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THREE-DISTANCE SEQUENCES WITH THREE SYMBOLS

By

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Abstract. We will show that every 3 dimensional cutting sequence is a three-distance sequence, and there are uncountable many periodic or aperiodic three-distance sequences (with 3-symbols) which are not 3 dimensional cutting sequences.

1 Introduction

W. F. Lunnon and P. A. B. Pleasants [1] defined two-distance sequences and proved that each 2 dimensional (2D) cutting sequence (see below, for the definition) is a two-distance sequence and the converse also holds. The basic framework of their research is traced back to the one by M. Morse and G. A. Hedlund [4].

In this paper, we will discuss the relationships between 3 dimensional (3D) cutting sequences and three-distance sequences. We will show that every 3D cutting sequence is a three-distance sequence, and there are uncountable many periodic or aperiodic three-distance sequences which are not 3D cutting sequences.

First, we recall the definition of 2D cutting sequences. Although the definition given below is slightly different from that described in [1] or [5], the equivalence of 2D cutting sequences and two-distance sequences ([1, theorem 1]) holds by the same proof.

The set of the real numbers and the rational integers, and the non-negative rational integers are denoted by $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{Z}_+$, respectively.

We consider the standard orthogonal coordinates x, y in the 2 dimensional Euclidean space $\mathbb{R}^2$, and take a line L in $\mathbb{R}^2$. We assume that the slope of the line L is non-negative, and L is not parallel to either axis. When the line L crosses a

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vertical grid line or a horizontal one, we mark the point of the intersection and label it as A and B, respectively.

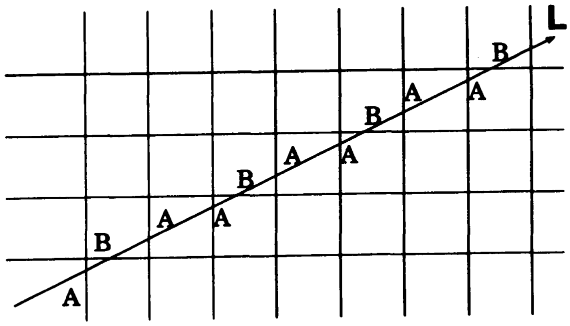

Figure 1

In the above labeling, we need to specify the way of labeling the intersection $\mathsf{L} \cap \mathbb{Z}^2$.

Type 1: $\#(\mathsf{L} \cap \mathbb{Z}^2) = 1$. Label the point of the intersection $\mathsf{L} \cap \mathbb{Z}^2$ by either of the two elements of $S_2 = \{\mathsf{AB}, \mathsf{BA}\}$.

Type 2: $\#(\mathsf{L} \cap \mathbb{Z}^2) \geq 2$. Observe that $\#(\mathsf{L} \cap \mathbb{Z}^2) = \infty$.

(1) Label all the points of the intersection $\mathsf{L} \cap \mathbb{Z}^2$ by one of the two elements of S_2.

In this way, we obtain two infinite periodic sequences associated with the line L.

(2) Fix an arbitrary point P on L. The point P divides L into two half-lines L_P^+ and L_P^-. We label the integer points on $\mathsf{L}_\mathsf{P}^+ \backslash \{\mathsf{P}\}$ by an element of S_2, and label the integer points on $\mathsf{L}_\mathsf{P}^- \backslash \{\mathsf{P}\}$ by another element of S_2. When P is an integer point, we label P by an element of S_2.

These give one or more two-way infinite sequences of symbols A and B. Such sequences are called the **2D cutting sequences** obtained from L.

REMARK 1.1. The labeling of Type 2 (2) is introduced to obtain the equivalence between 2D cutting sequences and two-distance sequences ([1]).

2 3D Cutting Sequence

In this section, we define 3D cutting sequences as a natural extension of 2D cutting sequences. We consider the standard orthogonal coordinates x, y, z in the 3 dimensional Euclidean space $\mathbb{R}^3$. Let $\mathsf{P}_{\mathsf{uv}}(\mathsf{L})$ be the projection of a line L in $\mathbb{R}^3$

on the uv-plane, where $u, v \in \{x, y, z\}$. We assume that each projection $P_{uv}(L)$ has a non-negative slope, and L does not lie in any uv-hyperplane. Let $\mathcal{H}_A$ (*resp.* $\mathcal{H}_B, \mathcal{H}_C$) be the collection of hyperplanes in $\mathbb{R}^3$ defined by

$$x = r_x, \quad (resp.\ y = r_y, \ z = r_z)$$

where $r_x, r_y, r_z \in \mathbb{Z}$.

