $(A_{I_{k,n}})^c\cap(B_{I^c_{k,n}}\cup C_{I^c_{k,n}})=((A_{I_{k,n}})^c\cap B_{I^c_{k,n}})\cup((A_{I_{k,n}})^c\cap C_{I^c_{k,n}})$ and, by $I_k = I_{k,n}\cup I^c_{k,n}$ (disjoint union),

$$(A_{I_{k,n}})^c\cap C_{I^c_{k,n}} = \bigcap_{\{i_1,\ldots,i_k\}\in I_k} \bigcup_{\{i_n,\ldots,i_d\}\subset\{i_1,\ldots,i_k\}^c} \{x_{i_n}=\cdots=x_{i_d}=0\}.$$

Moreover by (4.3) (the second claim in the previous proof) we have $(H \cap \mathbf{R}_+^d) \cap (A_{I_{k,n}})^c \cap C_{I_{k,n}^c} = \emptyset$. Therefore the above claim is reduced to

$$(H \cap \mathbf{R}_+^d) \cap (A_{I_{k,n}})^c \cap B_{I_{k,n}^c} = \emptyset.$$

However we can show that

$$(H \cap \mathbf{R}_+^d) \cap B_{I_{k,n}^c} = \emptyset,$$

more strongly, for any fixed $\{i_k, \dots, i_k\} \in I_{k,n}^c$, it holds that

(4.5)
$$(H \cap \mathbf{R}_+^d) \cap \left(\{x_{i_1} = \cdots = x_{i_k} = 0\} \cap \bigcup_{\substack{\{i_{n+1},\dots,i_d\} \\ \subset \{i_1,\dots,i_k\}^c}} \{x_{i_{n+1}} = \cdots = x_{i_d} = 0\} \right) = \emptyset.$$

In fact, if we assume there exists $x \in H$ such that

$$x \in \mathbf{R}_+^d \cap \{x_{i_1} = \cdots = x_{i_k} = 0\} \cap \bigcup_{\substack{\{i_{n+1},\dots,i_d\} \\ \subset \{i_1,\dots,i_k\}^c}} \{x_{i_{n+1}} = \cdots = x_{i_d} = 0\}. \tag{4.6}$$

By $x \in H$, we have $x = \sigma - \beta$ for some $\beta = \sum_{s=1}^k c_s \eta^{(s)} \in B$ with $c_s \geq 0$. Moreover by (4.6), we also have $x = \sum_{s=k+1}^d b_s e^{(i_s)}$ with $b_s \geq 0$, where at most $(n-k)$-number of $\{b_s\}$ are positive. Hence

$$\sigma = \beta + x = \sum_{s=1}^k c_s \eta^{(s)} + \sum_{s=k+1}^d b_s e^{(i_s)}. \tag{4.7}$$

On the other hand, by the definition of $I_{k,n}^c$, σ can not be expressed by the following form.

$$\sigma = \sum_{s=1}^k a_s \eta^{(s)} + \sum_{s=k+1}^d b_s e^{(i_s)} \quad \text{with } a_s > 0,\ b_s' \geq 0,$$

where just $(n-k)$-number of $\{b_s'\}$ are positive

(note that $\{\eta^{(1)}, \dots, \eta^{(k)}, e^{(i_{k+1})}, \dots, e^{(i_d)}\}$ is a basis of $\mathbf{R}^d$). By $\sigma \in T(n)$, this implies in (4.7) at least $(n-k+1)$-number of $\{b_s\}$ are positive. This contradicts. Therefore we have (4.5), and hence, (4.4) holds. ■

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