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THE CLASSIFICATION OF 3-DIMENSIONAL HYPERSURFACE PURELY ELLIPTIC SINGULARITIES OF (0, 1)-TYPE

By

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1. Introduction

We are interested in an isolated 3-dimensional hypersurface purely elliptic singularity defined by a nondegenerate polynomial, especially $(0,1)$-type. The notion of a purely elliptic singularity was introduced for a normal singularity by Watanabe [9], where this singularity coincide with a log-canonical, non log-terminal singularity. The type of the isolated purely elliptic singularity was defined by Ishii [3]. The isolated 3-dimensional hypersurface purely elliptic singularities are classified into 3 types: $(0,0)$-type, $(0,1)$-type, $(0,2)$-type.

Above all, a singularity of $(0,2)$-type is called a simple K3-singularity and studied in [5], [7] and [11]. As is well-known, Yonemura classifies in [11] isolated quasi-homogeneous hypersurface simple K3-singularities defined by nondegenerate polynomials into 95 classes by weights, while his list is bijective to a list of weighted Q-Fano 3-folds made by Fletcher [1]. However these 95 singularities are scattered. In fact, Ishii shows that they cannot connect to each other under any (FG)-deformation (see [4]).

On the other hand, the isolated n-dimensional hypersurface purely elliptic singularity was characterized by the Newton boundary of its defining polynomial by Watanabe [10] (see Section 2), where the defining polynomial is nondegenerate in the sense of [8]. Recall the Yonemura's classification method using this criterion. 95 classes are determined by taking a positive rational weight whose affine 3-dimensional hyperplane π passes through a point $(1,1,1,1)$, and such that this point is in the interior of a 3-dimensional face of the Newton diagram in $\boldsymbol{R}^4_{\geq 0}$ obtained from π. In the case of a singularity of $(0,1)$-type, we consider an affine 2-dimensional linear space with the same property instead of the 3-dimensional one.

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The notable point is the following fact. Certain affine 2-dimensional linear space of a singularity of $(0,1)$-type is contained by some affine 3-dimensional hyperplane of a singularity of $(0,2)$-type. To see this for all singularities of $(0,1)$-type, we will investigate the singularity of $(0,1)$-type. However the Yonemura's method for a singularity of $(0,2)$-type is not useful because the weight of a 2-dimensional linear space of a singularity of $(0,1)$-type is not determined uniquely.

In this paper, we introduce a new equivalence relation, called *leading equivalence relation*, on defining polynomials giving the isolated n-dimensional hypersurface purely elliptic singularities of the same type in Section 3. The aim of this paper is to classify the isolated 3-dimensional hypersurface purely elliptic singularities of $(0,1)$-type defined by nondegenerate polynomials under this leading equivalence relation. As a result, we classify them into 23 classes and provide a list of representative elements under the leading equivalence relation (see Section 4 Corollary 11). For singularities of $(0,1)$-type in 2-dimension and singularities of $(0,2)$-type in 3-dimension, we can see that the classification under the leading equivalence relation is equal to the one under the analytic equivalence realtion (see Section 4 Theorem 18 and 19). So the number of the classification of singularities under the leading equivalence relation is the same as the number of Saito's classification in [6] and Yonemura's classification in [11]. For an affine 2-dimensional linear space of a singularity of $(0,1)$-type, we also give all affine 3-dimensional hyperplane of $(0,2)$-type containing it in Section 5 Table 3.

Throughout this paper, the symbols $\boldsymbol{N}, \boldsymbol{Z}, \boldsymbol{Q}, \boldsymbol{R}$ denote the sets of natural numbers, integers, rational numbers and real numbers. For a topological space X, $\operatorname{int} X$ means the set of interior points of X. $\# A$ denotes the cardinality of a set A.

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2. The Criterion by the Newton Diagram

The hypersurface purely elliptic singularity is characterized in terms of the Newton boundary and its compact face. For the general definition, see [9] and [10]. First of all we recall some definitions of the Newton diagram.

Let $z = (z_0, \dots, z_n)$ be a variable and $f(z) = \sum_m a_m z^{\boldsymbol{m}} \in \boldsymbol{C}[z_0, \dots, z_n]$ where $m = (m_0, \dots, m_n) \in \boldsymbol{Z}_{\geq 0}^{n+1}$ and $z^m = z_0^{m_0} \cdots z_n^{m_n}$. The Newton diagram $\Gamma_+(f)$ is the convex hull of $\bigcup_{a_m \neq 0}(m + \boldsymbol{R}_{\geq 0}^{n+1})$ in $\boldsymbol{R}_{\geq 0}^{n+1}$ and the Newton boundary $\Gamma(f)$

is the union of the compact faces of $\Gamma_+(f)$. Set $f_\Delta(z) = \sum_{m\in\Delta} a_m z^m$ for a face Δ of $\Gamma(f)$. We say that the polynomial f is nondegenerate if $\partial f_\Delta/\partial z_0 = \cdots = \partial f_\Delta/\partial z_n = 0$ has no solution in $(\boldsymbol{C} - \{0\})^{n+1}$ for any face Δ of $\Gamma(f)$. Let $\delta = (1,\dots,1) \in \boldsymbol{R}^{n+1}$. Then Watanabe shows the following theorem which plays an important role in this paper.

Theorem 1 (Watanabe [10]). *Let f be a nondegenerate polynomial in $\boldsymbol{C}[z_0,\dots,z_n]$ and suppose that the hypersurface $X = \{z \in \boldsymbol{C}^{n+1} \mid f(z) = 0\}$ has an isolated singularity at $x = 0 \in \boldsymbol{C}^{n+1}$. Then,*

(i) *(X,x) is purely elliptic if and only if $\delta \in \Gamma(f)$.*

Let (X,x) be a purely elliptic singularity. Then there exists a unique compact face Δ_0 of $\Gamma(f)$ such that $\delta \in \operatorname{int}\Delta_0$. Let $s = \dim\Delta_0$. Then,

(ii) *(X,x) is of $(0,s-1)$-type if and only if $s \geq 2$ and (X,x) is of $(0,0)$-type if and only if $s = 0$ or 1.*

In this paper, the above corresponding compact face Δ_0 and the polynomial f_{Δ_0} are called leading face and leading term, respectively. For simplicity, we say that f is a $(0,s-1)$-type polynomial in $\boldsymbol{C}[z_0,\dots,z_n]$ if f is a nondegenerate polynomial defining a n-dimensional purely elliptic singularity of $(0,s-1)$-type at $x = 0$.

Yonemura showed 95 hypersurface simple K3 singularities by using the above theorem in [11]. In other words, he classified leading terms of defining polynomials giving simple K3 singularities by weights since they are quasi-homogeneous polynomials in this case.

3. The Leading Equivalence Class

To retrieve leading terms of defining polynomials giving purely elliptic singularities, we introduce a new equivalence relation.

In the following, we always assume that f is nondegenerate. Let S_{n+1} be a symmetric group of degree $n+1$. For $\sigma \in S_{n+1}$, the action of σ for $f(z) = \sum_m a_m z^m$ is as follows.

Definition 2. *$\sigma(f) = \sum_m a_m z^{\sigma(m)}$, where $\sigma(m) = (\sigma(m_0),\dots,\sigma(m_n))$.*

For $f \in \boldsymbol{C}[z_0,\dots,z_n]$, we denote $\Delta(f)$ a compact face of $\Gamma(f)$ such that $\delta \in \operatorname{int}\Delta(f)$, if it exists. For $s \in \boldsymbol{Z}_{\geq 0}$ with $0 \leq s \leq n$, we set

$\Phi_s^n = \{f \in \boldsymbol{C}[z_0, \dots, z_n] \mid \text{There exists } \Delta(f) \text{ of } \Gamma(f) \text{ such that } \dim \Delta(f) = s.\}$.

It is noted that a polynomial $f \in \Phi_s^n$ is a candidate for a $(0, s-1)$-type polynomial in $\boldsymbol{C}[z_0, \dots, z_n]$ in virtue of Theorem 1. Then we introduce an equivalent relation on Φ_s^n using the action σ for $f \in \Phi_s^n$.

DEFINITION 3. *For $f, g \in \Phi_s^n$, we say that f and g are leading equivalent if there exists $\sigma \in S_{n+1}$ such that $\Delta(f)$ and $\Delta(\sigma(g))$ lie on the same s-dimensional linear space. Then we denote it $f \sim g$ and call its equivalence class leading equivalence class.*

EXAMPLE 4. Let $f = x^2 + y^3 + z^6$, $g = x^3 + y^2 + z^6 + yz^3 \in \Phi_2^2$. For $\sigma(m) = (m_1, m_0, m_2)$, $\sigma(g) = x^2 + y^3 + z^6 + xz^3$. Then $\Delta(f)$ and $\Delta(\sigma(g))$ lie on the same 2-dimensional linear space whose normal vector is $(3, 2, 1)$. Hence $f \sim g$.

Consider the set

$$D\Phi_s^n = \{f \in \Phi_s^n \mid f \text{ has an isolated singularity at } 0\}.$$

Note that $D\Phi_s^n$ is the set of $(0, s-1)$-type polynomials in $\boldsymbol{C}[z_0, \dots, z_n]$ for $s \geq 2$ and $D\Phi_0^n \cup D\Phi_1^n$ is the set of $(0,0)$-type polynomials in $\boldsymbol{C}[z_0, \dots, z_n]$ by the definition of Φ_s^n and Theorem 1. Then our aim is the same as determining $D\Phi_s^n/\sim$, especially the case of $n = 3$ and $s = 2$.

In the latter half of this section, a method for determining $D\Phi_s^n/\sim$ is described. As stated above, $D\Phi_s^n$ has two properties: One is the property of Φ_s^n and the other is the property that f has an isolated singularity at 0. We consider $\Phi_s^n/\sim$ at first, and check isolatedness since Φ_s^n seems to be easier to be treated than $D\Phi_s^n$.