When L intersects with a hyperplane $H_A \in \mathcal{H}_A$ (*resp.* $H_B \in \mathcal{H}_B$, $H_C \in \mathcal{H}_C$), label the point of the intersection $H_A \cap L$ (*resp.* $H_B \cap L$, $H_C \cap L$) by A (*resp.* B, C).

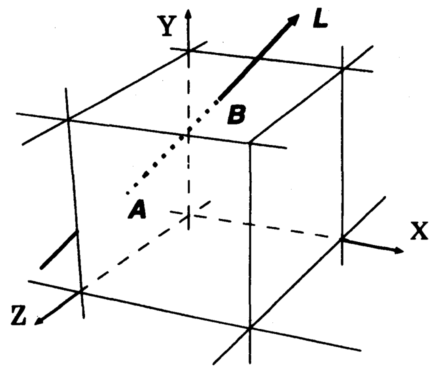

Figure 2

Let $\mathcal{L}_x$ (*resp.* $\mathcal{L}_y$, $\mathcal{L}_z$) be the collection of the lines defined by the equation

$$y = r_y \text{ and } z = r_z, \quad r_y, r_z \in \mathbb{Z}$$

$$(\text{resp. } x = r_x \text{ and } z = r_z, \quad r_x, r_z \in \mathbb{Z},$$

$$x = r_x \text{ and } y = r_y, \quad r_x, r_y \in \mathbb{Z}.)$$

We put $\mathcal{L} = \mathcal{L}_x \cup \mathcal{L}_y \cup \mathcal{L}_z$ and the set $\Lambda = \bigcup \mathcal{L}$ is called the grid of $\mathbb{R}^3$ in the present paper.

As we did in defining the 2D cutting sequences, we need to specify the way of labeling the points of the intersection of L and Λ or $\mathbb{Z}^3$. We divide our consideration into the following three cases. First notice that if $L \cap \mathbb{Z}^3 \neq \emptyset$ then $\#(L \cap \mathbb{Z}^3) = 1$ or ∞.

Case 1 $L \cap \mathbb{Z}^3 \neq \emptyset$ *and* $L \cap (\Lambda \backslash \mathbb{Z}^3) = \emptyset$,
Case 2 $L \cap \mathbb{Z}^3 = \emptyset$ *and* $L \cap (\Lambda \backslash \mathbb{Z}^3) \neq \emptyset$ and
Case 3 $L \cap \mathbb{Z}^3 \neq \emptyset$ *and* $L \cap (\Lambda \backslash \mathbb{Z}^3) \neq \emptyset$.

Case 1:

type 1: $\#(\mathsf{L}\cap\mathbb{Z}^3)=1$.

Label the point of the intersection $\mathsf{L}\cap\mathbb{Z}^3$ by an element of S_3, where

$$S_3=\{\mathsf{ABC},\mathsf{ACB},\mathsf{BAC},\mathsf{BCA},\mathsf{CAB},\mathsf{CBA}\}.$$

In this way, we obtain the six infinite sequences associated with the line L.

type 2: $\#(\mathsf{L}\cap\mathbb{Z}^3)=\infty$.

Fix an arbitrary point P on L. The point P divides L into two half-lines L_P^+ and L_P^-. Pick up two (possibly equal) elements X^+,X^- of S_3. Then label the points of the intersection $(\mathsf{L}_\mathsf{P}^+\backslash\{\mathsf{P}\})\cap\mathbb{Z}^3$ by X^+, and label the points of the intersection $(\mathsf{L}_\mathsf{P}^-\backslash\{\mathsf{P}\})\cap\mathbb{Z}^3$ by X^-.

In this way, we obtain the 36 infinite periodic sequences associated with the line L.

Case 2:

type 1: Suppose that there exists a unique $\ell\in\mathcal{L}$ which intersects with L.

We define $\mathcal{S}_\mathsf{u}$ $(\mathsf{u}=x,y,z)$ as follows.

$$\mathcal{S}_x=\{\mathsf{BC},\mathsf{CB}\},\quad \mathcal{S}_y=\{\mathsf{AC},\mathsf{CA}\},\quad \mathcal{S}_z=\{\mathsf{AB},\mathsf{BA}\}.$$

When $\ell\in\mathcal{L}_\mathsf{u}$, label the point of the intersection $\ell\cap\mathsf{L}$ by an element of $\mathcal{S}_\mathsf{u}$.

In this way, we obtain two infinite periodic sequences associated with the line L.

type 2: Suppose that there exist two lines $\ell,\ell'\in\mathcal{L}$ such that $\ell\cap\mathsf{L}\neq\emptyset$ and $\ell'\cap\mathsf{L}\neq\emptyset$, and recall that L does not lie in any uv-hyperplane. Fix an arbitrary point P on L. The point P divides L into two half-lines L_P^+ and L_P^-. Pick up two (possibly equal) elements $X_\mathsf{u}^+,X_\mathsf{u}^-$ of $\mathcal{S}_\mathsf{u}$. Then label the point of the intersection $(\mathsf{L}_\mathsf{P}^+\backslash\{\mathsf{P}\})\cap\ell$, $\ell\in\mathcal{L}_\mathsf{u}$ by X_u^+, and the point of the intersection $(\mathsf{L}_\mathsf{P}^-\backslash\{\mathsf{P}\})\cap\ell'$, $\ell'\in\mathcal{L}_\mathsf{u}$ by X_u^-. When $\{\mathsf{P}\}=\mathsf{L}\cap\ell$, $\ell\in\mathcal{L}_\mathsf{u}$, we label P by an element of $\mathcal{S}_\mathsf{u}$.

Case 3: First we observe that, $\#\{\ell\in\mathcal{L}:\mathsf{L}\cap(\ell\backslash\mathbb{Z}^3)\neq\emptyset\}=\infty$.

We define the following notation for the labeling in this case. Let W be the set of all finite sequences with symbols $\mathsf{A},\mathsf{B},\mathsf{C}$. A function

$$\mathcal{D}_\mathsf{u}:\mathsf{W}\to\mathsf{W}$$

$(\mathsf{u}=x,y,z)$ is defined as follows: for $\mathsf{w}\in\mathsf{W}$, $\mathcal{D}_\mathsf{u}(\mathsf{w})$ is a finite sequence with two symbols obtained by removing $\delta(\mathsf{u})$ from w, where

$$\delta(\mathsf{u}) = \begin{cases} \mathsf{A}, & \text{if } \mathsf{u} = x \\ \mathsf{B}, & \text{if } \mathsf{u} = y \\ \mathsf{C}, & \text{if } \mathsf{u} = z. \end{cases}$$

Also a function

$$\mathcal{F}_{\mathsf{u}} : \mathsf{W} \to \mathsf{W}$$

is defined as follows: for an element $\mathsf{w} = w_1 \cdots w_l$ of W ($\{w_1, \ldots, w_l\} \subset \{\mathsf{A}, \mathsf{B}, \mathsf{C}\}$), $\mathcal{F}_{\mathsf{u}}(\mathsf{w}) = w_l \cdots w_1$.

We fix an arbitrary point P on L. The point P divides L into two half-lines $\mathsf{L}_{\mathsf{P}}^+$ and $\mathsf{L}_{\mathsf{P}}^-$.

type 1: $\#(\mathsf{L} \cap \mathbb{Z}^3) = 1$.

Label the point of the intersection $\mathsf{L}_{\mathsf{P}}^+ \cap \mathbb{Z}^3$ by an element X of S_3. For the labeling the intersection $\ell \cap \mathsf{L}_{\mathsf{P}}^{\pm}$, we take the following two ways.

(1) Label the intersection $\ell \cap \mathsf{L}_{\mathsf{P}}^+$ and $\ell' \cap \mathsf{L}_{\mathsf{P}}^-$ with $\ell, \ell' \in \mathcal{L}_{\mathsf{u}}$ as $\mathcal{D}_{\mathsf{u}}(X)$.

(2) Label the intersection $\ell \cap \mathsf{L}_{\mathsf{P}}^+$ with $\ell \in \mathcal{L}_{\mathsf{u}}$ by $\mathcal{D}_{\mathsf{u}}(X)$, and the intersection $\ell' \cap \mathsf{L}_{\mathsf{P}}^-$ with $\ell' \in \mathcal{L}_{\mathsf{u}}$ by $\mathcal{F}_{\mathsf{u}} \circ \mathcal{D}_{\mathsf{u}}(X)$.

type 2: $\#(\mathsf{L} \cap \mathbb{Z}^3) = \infty$.