Consider a set $\{\Delta(f) \subset \boldsymbol{R}_{\geq 0}^{n+1} \mid f \in \Phi_s^n\}$. From the properties of the Newton boundary and $\delta \in \operatorname{int} \Delta(f)$, the compact face $\Delta(f)$ lies in the intersection of the first quadrant and the hyperplane with a positive rational weight which includes δ. Such a hyperplane can be expressed as

$$H^n(\alpha) = \{(u_1, \dots, u_{n+1}) \in \boldsymbol{R}_{\geq 0}^{n+1} \mid \alpha_1 u_1 + \cdots + \alpha_{n+1} u_{n+1} = 1\}$$

where $\alpha \in W_{n+1} = \{(\alpha_1, \dots, \alpha_{n+1}) \in \boldsymbol{Q}_{>0}^{n+1} \mid \alpha_1 + \cdots + \alpha_{n+1} = 1\}$. Let

$$D_s^n = \bigcup_{\alpha \in W_{n+1}} \left\{ \Delta \subset H^n(\alpha) \,\middle|\, \begin{array}{l} \Delta \text{ is a convex polyhedron, } \dim \Delta = s,\ \delta \in \operatorname{int} \Delta, \\ \text{all vertices of } \Delta \text{ are integral.} \end{array} \right\}.$$

Moreover we prepare a polynomial $h_\Delta = \sum_{m \in \Delta} z^m$ whose all coefficients are one. Then it is clear that $\{\Delta(f) \subset \boldsymbol{R}_{\geq 0}^{n+1} \mid f \in \Phi_s^n\} \subset D_s^n$. For $\Delta \in D_s^n$, it holds that

$h_\Delta \in \Phi_s^n$ since $\Gamma(h_\Delta) = \Delta(h_\Delta) = \Delta$ and $\dim \Delta = s$. Therefore the following proposition holds easily.

PROPOSITION 5. $\{\Delta(f) \subset \boldsymbol{R}_{\geq 0}^{n+1} \mid f \in \Phi_s^n\} = D_s^n$.

Here we introduce an equivalence relation on D_s^n, too.

DEFINITION 6. *For $\Delta_1, \Delta_2 \in D_s^n$, we say that Δ_1 and Δ_2 are leading equivalent if h_{Δ_1} and h_{Δ_2} are leading equivalent. Then we denote it $\Delta_1 \sim \Delta_2$ and its equivalence class $D_s^n/\sim$.*

REMARK 7. For $f \in \Phi_s^n$, it holds $f \sim h_{\Delta(f)}$ because $\Delta(f) = \Delta(h_{\Delta(f)})$.

Consider the mapping

$$H : \Phi_s^n/\sim \;\to D_s^n/\sim$$

defined by $H([f]) = [\Delta(f)]$ where the leading equivalence classes to which f and $\Delta(f)$ belong respectively are denoted by $[f]$ and $[\Delta(f)]$. Then the mapping H is well-defined. Indeed, if $[f] = [g]$, then it holds that $h_{\Delta(f)} \sim f \sim g \sim h_{\Delta(g)}$ by Remark 7, which implies $\Delta(f) \sim \Delta(g)$ by the definition, that is, $H([f]) = H([g])$. Then the following proposition holds.

PROPOSITION 8. *The mapping $H : \Phi_s^n/\sim \;\to D_s^n/\sim$ is bijective.*

PROOF. For any $[\Delta] \in D_s^n/\sim$, there exists $[h_\Delta] \in \Phi_s^n/\sim$ such that $H([h_\Delta]) = [\Delta(h_\Delta)] = [\Delta]$. On the other hand, if $H([f]) = H([g])$ for $[f], [g] \in \Phi_s^n/\sim$, then $\Delta(f) \sim \Delta(g)$, which implies $h_{\Delta(f)} \sim h_{\Delta(g)}$. From Remark 7, it holds $f \sim g$.
Q.E.D.

To determine $D\Phi_s^n/\sim$, we will follow next steps:

Step 1. Determine all elements $[\Delta]$ of $D_s^n/\sim$.
Step 2. Determine the element $[h_\Delta]$ of $\Phi_s^n/\sim$ corresponding to $[\Delta]$.
Step 3. Determine the element of $D\Phi_s^n/\sim$ by finding a polynomial which is leading equivalent to $[h_\Delta]$ and have an isolated singularity at 0.

Therefore we focus on D_s^n which is the set of some figures in the Euclidean space, instead of polynomials.

4. Classifications

In this section, we classify 3-dimensional hypersurface purely elliptic singularities of $(0,1)$-type, that is, determine $D\Phi_2^3/\sim$ using the leading equivalence relation. After that, apply it to other types in 2 and 3-dimensions.

First of all, we prepare the following lemma about the elements of $H^n(\alpha)$ for $\alpha \in W_{n+1}$, which is often used in this section.

Lemma 9. *If $\xi = (\xi_1, \dots, \xi_{n+1}) \in \boldsymbol{Z}_{\geq 0}^{n+1}$ is a point in $H^n(\alpha)$ for $\alpha \in W_{n+1}$ and $\xi \neq \delta$, then there exists some i such that $\xi_i = 0$.*

Proof. Assume that $\xi_i \geq 1$ for all i. From $\xi \in H^n(\alpha)$ and $\alpha \in W_{n+1}$, we have $\alpha_1(\xi_1 - 1) + \cdots + \alpha_{n+1}(\xi_{n+1} - 1) = 0$. Since $\alpha_i > 0$ and the assumption $\xi_i \geq 1$ for all i, we obtain $\xi_1 = \cdots = \xi_{n+1} = 1$. This contradicts $\xi \neq \delta$. *Q.E.D.*

4.1. The $(0,1)$-Type in 3-Dimension

This subsection is devoted to prove the following theorem mainly.

Theorem 10. *All representative elements of $\Phi_2^3/\sim$ are listed in Table 1.*

Table 1

No.	$\Phi_2^3/\sim$	W_4
1	$x^2 + y^3 + z^6w^6$	$(1/2, 1/3, \gamma, 1/6 - \gamma)$
2	$x^2 + y^4 + z^4w^4$	$(1/2, 1/4, \gamma, 1/4 - \gamma)$
3	$x^2 + y^6 + z^3w^3$	$(1/2, 1/6, \gamma, 1/3 - \gamma)$
4	$x^3 + y^3 + z^3w^3$	$(1/3, 1/3, \gamma, 1/3 - \gamma)$
5	$x^3 + y^6 + z^2w^2$	$(1/3, 1/6, \gamma, 1/2 - \gamma)$
6	$x^4 + y^4 + z^2w^2$	$(1/4, 1/4, \gamma, 1/2 - \gamma)$
7	$x^2 + yz^3 + y^4w^6$	$(1/2, \gamma, (1-\gamma)/3, (1-4\gamma)/6)$
8	$x^2 + yz^4 + y^3w^4$	$(1/2, \gamma, (1-\gamma)/4, (1-3\gamma)/4)$
9	$x^2 + y^2z^3 + y^2w^6$	$(1/2, \gamma, (1-2\gamma)/3, (1-2\gamma)/6)$
10	$x^2 + y^2z^4 + y^2w^4$	$(1/2, \gamma, (1-2\gamma)/4, (1-2\gamma)/4)$
11	$x^3 + yz^2 + y^3w^6$	$(1/3, \gamma, (1-\gamma)/2, (1-3\gamma)/6)$
12	$x^3 + yz^3 + y^2w^3$	$(1/3, \gamma, (1-\gamma)/3, (1-2\gamma)/3)$
13	$x^4 + yz^2 + y^2w^4$	$(1/4, \gamma, (1-\gamma)/2, (1-2\gamma)/4)$
14	$xy^2 + xz^3 + xw^6$	$(\gamma, (1-\gamma)/2, (1-\gamma)/3, (1-\gamma)/6)$
15	$xy^2 + xz^4 + xw^4$	$(\gamma, (1-\gamma)/2, (1-\gamma)/4, (1-\gamma)/4)$
16	$xy^3 + xz^3 + xw^3$	$(\gamma, (1-\gamma)/3, (1-\gamma)/3, (1-\gamma)/3)$
17	$xy^2 + x^2z^3 + yw^3 + zw^4$	$(\gamma, (1-\gamma)/2, (1-2\gamma)/3, (1+\gamma)/6)$
18	$x^2y + xz^2 + y^3w^4 + z^3w$	$(\gamma, 1-2\gamma, (1-\gamma)/2, (3\gamma-1)/2)$
19	$x^2y^2 + xz^2 + y^2w^4 + z^2w^2$	$(\gamma, 1/2-\gamma, (1-\gamma)/2, \gamma/2)$
20	$x^2y^2 + xz^3 + yw^3 + z^2w^2$	$(\gamma, 1/2-\gamma, (1-\gamma)/3, (1+2\gamma)/6)$
21	$x^3y^2 + xz^2 + y^2w^3 + z^2w$	$(\gamma, (1-3\gamma)/2, (1-\gamma)/2, \gamma)$
22	$x^3y^4 + xz^2 + yw^3 + zw^2$	$(\gamma, (1-3\gamma)/4, (1-\gamma)/2, (1+\gamma)/4)$
23	$x^2y^2 + x^2z^2 + yzw^2 + y^2w^2 + z^2w^2$	$(\gamma, 1/2-\gamma, 1/2-\gamma, \gamma)$

The right hand side of the elements of $\Phi_2^3/\sim$ *in Table* 1 *are the normal vectors of the hyperplanes which contain the corresponding elements of* $D_2^3/\sim$, *where* $\gamma \in \boldsymbol{Q}_{>0}$.

The proof of Theorem 10 is given after the next Corollary 11 which is the main result.

COROLLARY 11. $\#(D\Phi_2^3/\sim) = 23$.

PROOF. Note that, in Table 1 of Theorem 10, No. $17 \sim xy^2 + x^2z^3 + yw^3$, No. $18 \sim x^2y + xz^2 + y^3w^4$, No. $19 \sim x^2y^2 + xz^2 + y^2w^4$, No. $20 \sim x^2y^2 + xz^3 + yw^3$, No. $21 \sim x^3y^2 + xz^2 + y^2w^3$, No. $22 \sim x^3y^4 + xz^2 + yw^3$ and No. $23 \sim x^2y^2 + x^2z^2 + yzw^2$. Then we see that the polynomials in Table 2 are representative elements of $D\Phi_2^3/\sim$.

Table 2

No.	$D\Phi_2^3/\sim$
1	$x^2 + y^3 + z^6w^6 + z^{12} + w^{13}$
2	$x^2 + y^4 + z^4w^4 + z^8 + w^9$
3	$x^2 + y^6 + z^3w^3 + z^6 + w^7$
4	$x^3 + y^3 + z^3w^3 + z^6 + w^7$
5	$x^3 + y^6 + z^2w^2 + z^4 + w^5$
6	$x^4 + y^4 + z^2w^2 + z^4 + w^5$
7	$x^2 + yz^3 + y^4w^6 + y^5 + w^{31}$
8	$x^2 + yz^4 + y^3w^4 + w^{10} + y^6$
9	$x^2 + y^2z^3 + y^2w^6 + z^5 + w^{10} + y^6$
10	$x^2 + y^2z^4 + y^2w^4 + z^6 + w^6 + y^7$
11	$x^3 + yz^2 + y^3w^6 + y^4 + w^{25}$
12	$x^3 + yz^3 + y^2w^3 + w^6 + y^5$
13	$x^4 + yz^2 + y^2w^4 + y^4 + w^{12}$
14	$xy^2 + xz^3 + xw^6 + x^3 + y^4 + z^6$
15	$xy^2 + xz^4 + xw^4 + x^4 + y^3 + z^6$
16	$xy^3 + xz^3 + xw^3 + y^4 + x^5 + z^4$
17	$xy^2 + x^2z^3 + yw^3 + x^4 + z^9$
18	$x^2y + xz^2 + y^3w^4 + y^4 + w^{17}$
19	$x^2y^2 + xz^2 + y^2w^4 + x^3 + y^7 + w^6$
20	$x^2y^2 + xz^3 + yw^3 + x^4 + y^5$
21	$x^3y^2 + xz^2 + y^2w^3 + x^4 + y^9 + w^4$
22	$x^3y^4 + xz^2 + yw^3 + x^5 + y^{16}$
23	$x^2y^2 + x^2z^2 + yzw^2 + x^4 + y^5 + z^5 + w^4$

Q.E.D.

PROOF OF THEOREM 10. We give some devices for finding all elements of $D_2^3/\sim$. In this subsection, δ always means a point $(1, 1, 1, 1)$.

Recall the property of D_2^3. If $\Delta \in D_2^3$, then there exists $\alpha \in W_4$ such that $\delta \in \operatorname{int}\Delta \subset H^3(\alpha)$ and there exists a 2-dimensional linear space H_2 including Δ

(see Fig. 1). Consequently, if we find all of such a 2-dimensional linear space H_2, then we can easily take out the required convex polygon $\Delta \in D_2^3$. Therefore we will find H_2.

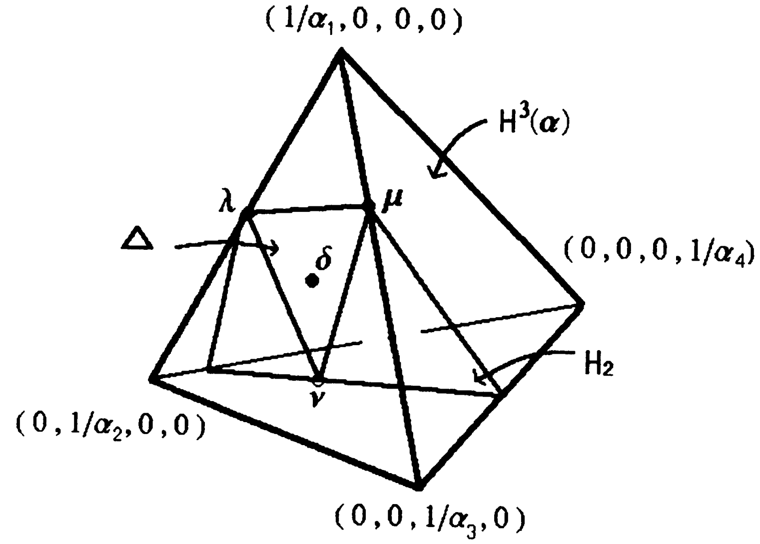

Figure 1

The plane H_2 is determined by 3 points λ, μ, ν with non-negative integral coordinates since $\dim H_2 = 2$. Let V, S and F be the sets of points having 4 coordinates whose only 1 coordinate is not zero, whose only 2 coordinates are not zero and whose only 3 coordinates are not zero, respectively. Lemma 9 implies that (λ, μ, ν) belongs to the one of the following cases:

(I)	$(\lambda, \mu, \nu) \in (V, V, V)$,	(II)	$(\lambda, \mu, \nu) \in (V, V, S)$,
(III)	$(\lambda, \mu, \nu) \in (V, V, F)$,	(IV)	$(\lambda, \mu, \nu) \in (V, S, S)$,
(V)	$(\lambda, \mu, \nu) \in (V, S, F)$,	(VI)	$(\lambda, \mu, \nu) \in (V, F, F)$,
(VII)	$(\lambda, \mu, \nu) \in (S, S, S)$,	(VIII)	$(\lambda, \mu, \nu) \in (S, S, F)$,
(IX)	$(\lambda, \mu, \nu) \in (S, F, F)$,	(X)	$(\lambda, \mu, \nu) \in (F, F, F)$.

Furthermore these cases are classified in more detail.

LEMMA 12. *It holds that (λ, μ, ν) belongs to one of the following cases*:

(II)	$\{(a,0,0,0), (0,b,0,0), (0,0,c,d)\}$	*where* $b \geq a$,
(III)	$\{(a,0,0,0), (0,b,0,0), (c,0,d,e)\}$	*where* $a > c$,
(IV.1)	$\{(a,0,0,0), (0,b,c,0), (0,d,0,e)\}$	*where* $e \geq c$,
(IV.2)	$\{(a,0,0,0), (b,c,0,0), (0,0,d,e)\}$	*where* $a > b$,
(V.1)	$\{(a,0,0,0), (b,c,0,0), (d,0,e,f)\}$	*where* $a > b, d$,

(V.2)	$\{(a,0,0,0),(b,c,0,0),(0,d,e,f)\}$	*where* $a > b$,
(V.3)	$\{(a,0,0,0),(0,b,0,c),(d,0,e,f)\}$	*where* $a > d$,
(V.4)	$\{(a,0,0,0),(0,b,0,c),(0,d,e,f)\}$	*where* $b > d$,
(VI.1)	$\{(a,0,0,0),(b,c,d,0),(e,0,f,g)\}$	*where* $a > b, e$,
(VI.2)	$\{(a,0,0,0),(b,c,d,0),(0,e,f,g)\}$	*where* $a > b$,
(VII.1)	$\{(a,b,0,0),(c,d,0,0),(0,0,e,f)\}$	*where* $a, d \geq 2$ *and* $a > c$ *and* $d > b$,
(VII.2)	$\{(a,b,0,0),(c,0,d,0),(e,0,0,f)\}$,	
(VII.3)	$\{(a,b,0,0),(c,0,d,0),(0,e,0,f)\}$	*where* $f \geq d$,
(VIII.1)	$\{(a,b,0,0),(c,d,0,0),(e,0,f,g)\}$	*where* $a > c$ *and* $d > b$,
(VIII.2)	$\{(a,b,0,0),(c,0,d,0),(0,e,f,g)\}$	*where* $d \geq b$,
(VIII.3)	$\{(a,b,0,0),(c,0,d,0),(e,0,f,g)\}$,	
(VIII.4)	$\{(a,b,0,0),(c,d,e,0),(0,0,f,g)\}$	*where* $a > e$,
(IX.1)	$\{(a,b,c,0),(d,e,f,0),(0,0,g,h)\}$,	
(IX.2)	$\{(a,b,0,0),(c,d,e,0),(f,0,g,h)\}$,	
(IX.3)	$\{(a,b,c,0),(d,0,e,0),(f,0,g,h)\}$,	
(IX.4)	$\{(a,b,c,0),(d,0,e,f),(0,g,0,h)\}$	*where* $h \geq g$,
(X.1)	$\{(a,b,c,0),(d,e,f,0),(g,0,h,i)\}$,	
(X.2)	$\{(a,b,c,0),(d,0,e,f),(g,h,0,i)\}$,	

with $a, b, \dots, i \in \boldsymbol{N}$.

PROOF. The case (I) can be excluded since H_2 determined by (I) cannot include δ. Consider the case (II). If we choose $\{(a,0,0,0),(0,b,0,0),(0,0,c,d)\}$ as $\{\lambda,\mu,\nu\}$ where $a,b,c,d \in \boldsymbol{N}$, then H_2 may contain δ. The other cases in (II) are reduced to this case, and we may assume $b \geq a$ by considering the leading equivalence relation, especially, the permutation. In the case (III), similarly, we may only consider the case $\{(a,0,0,0),(0,b,0,0),(c,0,d,e)\}$, where $a > c$ since the point $(c,0,d,e)$ is an internally dividing point of $(a,0,0,0)$ and some point on ZW-plane. The remaining cases are similar. *Q.E.D.*

On the other hand, δ is included in some convex polygon in H_2. By virtue of the following lemma 13, the set (λ,μ,ν) belongs to either (T) or (Q):

(T) (λ,μ,ν) constructs a triangle which includes δ in the interior.

(Q) (λ,μ,ν) constructs a quadrangle with another point where the intersection point of two diagonal lines is δ.

LEMMA 13. *Let* $p_1,\dots,p_r$ *be the vertices of a convex plane r-gon* X_r *with* $r \geq 4$ *and* x *be an interior point of* X_r. *Let* $\triangle_{p_ip_jp_k}$ *be a triangle spanned by* p_i, p_j, *and* p_k *for* $\{i,j,k\} \subset \{1,2,\dots,r\}$.

(i) *Let* $r = 4$. *There are no triangle* $\triangle_{p_ip_jp_k}$ *which includes* x *in the interior if and only if* x *is an intersection point of two diagonal lines of* X_4.

(ii) *Let* $r \geq 5$. *There exists a triangle* $\triangle_{p_ip_jp_k}$ *which includes* x *in the interior.*

PROOF. The case (i) is clear. For the case (ii), we show the way of finding a triangle which includes x in the interior. Set vertices $p_1, \dots, p_r$ of X_r clockwise and connect p_1 and p_j for any $j \neq 1, 2, r$. Since x is an interior point, either (a) or (b) holds:

(a) There exists $i \in \{2, \dots, r-1\}$ such that $x \in \operatorname{int} \triangle_{p_1p_ip_{i+1}}$.

(b) There exists $i \in \{3, \dots, r-1\}$ such that $x \in \overline{p_1p_i}$ where $\overline{p_1p_i}$ is a segment connected by p_1 and p_i.

The case (a) satisfies the lemma. Consider the case (b). Then x lies in the quadrangle spanned by $p_1, p_{i-1}, p_i, p_{i+1}$. Remembering the case (i), we may assume that $x \in \overline{p_1p_i} \cap \overline{p_{i-1}p_{i+1}}$. Assume that either p_{i-1} or p_{i+1} is next to p_1, for simplicity, p_{i-1} is next to p_1, that is, $i = 3$. Then there exists $j \in \{5, \dots, r\}$ such that $x \in \operatorname{int} \triangle_{p_2p_3p_j}$ because $r \geq 5$. Assume that both p_{i-1} and p_{i+1} are not next to p_1. It is clear that there exists $j \in \{i+2, \dots, r\}$ such that $x \in \operatorname{int} \triangle_{p_{i-1}p_ip_j}$. *Q.E.D.*

Further it is enough to consider only the case (T) by the following lemma.

LEMMA 14. *In each case of Lemma* 12, *the case* (Q) *does not occur or can be reduced into the case* (T).