Pick up two (possibly equal) elements X^+, X^- of S_3. Label the points of the intersection $\mathsf{L}_{\mathsf{P}}^+ \cap \mathbb{Z}^3$ by X^+ and $\mathsf{L}_{\mathsf{P}}^- \cap \mathbb{Z}^3$ by X^-. Then label $\mathsf{L}_{\mathsf{P}}^+ \cap \ell$ with $\ell \in \mathcal{L}_{\mathsf{u}}$ by $\mathcal{D}_{\mathsf{u}}(X^+)$ and $\mathsf{L}_{\mathsf{P}}^- \cap \ell'$ with $\ell' \in \mathcal{L}_{\mathsf{u}}$ by $\mathcal{D}_{\mathsf{u}}(X^-)$.

These give one or more bi-infinite sequences with symbols $\mathsf{A}, \mathsf{B}, \mathsf{C}$. Such sequences are called the **3D cutting sequences** obtained from L.

Remark 2.1. The function $\mathcal{D}_{\mathsf{u}}$ is naturally extended to a function $\mathcal{D}_{\mathsf{u}} : \Sigma \to \Sigma$ of the set Σ of all infinite sequences with symbols $\mathsf{A}, \mathsf{B}, \mathsf{C}$.
If S is a 3D cutting sequence associated with a line L, then $\mathcal{D}_{\mathsf{u}}(\mathsf{S})$ is a 2D cutting sequence associated with the line $\mathsf{P}_{\mathsf{uv}}(\mathsf{L})$, where $\{\mathsf{u}, \mathsf{v}\} \subset \{x, y, z\}$. In this way, 2D cutting sequences are obtained from 3D cutting sequences.

3 Three-Distance Sequence

In this section, we define the notion of three-distance sequences with three symbols. The following definitions are the natural extensions of those for two-distance sequences with two symbols A, B [1].

Let S be a bi-infinite sequence with three symbols $\mathsf{A}, \mathsf{B}, \mathsf{C}$.

Definition 3.1. *A word* w in S is a finite string of consecutive symbols from S.

DEFINITION 3.2. *The length* $|\mathsf{w}|$ of a word w is the total number of symbols which are contained in w.

DEFINITION 3.3. *The i-weight* $|\mathsf{w}|_i$ of a word w ($i \in \{\mathsf{A}, \mathsf{B}, \mathsf{C}\}$) is the number of the symbol i in the word w. Notice that $|\mathsf{w}| = |\mathsf{w}|_{\mathsf{A}} + |\mathsf{w}|_{\mathsf{B}} + |\mathsf{w}|_{\mathsf{C}}$.

DEFINITION 3.4. A sequence S is called *a three-distance sequence*, if, for each $l \in \mathbf{Z}_+$ and for each $i \in \{\mathsf{A}, \mathsf{B}, \mathsf{C}\}$, we have the inequality

$$\#\{|\mathsf{w}|_i : \mathsf{w} \text{ is a word of } \mathsf{S} \text{ and } |\mathsf{w}| = l\} \leq 3.$$

Similarly we define *m-distance sequences* for infinite sequences with n symbols ($n \geq 2$).

DEFINITION 3.5. An infinite sequence S with n symbols $x_1, x_2, \dots, x_n$ is called an *m-distance sequence* if, for each $l \in \mathbf{Z}_+$ and for each x_α ($1 \leq \alpha \leq n$), we have the inequality

$$\#\{|\mathsf{w}|_{x_\alpha} : |\mathsf{w}| = l\} \leq m.$$

By the definition, every $(m-1)$-distance sequence is an m-distance sequence.

LEMMA 3.1. *Let* S *be an infinite sequence with n symbols* $x_1, x_2, \dots, x_n$.
(1) *If* S *is m-distance, then, for each* $l \in \mathbf{Z}_+$ *and for each* x_α ($1 \leq \alpha \leq n$), *there exist* $\mu \in \mathbf{Z}_+$ *and* m' *with* $0 \leq m' \leq m-1$ *such that*

$$\{|\mathsf{w}|_{x_\alpha} : |\mathsf{w}| = l\} = \{\mu + \eta : 0 \leq \eta \leq m'\}.$$

(2) *If* S *is not m-distance, then there exist an* $l \in \mathbf{Z}_+$ *an* $\alpha \in \{1, \dots, n\}$ *and two words* $\mathsf{w}_1, \mathsf{w}_2$ *in* S *of length l, such that* $|\mathsf{w}_2|_{x_\alpha} - |\mathsf{w}_1|_{x_\alpha} = m$.