PROOF. The cases (II), (IV.1) and (VII.2) belong to the case (T). For the other cases, we show only two typical cases (III) and (V.1) since the other cases are proved in the similar way.

Consider the case (III). The condition (Q) means that an another point must be $(0, f, g, h)$ in the YZW-plane where $f, g, h \in \boldsymbol{N}$ and $b > f$ since the intersection point of two diagonal lines is δ. Then there exist $0 < \beta_i < 1$ for $i = 1, \dots, 4$ such that $\beta_1 + \beta_2 = 1$, $\beta_1(a, 0, 0, 0) + \beta_2(0, f, g, h) = \delta$, $\beta_3 + \beta_4 = 1$, $\beta_3(0, b, 0, 0) + \beta_4(c, 0, d, e) = \delta$. The equations imply that

$$\frac{1}{a} + \frac{1}{f} = 1, \quad \frac{1}{c} + \frac{1}{b} = 1, \quad f = g = h, \quad c = d = e.$$

Solve this, we have $a = \cdots = h = 2$, which contradicts $a > c$. Therefore the case (Q) does not occur.

Consider the case of (V.1). Then we may set an another point $(0, g, h, i)$ where $g, h, i \in \boldsymbol{N}$. Similarly we obtain $a = c = e = f = g = h = i = 2$ and $b = d = 1$. Since the quadrangle is included in $H^3(\alpha)$ where $\alpha = (1/2, 1/4, \gamma, 1/(4-\gamma))$

with $\gamma \in \boldsymbol{Q}_{>0}$, there is a vector $(0,0,4,4)$ on the same 2-dimensional plane H_2. Since the triangle constructed by $(2,0,0,0)$, $(1,2,0,0)$ and $(0,0,4,4)$ includes δ in the interior, this case reduces to the case (T). *Q.E.D.*

Summarize the above. We follow the next steps for each case of Lemma 12 to prove Theorem 10. Let $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in W_4$.

Step 1. Find H_2 by determining three points $\lambda, \mu, \nu \in \boldsymbol{Z}^4_{\geq 0}$ satisfying the following conditions (H) and (T):
(H) $\lambda, \mu, \nu \in H^3(\alpha)$,
(T) $\delta \in \operatorname{int}\Delta$ where Δ is a triangle determined by (λ, μ, ν), that is, there exist $0 < \beta_i < 1$ for $i = 1, 2, 3$ such that $\beta_1 + \beta_2 + \beta_3 = 1$ and $\beta_1\lambda + \beta_2\mu + \beta_3\nu = \delta$.

Step 2. Determine all elements $[\Delta]$ of $D^3_2/\sim$, where Δ is chosen as a maximum convex polygon in H_2.

Step 3. Determine an element $[h_\Delta]$ of $\Phi^3_2/\sim$ by using $[\Delta]$.

All elements of $\Phi^3_2/\sim$ have been already listed in Table 1 of Section 4 and are chosen so as to satisfy that

(i) All coefficients are 1,
(ii) The first three monomials include δ in the interior,
(iii) The monomials except the above are necessary for constructing the maximum convex polygon in H_2.

The remaining is devoted to follow the above steps for each case of Lemma 12. For this detailed proof, see [2].

Case (II)

(II) $\{\lambda, \mu, \nu\} = \{(a,0,0,0), (0,b,0,0), (0,0,c,d)\}$ where $b \geq a$ (see Fig. II).

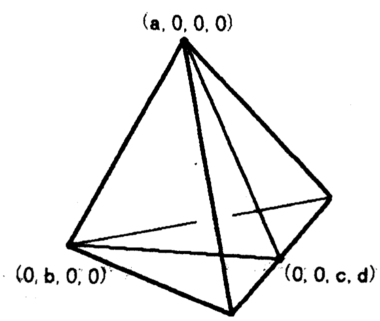

Figure II

The condition (H) implies that

$$\alpha_1 = \frac{1}{a}, \quad \alpha_2 = \frac{1}{b}, \quad c\alpha_3 + d\alpha_4 = 1. \tag{1}$$

Considering $\alpha_i < 1$, we have $a, b \geq 2$. From the condition (T),

$$\beta_1 = \frac{1}{a}, \quad \beta_2 = \frac{1}{b}, \quad \beta_3 = \frac{1}{c}, \quad c = d.$$

Considering $\beta_i < 1$, we have $a, b, c \geq 2$. It follows from (1) and $c = d$ that $1 = \sum_{i=1}^{4} \alpha_i = 1/a + 1/b + 1/c$. Put together these conditions as:

$$a, b, c \geq 2, \quad b \geq a, \quad c = d, \quad \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1.$$

Solutions of the above equation are

$$(a, b, c, d) = (2,3,6,6), (2,4,4,4), (2,6,3,3), (3,3,3,3), (3,6,2,2), (4,4,2,2).$$

Consider the 2-dimensional linear space H_2 determined by $(a,b,c,d) = (2,3,6,6)$, then a triangle Δ obtained by $(2,0,0,0)$, $(0,3,0,0)$ and $(0,0,6,6)$ in H_2 satisfies the condition (T). Therefore $x^2 + y^3 + z^6 w^6$ given by Δ is an element of $\Phi_2^3/\sim$. Similarly, we obtain 6 elements as follows:

No. 1 $x^2 + y^3 + z^6 w^6$, No. 2 $x^2 + y^4 + z^4 w^4$, No. 3 $x^2 + y^6 + z^3 w^3$,
No. 4 $x^3 + y^3 + z^3 w^3$, No. 5 $x^3 + y^6 + z^2 w^2$, No. 6 $x^4 + y^4 + z^2 w^2$.

Case (III)

(III) $\{(a,0,0,0), (0,b,0,0), (c,0,d,e)\}$ where $a > c$.

We have $a, b \geq 2$ from the condition (H). The condition (T) implies that

$$\beta_1 = \frac{d-c}{ad}, \quad \beta_2 = \frac{1}{b}, \quad \beta_3 = \frac{1}{d}, \quad d = e. \tag{2}$$

Considering $0 < \beta_i < 1$, we have $d > c$ and $b, d \geq 2$. If $(c,0,d,e)$ is an internally dividing point of $(a,0,0,0)$ and some integral point on ZW-plane, then this case is reduced to (II). Therefore we assume $ad \notin (a-c)\boldsymbol{Z}$. If $a = 2$, then we have $c = 1$ by $a > c$, which contradicts $ad \notin (a-c)\boldsymbol{Z} = \boldsymbol{Z}$. Hence we assume $a \geq 3$. We obtain the following conditions:

$$a \geq 3, \quad b, d \geq 2, \quad a, d > c, \quad d = e, \quad ad \notin (a-c)\boldsymbol{Z}.$$

Since $\beta_1 + \beta_2 + \beta_3 = 1$ and (2),

$$\beta_2 = 1 - \beta_1 - \beta_3 = 1 - \frac{d-c}{ad} - \frac{1}{d} = \frac{d(a-1) + (c-a)}{ad}.$$

On the other hand, from $\beta_2 = 1/b$,

$$b = \frac{1}{\beta_2} = \frac{ad}{d(a-1) + (c-a)} \geq 2. \tag{3}$$

Assume $c = 1$, then $b = (ad)/\{(a-1)(d-1)\}$ and $ad \notin (a-c)\boldsymbol{Z} = (a-1)\boldsymbol{Z}$, which contradicts $b \in \boldsymbol{N}$. Hence $c \geq 2$. From (3), $a \geq 3$ and $c \geq 2$,

$$2 = \frac{2(a-2)}{a-2} \geq \frac{2(a-c)}{a-2} \geq d.$$

Therefore we obtain $d = 2$, so that $c = 1$ from $d > c$, which contradicts $c \geq 2$.

Case (IV)

(IV.1) $\{(a,0,0,0), (0,b,c,0), (0,d,0,e)\}$ where $e \geq c$ (see Fig. IV.1).

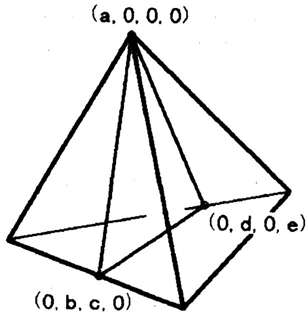

Figure IV.1

By an argument similar to (II), we obtain 7 elements as follows:

No. 7 $x^2 + yz^3 + y^4w^6$, No. 8 $x^2 + yz^4 + y^3w^4$,
No. 9 $x^2 + y^2z^3 + y^2w^6$, No. 10 $x^2 + y^2z^4 + y^2w^4$,
No. 11 $x^3 + yz^2 + y^3w^6$, No. 12 $x^3 + yz^3 + y^2w^3$,
No. 13 $x^4 + yz^2 + y^2w^4$.

(IV.2) $\{(a,0,0,0), (b,c,0,0), (0,0,d,e)\}$ where $a > b$.

This case does not occur by an argument similar to (III).

Case (V)

(V.1) $\{(a,0,0,0),(b,c,0,0),(d,0,e,f)\}$ where $a > b,d$.

From $\beta_1+\beta_2+\beta_3 = 1 = \beta_1 a+\beta_2 b+\beta_3 d$, it holds that $\beta_1(a-1)+\beta_2(b-1)+\beta_3(d-1)=0$. Then we obtain $a=b=d=1$ since $\beta_i>0$ and $a,b,d\in \boldsymbol{N}$, which contradicts $a>b,d$.

(V.2) $\{(a,0,0,0),(b,c,0,0),(0,d,e,f)\}$ where $a>b$.

The condition (T) implies that

$$\beta_1=\frac{e(c-b)+bd}{aec},\quad \beta_2=\frac{e-d}{ec},\quad \beta_3=\frac{1}{e}. \tag{4}$$

We obtain the following conditions:

$$a\geq 3,\quad e\geq 2,\quad a>b,\quad e>d,\quad e=f,\quad ac\notin(a-b)\boldsymbol{Z},$$

where $ac\notin(a-b)\boldsymbol{Z}$ means that $(b,c,0,0)$ is not an internally dividing point of $(a,0,0,0)$ and some integral point on Y-plane. Then $\beta_1=1-\beta_2-\beta_3=\{e(c-1)+(d-c)\}/(ec)$. From (4),

$$a=\frac{e(c-b)+bd}{ec\beta_1}=\frac{e(c-b)+bd}{e(c-1)+(d-c)}\geq 3, \tag{5}$$

$$(3-b)(e-d)\geq c(2e-3). \tag{6}$$

It follows from $e\geq 2$ and $e>d$ that $b=1$ or 2. If $b=1$, we have $c\geq 2$ because $(b,c,0,0)\in H^3(\alpha)$. From (6),

$$2(e-d)\geq c(2e-3)\geq 2(2e-3)$$

$$e-d\geq 2e-3$$

$$3\geq e+d\geq 2+d\geq 3$$

Therefore $d=1$ and $e=2$. Then $a=(2c-1)/(c-1)\geq 3$ implies that $2\geq c$. Since $c\geq 2$, we have $c=2$, so that $a=3$, which contradicts $ac\notin(a-b)\boldsymbol{Z}$. If $b=2$, it holds that $e-d\geq c(2e-3)\geq 2e-3$ by (6). Therefore we have $e=2$ and $d=1$, then $a=2$, which contradicts $a>b$.

(V.3) $\{(a,0,0,0),(0,b,0,c),(d,0,e,f)\}$ where $a>d$.

This case does not occur by an argument similar to (III).

(V.4) $\{(a,0,0,0),(0,b,0,c),(0,d,e,f)\}$ where $b > d$.

This case does not occur by an argument similar to (III).

Case (VI)

(VI.1) $\{(a,0,0,0),(b,c,d,0),(e,0,f,g)\}$ where $a > b,e$.

This case does not occur by an argument similar to (V.1).

(VI.2) $\{(a,0,0,0),(b,c,d,0),(0,e,f,g)\}$ where $a > b$.

This case does not occur by an argument similar to (V.2).

Case (VII)

(VII.1) $\{(a,b,0,0),(c,d,0,0),(0,0,e,f)\}$ where $a,d \geq 2$ and $a > c$ and $d > b$.

At first, we need some consideration about a and b. The condition (H) implies that

$$a\alpha_1 + b\alpha_2 = 1, \quad e\alpha_3 + f\alpha_4 = 1. \tag{7}$$

The condition (T) implies that

$$a\beta_1 + c\beta_2 = 1, \quad b\beta_1 + d\beta_2 = 1, \quad \beta_3 = \frac{1}{e}, \quad e = f. \tag{8}$$

Considering $\beta_3 < 1$, we have $e \geq 2$. Suppose $a = b$, then $\alpha_1 + \alpha_2 = 1/a$ from (7). Since $e = f$, we have $\alpha_3 + \alpha_4 = 1/e$. Therefore $1 = \sum_{i=1}^{i=4} \alpha_i = 1/a + 1/e$, so that we have $a = e = 2$ since $a, e \geq 2$. Then δ is on the side which connects $(2,2,0,0)$ and $(0,0,2,2)$, which is a contradiction. Therefore $a \neq b$. On the other hand, from (8), $\beta_1(a-b) + \beta_2(c-d) = 0$. If $a > b$, then it must be $c < d$ because $\beta_i > 0$. If $a < b$, it must be $c > d$, so that $b > a > c > d$ from $a > c$, which contradicts $b < d$. Therefore $a, d > b, c$.

From (7), $1 = e(\alpha_3 + \alpha_4) = e(1 - \alpha_1 - \alpha_2)$, that is, $e\alpha_1 + e\alpha_2 = e - 1$. Since $e \geq 2$, we have $\{e/(e-1)\}\alpha_1 + \{e/(e-1)\}\alpha_2 = 1$. From this equation and (7), $\{e/(e-1) - a\}\alpha_1 + \{e/(e-1) - b\}\alpha_2 = 0$. Considering $\alpha_i > 0$ and $a \neq b$, if $a < e/(e-1)$, then $2 \leq a < e/(e-1) = 1 + 1/(e-1) \leq 2$, which is a contradiction. If $a > e/(e-1)$, then $2 \geq 1 + 1/(e-1) = e/(e-1) > b \geq 1$, so that $b = 1$. Then this case does not occur by an argument similar to (III).

(VII.2) $\{(a,b,0,0),(c,0,d,0),(e,0,0,f)\}$ (see Fig. VII.2).

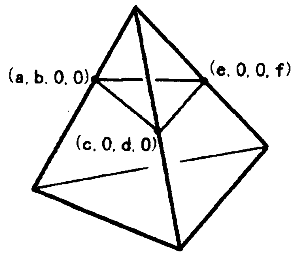

Figure VII.2

By an argument similar to (V.1), we obtain 3 elements as follows:

No. 14 $xy^2 + xz^3 + xw^6$, No. 15 $xy^2 + xz^4 + xw^4$,
No. 16 $xy^3 + xz^3 + xw^3$.

(VII.3) $\{(a,b,0,0),(c,0,d,0),(0,e,0,f)\}$ where $f \geq d$ (see Fig. VII.3).

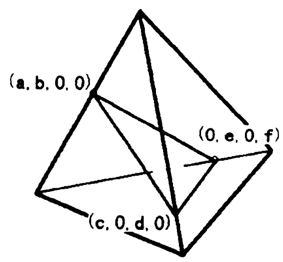

Figure VII.3

The condition (T) implies that $\beta_1 = (d-c)/(ad) = (f-e)/(bf)$, $\beta_2 = 1/d$, $\beta_3 = 1/f$. We obtain the following conditions: $d > c$, $f > e$, $f \geq d \geq 2$. Using $\beta_2 = 1 - \beta_1 - \beta_3 = (fb - f + e - b)/(bf)$, we have

(9) $$d = \frac{1}{\beta_2} = \frac{bf}{fb - f + e - b} \geq 2,$$

$$2(f-e) \geq b(f-2).$$

If $f = 2$, then $d = 2$ and $c = e = 1$ by the above condition. Then, we have $d = 2b/(b-1)$, which contradicts $d = 2$. Hence $f \geq 3$. Then,

(10) $$\frac{2(f-e)}{f-2} \geq b.$$

Divide into two cases: $e = 1$ and $e \geq 2$.

(i) Suppose $e = 1$. We have $b \geq 2$ by $b\beta_1 + \beta_3 = 1$. Then $4 \geq 2 + 2/(f-2) = 2(f-1)/(f-2) \geq b \geq 2$, that is, $b = 2, 3, 4$.

(i.1) If $b=2$, from (9) and $f \geq 3$, it holds that $2 < d = 2f/(f-1) = \{2(f-1)+2\}/(f-1) = 2 + 2/(f-1) \leq 3$. Hence we have $d = f = 3$, and $c = 1$ or 2 by $d > c$. If $c = 1$, it holds that $a = 2$ from $1/3 = (f-e)/(bf) = \beta_1 = (d-c)/(ad) = 2/(3a)$. Therefore we obtain $(a,b,c,d,e,f) = (2,2,1,3,1,3)$, which corresponds to

$$\text{No. 20} \quad x^2y^2 + xz^3 + yw^3 + z^2w^2.$$

If $c = 2$, we have $(a,b,c,d,e,f) = (1,2,2,3,1,3)$, which corresponds to

$$\text{No. 17} \quad xy^2 + x^2z^3 + yw^3 + zw^4.$$

(i.2) If $b = 3$, similarly, we have $(a,b,c,d,e,f) = (2,3,1,2,1,4)$, which is leading equivalent to No. 17.

(i.3) If $b = 4$, we have $(a,b,c,d,e,f) = (3,4,1,2,1,3)$, which corresponds to

$$\text{No. 22} \quad x^3y^4 + xz^2 + yw^3 + zw^2.$$

(ii) Suppose $e \geq 2$. It follows from (10) that $2 \geq \{2(f-e)\}/(f-2) \geq b$, so that $b = 1$ and 2.

(ii.1) If $b = 1$, we have $a, e \geq 2$ from $(a,b,0,0) \in H^3(\alpha)$ and $b\beta_1 + e\beta_3 = 1$. Then $d = f/(e-1)$ from (9). Hence we have $\beta_1 = (f-e)/f$, $\beta_2 = (e-1)/f$. It follows from $a\beta_1 + c\beta_2 = 1$ and $f > e$ that $a(f-e)/f + c(e-1)/f = 1$, that is,

$$a = \frac{f + c - ec}{f - e}. \tag{11}$$

We have $e + c - ec \geq f - e > 0$ because $a \geq 2$ and $f > e$, that is, $0 \leq (e-1) \cdot (c-1) < 1$. Therefore it holds that $c = 1$ by $e \geq 2$. Then, from (11) and $f > e$, it holds that $2 \leq a = (f-e+1)/(f-e) \leq 2$. Hence $a = 2$ and $f = e+1$. Then $2 \leq d = f/(e-1) = (e+1)/(e-1)$, that is, $3 \geq e$, therefore $e = 2$ and 3 from $e \geq 2$. If $e = 2$, we have $(a,b,c,d,e,f) = (2,1,1,3,2,3)$, which is leading equivalent to No. 17. If $e = 3$, we have $(a,b,c,d,e,f) = (2,1,1,2,3,4)$, which corresponds to

$$\text{No. 18} \quad x^2y + xz^2 + y^3w^4 + z^3w.$$

(ii.2) If $b = 2$, it follows from (9) that $d = 2f/(f+e-2) \geq 2$. We have $e \leq 2$, then $e = 2$ by $e \geq 2$, so that $d = 2$. It holds that $c = 1$ by $d > c$. Therefore $\beta_1 = 1/(2a)$, $\beta_2 = 1/2$ and $\beta_3 = 1/f$. Since $\beta_1 + \beta_2 + \beta_3 = 1$, we have $(f-2)(a-1) = 2$. Solve this equation, we obtain $(a,b,c,d,e,f) = (3,2,1,2,2,3)$ and $(2,2,1,2,2,4)$, which correspond to

$$\text{No. 21} \quad x^3y^2 + xz^2 + y^2w^3 + z^2w, \qquad \text{No. 19} \quad x^2y^2 + xz^2 + y^2w^4 + z^2w^2.$$

Case (VIII)

(VIII.1) $\{(a,b,0,0),(c,d,0,0),(e,0,f,g)\}$ where $a>c$ and $d>b$.

This case does not occur by an argument similar to (V.1).

(VIII.2) $\{(a,b,0,0),(c,0,d,0),(0,e,f,g)\}$ where $d\geq b$ (see Fig. VIII.2).

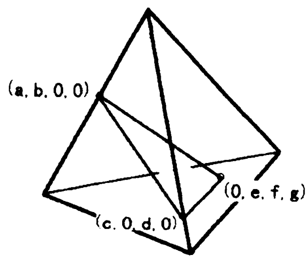

Figure VIII.2

Divide into two cases: $a=1$ and $a\geq 2$.