PROOF. Fix an arbitrary $l \in \mathbf{Z}_+$ and $\alpha \in \{1, \dots, n\}$. We put $\mu = min\{|\mathsf{w}|_{x_\alpha} : |\mathsf{w}| = l\}$ and $M = max\{|\mathsf{w}|_{x_\alpha} : |\mathsf{w}| = l\}$. Then for each word w such that $|\mathsf{w}| = l$, $\mu \leq |\mathsf{w}|_{x_\alpha} \leq M$. When $M - \mu \leq 1$, there is nothing to prove. In what follows, we consider the case $M - \mu \geq 2$. The sequence S is written as

$$\mathsf{S} = \cdots w_{-1} w_0 w_1 \cdots w_l w_{l+1} w_{l+2} \cdots$$

Take two words $\mathsf{w}_1, \mathsf{w}_1^+$ in S, such that $|\mathsf{w}_1|_{x_\alpha} = \mu$, $|\mathsf{w}_1^+|_{x_\alpha} = M$. We assume, without loss of generality, that $\mathsf{w}_1 = w_1 w_2 \cdots w_{l-1} w_l$, $\mathsf{w}_1^+ = w_{1+d} w_{2+d} \cdots w_{l-1+d} w_{l+d}$, $d > 0$. We define

$$\chi(\mathsf{w}_1) = w_2 \cdots w_{l+1},$$

and

$$\chi^c(\mathsf{w}_1) = \chi(\chi^{c-1}(\mathsf{w}_1)) = w_{1+c} \cdots w_{l+c}, \quad (c \in \mathbb{Z}_+).$$

If $|\chi^c(\mathsf{w}_1)|_{x_\alpha} = |\mathsf{w}_1|_{x_\alpha}$, for each $c \geq 0$, then S is three-distance. If it is not the case, let

$$c_1 = max\{c : |\chi^c(\mathsf{w}_1)|_{x_\alpha} = |\mathsf{w}_1|_{x_\alpha}\}.$$

By the definition, it follows that

$$|\chi^{c_1+1}(\mathsf{w}_1)|_{x_\alpha} = |\mathsf{w}_1|_{x_\alpha} + 1.$$

If $|\chi^c(\mathsf{w}_1)|_{x_\alpha} \leq |\mathsf{w}_1|_{x_\alpha} + 1$, for each $c \geq c_1$, then S is three-distance. If it is not the case, we put

$$c_2 = max\{c : |\chi^c(\mathsf{w}_1)|_{x_\alpha} \leq |\mathsf{w}_1|_{x_\alpha} + 1, c \geq c_1\}.$$

Then

$$|\chi^{c_2+1}(\mathsf{w}_1)|_{x_\alpha} = |\mathsf{w}_1|_{x_\alpha} + 2.$$

If $|\chi^c(\mathsf{w}_1)|_{x_\alpha} \leq |\mathsf{w}_1|_{x_\alpha} + 2$, for each $c \geq c_2$, then S is three-distance. If it is not the case, let

$$c_3 = max\{c : |\chi^c(\mathsf{w}_1)|_{x_\alpha} \leq |\mathsf{w}_1|_{x_\alpha} + 2, c \geq c_2\}.$$

Then

$$|\chi^{c_3+1}(\mathsf{w}_1)|_{x_\alpha} = |\mathsf{w}_1|_{x_\alpha} + 3.$$

We repeat this process up to $M - \mu$ steps. If S is m-distance, then $M - \mu < m$. Then μ and $m' := M - \mu$ satisfy the conclusion of (1). If S is not m-distance, then there exist an $l \in \mathbb{Z}_+$ and an α such that $\#\{|\mathsf{w}|_{x_\alpha} : |\mathsf{w}| = l\} > m$. Arguing as above, we may find two words $\mathsf{w}_1, \mathsf{w}_2$ in S of length l, such that $|\mathsf{w}_2|_{x_\alpha} - |\mathsf{w}_1|_{x_\alpha} = m$.

This completes the proof.

Some examples of three-distance sequences with three symbols will be given in the next section.

4 3D Cutting Sequences and Three-Distance Sequences

Example 4.1. The line in $\mathbb{R}^3$ defined by the equation "$x = y = z$" yields a periodic 3D cutting sequence

$$(\mathsf{ABC})^\infty = \cdots \mathsf{ABCABCABCABC} \cdots \mathsf{ABCABCABCABC} \cdots.$$

It is easy to see that the above sequence is two-distance.

Table 1 is a list of the words in the above sequence of length up to 5, and their weights.

Table 1

Length	Words	Weights		
$\|w\|$	w	$\|w\|_A$	$\|w\|_B$	$\|w\|_C$
1	A, B, C	0, 1	0, 1	0, 1
2	AB, BC, CA	0, 1	0, 1	0, 1
3	ABC, BCA, CAB	1	1	1
4	ABCA, BCAB, CABC	1, 2	1, 2	1, 2
5	ABCAB, BCABC, CABCA	1, 2	1, 2	1, 2

□

EXAMPLE 4.2. The line L which passes through the points $(1+\sqrt{2}, (1+\sqrt{5})/2, 1)$ and $(0,0,0)$ yields an aperiodic 3D cutting sequence $\cdots$ BACB-BCABCBBACBCBABCBCBABCBACBBCABCBBACBCBABCBCBABCBACBBC-ABCBBCABCBABCBCBABABCBBACBBCBACB $\cdots$. Theorem 4.1 below shows that the above sequence is three-distance.

Table 2 is a list of the words in the above sequence of length up to 4, and their weights.

Table 2

Length	Words	Weights		
$\|w\|$	w	$\|w\|_A$	$\|w\|_B$	$\|w\|_C$
1	A, B, C	0, 1	0, 1	0, 1
2	AB, BA, BB, AC, CB, CA, BC	0, 1	0, 1, 2	0, 1
3	ABC, CBB, BAB, BBA, BCB, CBC, BAC, CAB, CBA,	0, 1	1, 2	0, 1, 2
.	BBC, BCA, ACB, ABB			
4	ACBB, ABCB, ACBC, ABCB, BACB, BBCA, BCAB, BCBB,	0, 1	1, 2, 3	1, 2
.	BBAC, BCBA, BABC, BCBC, BBCB, CBBC, CABC, CBCA,			
.	CBBA, CABB, CBAC, CBAB, CBCB			

□

We show that each 3D cutting sequence is three-distance.

The orthogonal projection on the u-axis ($u \in \{x, y, z\}$) in $\mathbb{R}^3$ is denoted by P_u. Let S be a 3D cutting sequence associated with a line L in $\mathbb{R}^3$. Take an arbitrary word $w = w_1 \cdots w_l$ in S, $\{w_1, \ldots, w_l\} \subset \{A, B, C\}$. And take the points

m, m' which correspond to w_1 and w_l respectively, as the point of the intersection $\mathsf{L} \cap H_i$ ($H_i \in \mathcal{H}_i, i \in \{\mathsf{A}, \mathsf{B}, \mathsf{C}\}$), or $\mathsf{L} \cap \ell$ ($\ell \in \mathcal{L}$), or $\mathsf{L} \cap \mathbb{Z}^3$. Let M be the segment of L whose end-points are m and m'. The length of the projection of M on the u-axis is denoted by $\overline{\mathsf{P}_\mathsf{u}(\mathsf{M})}$. Then we obtain the following inequalities.

$$\begin{cases} |\mathsf{w}|_\mathsf{A} - 1 \le \overline{\mathsf{P}_x(\mathsf{M})} \le |\mathsf{w}|_\mathsf{A} + 1 \\ |\mathsf{w}|_\mathsf{B} - 1 \le \overline{\mathsf{P}_y(\mathsf{M})} \le |\mathsf{w}|_\mathsf{B} + 1 \\ |\mathsf{w}|_\mathsf{C} - 1 \le \overline{\mathsf{P}_z(\mathsf{M})} \le |\mathsf{w}|_\mathsf{C} + 1 \end{cases} \tag{4.0}$$

The symbols $\mathsf{A}, \mathsf{B}, \mathsf{C}$ correspond to x, y, z, respectively via the above inequality.

THEOREM 4.1. *Each 3D cutting sequence is three-distance.*

PROOF. Let S be a 3D cutting sequence associated with a line L in $\mathbb{R}^3$. We assume that there exist an $i \in \{\mathsf{A}, \mathsf{B}, \mathsf{C}\}$ and two words $\mathsf{w}_1, \mathsf{w}_2$ in S, such that $|\mathsf{w}_1| = |\mathsf{w}_2|$ and $|\mathsf{w}_1|_i + 2 < |\mathsf{w}_2|_i$. Then we obtain

$$0 < |\mathsf{w}_1|_i + 1 < |\mathsf{w}_2|_i - 1. \tag{4.1}$$

Let u be the coordinate corresponding to i via (4.0). And let $\mathsf{M}_1, \mathsf{M}_2$ be the segments of L whose end-points are the points corresponding to the first and last symbols of $\mathsf{w}_1, \mathsf{w}_2$ respectively. Then the slope of $\mathsf{P}_{\mathsf{uv}}(\mathsf{L})$ is

$$\frac{\overline{\mathsf{P}_\mathsf{v}(\mathsf{M}_1)}}{\overline{\mathsf{P}_\mathsf{u}(\mathsf{M}_1)}} = \frac{\overline{\mathsf{P}_\mathsf{v}(\mathsf{M}_2)}}{\overline{\mathsf{P}_\mathsf{u}(\mathsf{M}_2)}}.$$