(i) Suppose $a=1$. The condition (H) implies $b\geq 2$. The condition (T) implies that $\beta_1=(g-e)/(bg)$, $\beta_2=(g-f)/(dg)$, $\beta_3=1/g$. We obtain the following conditions: $a=1$, $b,c,d,g\geq 2$, $d\geq b$, $g>e,f$. Using $\beta_1=1-\beta_2-\beta_3=\{g(d-1)+(f-d)\}/(dg)$, we have

$$b=\frac{d(g-e)}{g(d-1)+(f-d)}\geq 2, \tag{12}$$

$$2(g-f)\geq 2(g-f)-de\geq d(g-2). \tag{13}$$

Divide into two cases: $g=2$ and $g\geq 3$.

(i.1) If $g=2$, then $g>f,e$ implies $f=e=1$. It follows from $\beta_1+\beta_2+\beta_3=1$ that $1/(2b)+1/(2d)+1/2=1$, so that $b=d=2$. Using $\beta_1=\beta_2=1/4$, we have $c=3$ from $\beta_1+c\beta_2=1$. Therefore $(a,b,c,d,e,f,g)=(1,2,3,2,1,1,2)$, which is leading equivalent to No. 21.

(i.2) If $g\geq 3$, from (13),

$$\frac{2(g-f)}{g-2}\geq d. \tag{14}$$

Divide into two cases: $f=1$ and $f\geq 2$.

Assume $f=1$, using $g\geq 3$ and $d\geq 2$,

$$4\geq 2+\frac{2}{g-2}=\frac{2(g-1)}{g-2}\geq d\geq 2. \tag{15}$$

Therefore $d = 2, 3, 4$. If $d = 2$, we have $(a, b, c, d, e, f, g) = (1, 2, 2, 2, 1, 1, 3)$ from the condition (T), which is leading equivalent to No. 19. If $d = 3$, from (15) and $g \geq 3$, we have $g = 3$ and 4. However, from the condition (T), they does not occur. If $d = 4$, similarly, it does not occur.

Assume $f \geq 2$, from (14) and $d \geq 2$, we have $d = 2$. Then $d \geq b \geq 2$ implies $b = 2$. Then $\beta_1 = 1 - \beta_2 - \beta_3 = (g + f - 2)/(2g)$. From $\beta_1 + c\beta_2 = 1$, it holds that $(g - f)(c - 1) = 2$. Therefore $g = f + 2$, $c = 2$ or $g = f + 1$, $c = 3$. If $g = f + 2$ and $c = 2$, we have $\beta_1 = f/(f + 2)$, $\beta_2 = \beta_3 = 1/(f + 2)$. It follows from $2\beta_1 + e\beta_3 = 1$ that $2\{f/(f + 2)\} + \{1/(f + 2)\}e = 1$. Then $f = 2 - e$, which contradicts $e \in \boldsymbol{N}$ from $f \geq 2$. Similarly, this case of $g = f + 1$ and $c = 3$ is a contradiction.

(ii) Suppose $a \geq 2$. The condition (T) implies that $\beta_1 = (g - e)/(bg) = \{g(d - c) + fc\}/(adg)$, $\beta_2 = (g - f)/(dg)$, $\beta_3 = 1/g$. We obtain the following conditions: $d \geq b$ and $a, g \geq 2$ and $g > e, f$. Using $\beta_1 = 1 - \beta_2 - \beta_3 = \{g(d - 1) + (f - d)\}/(dg)$, we have

$$a = \frac{g(d - c) + fc}{g(d - 1) + (f - d)} \geq 2, \tag{16}$$

$$(2 - c)(g - f) \geq d(g - 2) \geq 0. \tag{17}$$

Hence $c = 1, 2$ by $g > f$ and $g \geq 2$.

(ii.1) If $c = 1$, it holds that

$$g - f \geq d(g - 2). \tag{18}$$

Assume $g = 2$, then $g > f, e$ implies $f = e = 1$. From (16) and $a \in \boldsymbol{N}$, we have $a = 3$ and $d = 2$. The condition (T) implies that $b = 2$. Therefore $(a, b, c, d, e, f, g) = (3, 2, 1, 2, 1, 1, 2)$, which is leading equivalent to No. 21.

Assume $g \geq 3$, from (18), it holds that $1 + (2 - f) \geq 1 + (2 - f)/(g - 2) = (g - f)/(g - 2) \geq d \geq 1$. Therefore $f = 1$ or 2. If $f = 1$, we have $d = 1$ or 2. If $d = 1$, then $d \geq b$ implies $b = 1$. Then $\alpha_1 + \alpha_3 = 1$ by $(1, 0, 1, 0) \in H^3(\alpha)$, which is a contradiction. If $d = 2$, from the condition (T), we have $b = 2$, $e = 1$ or $b = 1$, $e = 2$. The case $(a, b, c, d, e, f, g) = (2, 2, 1, 2, 1, 1, 3)$ is leading equivalent to No. 19, and the case $(a, b, c, d, e, f, g) = (2, 1, 1, 2, 2, 1, 3)$ which is leading equivalent to No. 18. If $f = 2$, similary, this case is a contradiction.

(ii.2) If $c = 2$, we have $g = 2$ from (17). Then $g > e, f$ implies $e = f = 1$. From (16), we have $a = 2$. The condition (T) implies that $b = d = 2$. Therefore $(a, b, c, d, e, f, g) = (2, 2, 2, 2, 1, 1, 2)$, which corresponds to

No. 23 $\quad x^2y^2 + x^2z^2 + yzw^2 + y^2w^2 + z^2w^2$.

(VIII.3) $\{(a,b,0,0),(c,0,d,0),(e,0,f,g)\}$.

This case does not occur by an argument similar to (III) and (V.1).

(VIII.4) $\{(a,b,0,0),(c,d,e,0),(0,0,f,g)\}$ where $a>e$.

This case does not occur by an argument similar to (VI.2).

Case (IX)

(IX.1) $\{(a,b,c,0),(d,e,f,0),(0,0,g,h)\}$.

At first we consider the relation between a and b. The condition (T) implies that

$$(19)\qquad c=f=g=1,\quad a\beta_1+d\beta_2=1,\quad b\beta_1+e\beta_2=1,\quad \beta_3=\frac{1}{h}.$$

We may assume $e>b$ and $a>d$, and $ea-bd\notin(e-b)\boldsymbol{Z}$, $ea-bd\notin(a-d)\boldsymbol{Z}$ and $a,e\geq 3$ where $ea-bd\notin(e-b)\boldsymbol{Z}$ means that $(a,b,c,0)$ is not an internally dividing point of $(d,e,f,0)$ and some integral point on X-plane and $ea-bd\notin(a-d)\boldsymbol{Z}$ means that $(d,e,f,0)$ is not an internally dividing point of $(a,b,c,0)$ and some integral point on YZ-plane. The condition (H) implies that

$$(20)\qquad a\alpha_1+b\alpha_2+\alpha_3=1,\quad d\alpha_1+e\alpha_2+\alpha_3=1,\quad \alpha_3+h\alpha_4=1,\quad h\geq 2.$$

If $a=b$, from (19), we have $\beta_2(d-e)=0$, so that $d=e$. From (20), $a(\alpha_1+\alpha_2)+\alpha_3=1$. Using $\sum_{i=1}^{i=4}\alpha_i=1$, it holds that $(a-1)\alpha_3+a\alpha_4=a-1$. From $a\geq 2$, $\alpha_3+a/(a-1)\alpha_4=1$. From (20), we have $\{a/(a-1)-h\}\alpha_4=0$. Since $\alpha_4>0$, $h=a/(a-1)\leq 2$, so that $h=2$ and $a=2$ because $h\geq 2$, and $\beta_3=1/2$. From $\beta_1+\beta_2=1/2$ and (19), we have $(2-d)\beta_2=0$, so that $d=2$ by $\beta_2>0$, which contradicts $a>d$. Therefore $a\neq b$. From (19), it holds that $\beta_1(a-b)+\beta_2(d-e)=0$. If $a<b$, we have $d>e$, which contradicts $b<e$, $d<a$. Hence $a>b$, $e>d$.

It follows from (20) and $\alpha_4=1-\sum_{i=1}^{3}\alpha_i$ that $h\alpha_1+h\alpha_2+(h-1)\alpha_3=h-1$. Since $h\geq 2$, $h/(h-1)\alpha_1+h/(h-1)\alpha_2+\alpha_3=1$. From (20), $\{h/(h-1)-a\}\alpha_1+\{h/(h-1)-b\}\alpha_2=0$. Considering $a>b$ and $a,h\geq 2$, we have $a>h/(h-1)>b$. Then $2\geq 1+1/(h-1)>b\geq 1$, hence $b=1$. We obtain the following conditions: $b=1$, $h\geq 2$, $a,e\geq 3$, $a,e>d$, $ea-d\notin(e-1)\boldsymbol{Z}$, $ea-d\notin(a-d)\boldsymbol{Z}$. Similarly, we have $\beta_1=\{h(e-1)-e\}/\{h(e-1)\}$. From (19), $a=(he-h-d)/(he-h-e)$. Since $a\geq 3$, it holds that

$$(21)\qquad 2h-d\geq e(2h-3).$$

From $e \geq 3$ and $h \geq 2$, we have $4 - d \geq 1 + (3 - d)(2h - 3) = (2h - d)(2h - 3) \geq e \geq 3$. Hence we have $d = 1$, so that $e = 3$. From (21) and $h \geq 2$, it holds $h = 2$. Then $a = 3$, which contradicts $ea - d \notin (e - 1)\boldsymbol{Z}$.

(IX.2) $\{(a, b, 0, 0), (c, d, e, 0), (f, 0, g, h)\}$.

This case does not occur by an argument similar to (V.2).

(IX.3) $\{(a, b, c, 0), (d, 0, e, 0), (f, 0, g, h)\}$.

This case does not occur by an argument similar to (V.1).

(IX.4) $\{(a, b, c, 0), (d, 0, e, f), (0, g, 0, h)\}$ where $h \geq g$.

Divide into two cases: $g = 1$ and $g \geq 2$.

(i) Suppose $g = 1$. The condition (T) implies that

$$\beta_1 = \frac{1 - \beta_3}{b}, \quad \beta_2 = \frac{b - (1 - \beta_3)c}{be}, \quad \beta_3 = \frac{(e - f)b + cf}{ehb + cf}. \tag{22}$$

We obtain the following conditions: $g = 1$, $b, h \geq 2$. Similarly, from $\beta_3 = 1 - \beta_2 - \beta_1$, we have $(be + c - e)\beta_3 = be - b + c - e$. If $be + c - e = 0$, we have $be - b + c - e = -b \neq 0$, which is a contradiction. Therefore $be + c - e \neq 0$. Then it holds that $\beta_3 = (be - b + c - e)/(eb + c - e)$. From (22),

$$h = \frac{be + c - e + f - bf}{be - b + c - e} \geq 2, \tag{23}$$

$$(b - 1)(f - 1) \leq (b - 1)(1 - e) + 2 - c.$$

It follows from $b \geq 2$ that $f \leq 2 - e + (2 - c)(b - 1) \leq 4 - e - c \leq 4 - 1 - 1 = 2$, that is, $f = 1$ or 2.