Let k be a symbol, $k \in \{\mathsf{A}, \mathsf{B}, \mathsf{C}\} \backslash \{i\}$ and v the coordinate corresponding to k, $\mathsf{v} \in \{x, y, z\} \backslash \{\mathsf{u}\}$. By using the inequalities (4.0) and (4.1), it follows that

$$\frac{|\mathsf{w}_1|_k - 1}{|\mathsf{w}_1|_i + 1} \le \frac{\overline{\mathsf{P}_\mathsf{v}(\mathsf{M}_1)}}{\overline{\mathsf{P}_\mathsf{u}(\mathsf{M}_1)}} = \frac{\overline{\mathsf{P}_\mathsf{v}(\mathsf{M}_2)}}{\overline{\mathsf{P}_\mathsf{u}(\mathsf{M}_2)}} \le \frac{|\mathsf{w}_2|_k + 1}{|\mathsf{w}_2|_i - 1}.$$

Therefore, we have

$$\frac{|\mathsf{w}_1|_k - 1}{|\mathsf{w}_1|_i + 1} \le \frac{|\mathsf{w}_2|_k + 1}{|\mathsf{w}_2|_i - 1}. \tag{4.2}$$

From (4.1) and (4.2), we obtain

$$|\mathsf{w}_1|_k - 1 < |\mathsf{w}_2|_k + 1. \tag{4.3}$$

Let j be the symbol other then i, k. Namely $\{i, j, k\} = \{\mathsf{A}, \mathsf{B}, \mathsf{C}\}$. Then,

$$\begin{aligned} |\mathsf{w}_1| &= |\mathsf{w}_1|_i + |\mathsf{w}_1|_j + |\mathsf{w}_1|_k = |\mathsf{w}_2|_i + |\mathsf{w}_2|_j + |\mathsf{w}_2|_k \\ &< |\mathsf{w}_2|_i - 2 + |\mathsf{w}_1|_j + |\mathsf{w}_2|_k + 2 = |\mathsf{w}_2|_i + |\mathsf{w}_1|_j + |\mathsf{w}_2|_k. \end{aligned}$$

Hence

$$|\mathsf{w}_2|_j < |\mathsf{w}_1|_j. \tag{4.4}$$

By the symmetric argument, from (4.2), we have

$$\frac{|\mathsf{w}_1|_j - 1}{|\mathsf{w}_1|_i + 1} \le \frac{|\mathsf{w}_2|_j + 1}{|\mathsf{w}_2|_i - 1}, \tag{4.5}$$

and thus

$$|\mathsf{w}_1|_j - 1 < |\mathsf{w}_2|_j + 1. \tag{4.6}$$

The inequalities (4.4) and (4.6) imply $|\mathsf{w}_1|_j - 1 < |\mathsf{w}_2|_j + 1 < |\mathsf{w}_1|_j + 1$. Hence, we have

$$|\mathsf{w}_2|_j + 1 = |\mathsf{w}_1|_j. \tag{4.7}$$

Then, $|\mathsf{w}_1|_i + |\mathsf{w}_1|_j = |\mathsf{w}_1|_i + |\mathsf{w}_2|_j + 1 < |\mathsf{w}_2|_i + |\mathsf{w}_2|_j - 1$. Therefore, we obtain

$$|\mathsf{w}_1|_k > |\mathsf{w}_2|_k. \tag{4.8}$$

The inequalities (4.8) and (4.3) imply $|\mathsf{w}_1|_k - 1 < |\mathsf{w}_2|_k + 1 < |\mathsf{w}_1|_k + 1$. Hence we have

$$|\mathsf{w}_2|_k + 1 = |\mathsf{w}_1|_k. \tag{4.9}$$

From (4.7) and (4.9),

$$\begin{aligned} |\mathsf{w}_1| &= |\mathsf{w}_1|_i + |\mathsf{w}_1|_j + |\mathsf{w}_1|_k \\ &= |\mathsf{w}_1|_i + |\mathsf{w}_2|_j + |\mathsf{w}_2|_k + 2 < |\mathsf{w}_2|_i + |\mathsf{w}_2|_j + |\mathsf{w}_2|_k = |\mathsf{w}_2|. \end{aligned}$$

This is the contradiction. Hence for each $i \in \{\mathsf{A}, \mathsf{B}, \mathsf{C}\}$, there exist no words $\mathsf{w}_1, \mathsf{w}_2$ such that $||\mathsf{w}_2|_i - |\mathsf{w}_1|_i| > 2$. So S is a three-distance sequence. Q.E.D

There exists a three-distance sequence which is not a 3D cutting sequence. We give such an example.