(i.1) If $f = 1$, from (23), $2 \geq (b - 1)(e - 1) + c \geq 1$. If $(b - 1)(e - 1) + c = 1$, we have $c = e = 1$ by $b \geq 2$. Then $\beta_1 + \beta_2 = 1$, which is a contradiction. If $(b - 1)(e - 1) + c = 2$. Considering $b \geq 2$ and $c, e \geq 1$, we have $(b - 1)(e - 1) = 0$, $c = 2$ or $(b - 1)(e - 1) = c = 1$. If $(b - 1)(e - 1) = 0$ and $c = 2$, then $e = 1$ since $b \geq 2$. From (23), we have $h = 2$. Moreover $d \geq 2$ since $(d, 0, 1, 1) \in H^3(\alpha)$. Then $\beta_1 = 1/(b + 1)$ and $\beta_2 = (b - 1)/(b + 1)$. It follows from $a\beta_1 + d\beta_2 = 1$ that $a/(b + 1) + \{d(b - 1)\}/(b + 1) = 1$. Considering $b, d \geq 2$, we have $(a, b, c, d, e, f, g, h) = (1, 2, 2, 2, 1, 1, 1, 2)$, which is leading equivalent to No. 21. If $(b - 1) \cdot (e - 1) = c = 1$, similarly, we have $(a, b, c, d, e, f, g, h) = (1, 2, 1, 2, 2, 1, 1, 2)$ which is leading equivalent to No. 4 and $(a, b, c, d, e, f, g, h) = (2, 2, 1, 1, 2, 1, 1, 2)$ which is leading equivalent to No. 21.

(i.2) If $f = 2$, similarly, this case does not occur.

(ii) Suppose $g \geq 2$. Then $h \geq g$ implies $h \geq 2$. Similarly, from $\beta_3 = 1 - \beta_1 - \beta_2$, we have $(fc - hc + he)\beta_3 = fc - c - f + e$. Suppose $fc - c - f + e = f(c-1) - (c-e) = 0$, we have $fc - hc + he = fc - h(c-e) = 0$. Since $c - e = f(c-1)$, we have $(c-1)(h-1) = 1$ by $f \neq 0$. Considering $h \geq 2$, we have $h = c = 2$, and $f = e = 1$. The condition (H) implies $d \geq 2$. Then $\beta_1 = 1/(b+g)$ and $\beta_2 = (b+g-2)/(b+g)$. It follows from $d \geq 2$ and $a\beta_1 + d\beta_2 = 1$ that $4 - a \geq b + g \geq 3$. Then $(a,b,c,d,e,f,g,h) = (1,1,2,2,1,1,2,2)$, which is leading equivalent to No. 23. Suppose $fc - c - f + e \neq 0$, we have

$$g = \frac{fc - fb + be - hc + hb + he - beh}{fc - c - f + e} \geq 2, \tag{24}$$

$$(2-b)(f-e) \geq c(f-2). \tag{25}$$

Divide into three cases: $f = 1$, $f = 2$ and $f \geq 3$.

(ii.1) If $f = 1$. From $0 \neq fc - c - f + e = -1 + e$, we have $e \geq 2$. Since (24) and $e, h \geq 2$,

$$(b-1)(1-h)(e-1) \geq (h-1)(c-1) + e - 2 \geq 0,$$

$$0 \leq (b-1)(h-1) \leq 0.$$

Hence we have $b = 1$. From (24), $g = (e - 1 + h + c - hc)/(e-1) \in \boldsymbol{N}$. Since $g \geq 2$ and $e \geq 2$, we have $h + c - hc > 0$. From $c - 1 \leq (h-1)(c-1) < 1$ and $h \geq 2$, we get $c = 1$. Then $g = e/(e-1) \geq 2$ and $e \geq 2$ imply $e = 2$, therefore $g = 2$. From the condition (H) and (T), we have $(a,b,c,d,e,f,g,h) = (2,1,1,2,2,1,2,1)$, which is leading equivalent to No. 23.

(ii.2) If $f = 2$, from (25), it holds that $(2-b)(2-e) \geq 0$. Divide into two cases: $2 \geq b, e$ and $2 \leq b, e$. The case of $2 \geq b, e$ does not occur by using (25). The case of $2 \leq b, e$, from $b\beta_1 + g\beta_3 = 1$ and $b, g \geq 2$, we have $\beta_1 + \beta_3 = 1/2 - \{\beta_1(b-2) + \beta_3(g-2)\}/2 \leq 1/2$. Then $\beta_2 = 1 - (\beta_1 + \beta_3) \geq 1/2$. From $2\beta_2 + h\beta_3 = 1$ and $\beta_2 \geq 1/2$, it holds that $1 + h\beta_3 \leq 2\beta_2 + h\beta_3 = 1$, hence $h\beta_3 \leq 0$, which is a contradiction.

(ii.3) If $f \geq 3$, from (25),

$$(2-b)(2-e) \geq c(f-2) \geq 1. \tag{26}$$

Therefore $b \neq 2$. Divide into two cases: $b = 1$ and $b \geq 3$.

Assume $b = 1$, from (26),

$$1 + \frac{2-e}{f-2} = \frac{f-e}{f-2} \geq c \geq 1. \tag{27}$$

Since $f \geq 3$, we have $e = 1$ or 2. If $e = 1$, this case does not occur by an argument similar to (ii.2). If $e = 2$, we have $c = 1$ by (27). Since $(a, 1, 1, 0) \in H^3(\alpha)$, we have $a \geq 2$. Using the condition (T), this case does not occur. Similarly, the case of $b \geq 3$ does not occur.

Case (X)

(X.1) $\{(a, b, c, 0), (d, e, f, 0), (g, 0, h, i)\}$.

This case does not occur by an argument similar to (V.1).

(X.2) $\{(a, b, c, 0), (d, 0, e, f), (g, h, 0, i)\}$.

Divide into two cases: $b = 1$ and $b \geq 2$. The condition (T) implies that $a = d = g = 1$.

(i) Suppose $b = 1$. The condition (H) implies $c \geq 2$. Then

$$(cfh + ei)\beta_1 = h(f - e) + ei. \tag{28}$$

Considering $\beta_1 + h\beta_3 = 1$, we see $h \geq 2$. On the other hand, from $\beta_1 + \beta_2 + \beta_3 = 1$, it holds that $(fh + i - f)\beta_1 = h(f-1) + (i - f)$. If $fh + i - f = 0$, then $0 = h(f-1) + (i-f) = -h$, which is a contradiction. Therefore $fh + i - f \neq 0$, that is, $\beta_1 = \{h(f-1) + (i-f)\}/(fh + i - f)$. From (28),

$$c = \frac{fh - he + i - f + e}{h(f-1) + (i-f)} \geq 2. \tag{29}$$

$$(h-1)(e-1) \leq -(h-1)(f-1) + 2 - i.$$

Since $h \geq 2$,

$$e - 1 \leq -(f-1) + \frac{2-i}{h-1} \leq -f + 1 + 2 - i,$$

$$e \leq 4 - f - i \leq 2.$$

Hence we have $e = 1$ or 2.

(i.1) If $e = 1$, then $f \geq 2$ by $\alpha_1 + \alpha_3 + f\alpha_4 = 1$. From (29),

$$c = \frac{fh - h + i - f + 1}{h(f-1) + (i-f)} \geq 2. \tag{30}$$

Since $h, f \geq 2$, it holds that $2 - i \geq (h-1)(f-1) \geq 1$, so that $i = 1$. Then we have $(a, b, c, d, e, f, g, h, i) = (1, 1, 2, 1, 1, 2, 1, 2, 1)$, which is leading equivalent to No. 16.

(i.2) If $e = 2$, from (29), we have $2 - i \geq f(h-1) \geq 1$, that is, $i = 1$ and $f(h-1) = 0$. We have $h = 2$ and $f = 1$ by $h \geq 2$. Then $c = 1$, which contradicts $c \geq 2$.

(ii) Suppose $b \geq 2$. We may set $c \geq b$. Then $c \geq b$ and $b \geq 2$ imply $c \geq 2$. Then $(ebi + hfc)\beta_1 = (i-h)e + hf$. We have $(ei - ic + fc)\beta_1 = ei - i - e + f$. If $ei - ic + fc = 0$, then we have $ei - i - e + f = 0$. We have $c = i = 2$ by $c \geq 2$, so that $e = f = 1$. Then $\alpha_1 + \alpha_3 + \alpha_4 = 1$, which is a contradiction. Therefore $ei - ic + fc \neq 0$, that is, $\beta_1 = (ei - i - e + f)(ei - ic + fc)$. Moreover

$$b = \frac{ei - eh + hf - ic + ch + fc - hfc}{ei - i - e + f} \geq 2, \tag{31}$$

$$(2-h)(e-f) - c\{(i-1) + (h-1)(f-1)\} \geq i(e-2),$$

$$(2-h)(e-h) \geq i(e-2). \tag{32}$$

Divide into three cases: $e = 1$, $e = 2$ and $e \geq 3$.

(ii.1) If $e = 1$, then $f \geq 2$ by $\alpha_1 + \alpha_3 + f\alpha_4 = 1$. Since (31) and $e = 1$, it holds that $(h-1)(1-c)(f-1) \geq (i-1)(c-1) + f - 2 \geq 0$. Since $f \geq 2$, we have $0 \leq (h-1)(c-1) \leq 0$. We have $h = 1$ by $c \geq 2$. Then $b = (i - 1 + f - ic + c)/(f-1) \geq 2$ from (31), and $f \geq 2$ implies that $c + i - ic > 0$. Then $i - 1 \leq (i-1)(c-1) < 1$ by $c \geq 2$. It holds $i = 1$, then $\alpha_1 + \alpha_2 + \alpha_4 = 1$, which is a contradiction.

(ii.2) If $e = 2$, from (32), it holds that $(2-h)(2-f) \geq 0$. Divide into two cases: $2 > f, h$ and $2 \leq f, h$. If $2 > f, h$, it means that $f = h = 1$. We have $i \geq 2$ by $\alpha_1 + h\alpha_2 + i\alpha_4 = 1$. From (31), $1/(i-1) \geq c \geq 2$, which contradicts $i \geq 2$. If $2 \leq f, h$, this case does not occur by an argument similar to (IX.4) (ii.2).

(ii.3) If $e \geq 3$, from (32),

$$(2-h)(e-f) \geq i(e-2) \geq 1. \tag{33}$$

Therefore $h \neq 2$. Divide into two cases: $h = 1$ and $h \geq 3$. If $h = 1$, from (33),

$$1 + \frac{2-f}{e-2} = \frac{e-f}{e-2} \geq i \geq 1. \tag{34}$$

Since $e \geq 3$, we have $f = 1$ or 2. If $f = 1$, this case does not occur by an argument similar to (IX.4) (ii.2). If $f = 2$, we obtain $i = 1$ by (34), then $\alpha_1 + \alpha_2 + \alpha_4 = 1$, which is a contradiction. If $h \geq 3$, similarly, this case does not occur.

Therefore we obtained all elements of $\Phi_2^3/\sim$ listed in Table 1 of Section 4. *Q.E.D.*

4.2. The Other Types in 2 and 3-Dimensions

For a singularity of the $(0,0)$-type, it is easy to see the following theorems.

THEOREM 15. $(\Phi_0^2/\sim) \cup (\Phi_1^2/\sim) = \{[xyz], [x^2 + y^2z^2], [x^2y + yz^2]\}$. *Moreover* $\#\{(D\Phi_0^2/\sim) \cup (D\Phi_1^2/\sim)\} = 3$.

PROOF. We follow Steps 1, 2 and 3 of Section 3. It is clear that $\Phi_0^2/\sim$ is $[xyz]$. For $\Phi_1^2/\sim$, set two points $m, m' \in \boldsymbol{Z}_{\geq 0}^3$ which construct such a compact line including $(1,1,1)$ in the interior. Lemma 9 implies that $(m,m') \in (V,V)$ or (V,S) or (S,S) where V and S are the sets of points having 3 coordinates whose only 1 coordinate is not zero and whose only 2 coordinates are not zero, respectively. The case $(m,m') \in (V,V)$ does not occur because the line determined by m and m' does not include $(1,1,1)$. Under the leading equivalence relation and the equation $(1,1,1) = tm + (1-t)m'$ for some $t \in \boldsymbol{Q}_{>0}$ with $0 < t < 1$, we see that $\Phi_1^2/\sim$ are $[x^2 + y^2z^2]$ and $[x^2y + yz^2]$.

At last, it is clear that $T_{pqr} : x^p + y^q + z^r + axyz$ ($a \neq 0$ and $1/p + 1/q + 1/r < 1$), $x^2 + y^2z^2 + y^4 + z^5$ and $x^2y + yz^2 + y^3 + z^4$ are representative elements of $(D\Phi_0^2/\sim) \cup (D\Phi_1^2/\sim)$. *Q.E.D.*

THEOREM 16. $(\Phi_0^3/\sim) \cup (\Phi_1^3/\sim) = \{[xyzw], [x^2 + y^2z^2w^2], [x^2y^2 + z^2w^2], [xy^2 + xz^2w^2], [xy^2z + xzw^2]\}$. *Moreover* $\#\{(D\Phi_0^3/\sim) \cup (D\Phi_1^3/\sim)\} = 5$.

PROOF. Similary, it is clear that $T_{pqrs} : x^p + y^q + z^r + w^s + axyzw$ ($a \neq 0$ and $1/p + 1/q + 1/r + 1/s < 1$), $x^2 + y^2z^2w^2 + y^6 + z^6 + w^7$, $x^2y^2 + z^2w^2 + x^4 + y^4 + z^4 + w^5$, $xy^2 + xz^2w^2 + x^2 + z^8 + w^9$ and $xy^2z + xzw^2 + x^3 + y^6 + w^6 + z^4$ are representative elements of $(D\Phi_0^3/\sim) \cup (D\Phi_1^3/\sim)$. *Q.E.D.*

REMARK 17. For the $(0,0)$-type polynomial in $\boldsymbol{C}[z_0, \ldots, z_n]$, the dimension of its leading face Δ_0 is equal to one or zero in virtue of Theorem 1. Then we see that some case of $\dim \Delta_0 = 1$ is reduced to the case of $\dim \Delta_0 = 0$ by a suitable coordinate transformation as follows: Let f be a $(0,0)$-type polynomial in $\boldsymbol{C}[z_0, \ldots, z_n]$ with $\dim\Delta_0 = 1$ where Δ_0 is a leading face of f. Then the leading terms f_{Δ_0} is leading equivalent to $z_{k_1} \cdots z_{k_l}\{(z_{p_1} \cdots z_{p_r})^2 + (z_{q_1} \cdots z_{q_s})^2\}$ where $\{0, \ldots, n\} = \{k_1, \ldots, k_l\} \amalg \{p_1, \ldots, p_r\} \amalg \{q_1, \ldots, q_s\}$ by the similar way of Theorems 15 and 16 and simple coordinate transformations. If $r = 1$, then

$$f_{\Delta_0} \sim z_{k_1} \cdots z_{k_l}(z_{p_1}^2 + z_{q_1}^2 \cdots z_{q_s}^2)$$
$$= z_{k_1} \cdots z_{k_l}\{(z_{p_1} + iz_{q_1} \cdots z_{q_s})^2 - 2i(z_{p_1} + iz_{q_1} \cdots z_{q_s})z_{q_1} \cdots z_{q_s}\}.$$

The transformation

$$z_{p_1} \longrightarrow z_{p_1} - iz_{q_1} \cdots z_{q_s}$$

implies $z_{k_1} \cdots z_{k_l} z_{p_1}^2 - 2iz_0 \cdots z_n$ in the new coordinates. Then the leading face is changed into a point.

For the n-dimensional isolated hypersurface purely elliptic singularity of $(0, n-1)$-type, it is easy to see that the classification using leading equivalence relation is the same as the one using the weight since the weight of the leading face of defining polynomial giving purely elliptic singularity is determined uniquely. Therefore, in the 3-dimension, the classification using leading equivalence relation coincides with the Yomemura's classification. Therefore we obtain the following theorem immediately.

THEOREM 18 (Yonemura [11]). *All representative elements of* $\Phi_3^3/\sim$ *are listed in Table* 2.2 *of* [11]. *Moreover* $\#(D\Phi_3^3/\sim) = 95$.

Moreover, in general for the n-dimensional isolated hypersurface purely elliptic singularity of $(0, n-1)$-type, the following fact was known. The singularities which belong to the same analytical equivalence class have a canonical model up to isomorphisms, which are obtained from the weighted blowing-ups by the weights of leading faces of each singularities. Then, by a result of Tomari [7] Theorem 4.16, we see that their weights are equal because the Konöller invariant γ_m is determined by the canonical model uniquely. Therefore, as is well known, Saito's classification of the 2-dimensional hypersurface purely elliptic singularities under the analytic equivalence relation in [6] is the same as the classification under the leading equivalence relation. Therefore the following theorem holds immediately.

THEOREM 19 (Saito [6]). $\Phi_2^2/\sim = \{[x^3 + y^3 + z^3], [x^2 + y^4 + z^4], [x^2 + y^3 + z^6]\}$. *Moreover* $\#(D\Phi_2^2/\sim) = 3$.

5. On the (0, 2)-Type

Through a singularity of $(0, 1)$-type, we review a singularity of $(0, 2)$-type in the 3-dimension. For instance, let $f = x^2 + y^3 + z^6w^6 + z^{12} + w^{13}$. Then f is a

$(0,1)$-type polynomial and its leading term is No. 1 $f_0 = x^2 + y^3 + z^6w^6$ in Table 1 of Section 4. Furthermore, it is obvious that the leading face of f is included in a leading face of the $(0,2)$-type polynomial $g = x^2 + y^3 + z^6w^6 + z^{12} + w^{12}$. In general, we have the following theorem.

THEOREM 20. *The leading face of a* $(0,1)$*-type polynomial in* $\boldsymbol{C}[x,y,z,w]$ *is included in the leading face of some* $(0,2)$*-type polynomial in* $\boldsymbol{C}[x,y,z,w]$.

PROOF. To show this, we give the list of Table 3 of all corresponding weights of leading faces of $(0,2)$-type polynomials for each elements in Table 1 of Section 4, where, for example, YN10 means No. 10 $(1/2, 1/3, 1/12, 1/12)$ in Table 2.2 of [11]. As a remark, for the another representative element, it is only the permutation of the weight differences.

Table 3

The number of $(0,1)$-type	The number of $(0,2)$-type
No. 1	YN10, YN11, YN12, YN13, YN14, YN46, YN47, YN48, YN49, YN50, YN51, YN83
No. 2	YN7, YN8, YN9, YN35, YN36, YN37
No. 3	YN5, YN8, YN29, YN31, YN33, YN39
No. 4	YN3, YN4, YN17, YN18
No. 5	YN2, YN3, YN24, YN53
No. 6	YN1, YN2, YN19
No. 7	YN38, YN39, YN40, YN41, YN42, YN43, YN44, YN45, YN77, YN78, YN79, YN80, YN81, YN82, YN92, YN93
No. 8	YN6, YN31, YN32, YN33, YN34, YN37, YN74, YN75, YN76, YN78, YN90, YN91
No. 9	YN6, YN7, YN8, YN12, YN33, YN40, YN44, YN75
No. 10	YN5, YN6, YN7, YN10, YN32, YN42
No. 11	YN20, YN22, YN24, YN25, YN27, YN28, YN59, YN60, YN65, YN67, YN68, YN72, YN88
No. 12	YN2, YN15, YN16, YN18, YN22, YN53, YN54, YN84
No. 13	YN4, YN19, YN23, YN26, YN55, YN58, YN61, YN62, YN69
No. 14	YN8, YN18, YN19, YN63, YN66, YN89
No. 15	YN3, YN7, YN21, YN66
No. 16	YN1, YN5, YN21
No. 17	YN18, YN21, YN57, YN69, YN85, YN87, YN89, YN94
No. 18	YN58, YN63, YN64, YN66, YN70, YN71, YN87, YN89, YN95
No. 19	YN3, YN19, YN24, YN63, YN85
No. 20	YN1, YN53, YN63, YN85
No. 21	YN19, YN21, YN23, YN25, YN66
No. 22	YN58, YN63, YN64, YN66, YN70, YN71, YN87, YN89, YN95
No. 23	YN1, YN3

Q.E.D.

REMARK 21. The converse of the above theorem does not hold. In fact, there exist leading faces of $(0,2)$-type polynomials which never contain any leading faces of $(0,1)$-type polynomials: YN30, YN52, YN56, YN73, YN86 in Table 2.2 of [11].

REMARK 22. Under a certain condition on a deformation of a purely elliptic singularity, Ishii shows that a singularity of $(0,1)$-type deforms to a singularity of $(0,1)$-type or $(0,2)$-type (see [4]).

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