EXAMPLE 4.3. A periodic infinite sequence which repeats the word AACABCAB

$$S = \cdots \mathsf{CABAACABCABAACAB} \cdots = (\mathsf{AACABCAB})^{\infty}$$

is three-distance. We show that S is not a 3D cutting sequence. If S is a 3D cutting sequence associated with a line L in $\mathbb{R}^3$, then by Remark 2.1, for each u, $\mathcal{D}_{\mathsf{u}}(S)$ is a 2D cutting sequence associated with $\mathsf{P}_{\mathsf{uv}}(\mathsf{L})$ ($\{\mathsf{u}, \mathsf{v}\} \subset \{x, y, z\}$). Here by [1, Theorem 1], $\mathcal{D}_{\mathsf{u}}(S)$ is a two-distance sequence. However,

$$\mathcal{D}_y(S) = \cdots \mathsf{CAAACACAAACA} \cdots = (\mathsf{CAAACA})^{\infty}$$

is not two-distance with two symbols A, C, since the C-weight of the words AAA, ACA, CAC of length 3 in $\mathcal{D}_y(S)$ is 0, 1, 2 respectively. Thus $\mathcal{D}_y(S)$ cannot be a 2D cutting sequence. Accordingly, S is a three-distance sequence which is a not 3D cutting sequence.

5 Three-Distance Sequences which are not 3D Cutting Sequences

In this section, we show that there exist infinitely many three-distance sequences which are not 3D cutting sequences. Let $x_1, \dots, x_n$ be the n symbols. We fix a bijection

$$\mathsf{f}_n : \{1, 2, \dots, n!\} \to \mathbb{S}_n,$$

where

$$\mathbb{S}_n = \{x_{\sigma_1} \cdots x_{\sigma_n} : \{\sigma_1, \dots, \sigma_n\} = \{1, \dots, n\}\}.$$

Note that $\#\{\mathbb{S}_n\} = n!$. For each bi-infinite sequence $R_n = \cdots \rho_{-1}\rho_0\rho_1\rho_2 \cdots$ with $\rho_\nu \in \{1, 2, \dots, n!\}$ $(\nu \in \mathbb{Z})$, we define a bi-infinite sequence with n symbols $x_1, \dots, x_n$ as follows.

$$\mathsf{f}_n(R_n) = \cdots \mathsf{f}_n(\rho_{-1})\mathsf{f}_n(\rho_0)\mathsf{f}_n(\rho_1)\mathsf{f}_n(\rho_2) \cdots.$$

The set of all such sequences is denoted by Σ_{f_n}.

Proposition 5.1.
(1) *If* $n \le 3$, *then each sequence of* Σ_{f_n} *is three-distance.*
(2) *If* $n \ge 4$, *then each sequence of* Σ_{f_n} *is four-distance.*

Proof. When $n = 1$, there is nothing to prove. Assume $n \ge 2$. Let S be an element of Σ_{f_n}. Fix an arbitrary $l \in \mathbb{Z}_+$. We put $l = nt + r$ with $t \in \mathbb{Z}_+$ and $0 \le r < n$. Let w be a word of S such that $|\mathsf{w}| = l$. When $l = |\mathsf{w}| < n$, we obtain $|\mathsf{w}|_{x_\alpha} \le 2$ $(x_\alpha \in \{x_1, \dots, x_n\})$. Now suppose $l \ge n$. We write w as $\mathsf{w} = \mathsf{w}_1\overline{\mathsf{w}}\mathsf{w}_2$, where $\overline{\mathsf{w}} = \mathsf{f}_n(\rho_\nu) \cdots \mathsf{f}_n(\rho_{\nu+h})$, $\nu \in \mathbb{Z}$, $h \in \mathbb{Z}_+$, and $\mathsf{w}_1, \mathsf{w}_2$ are the words of S such that w_1 is a proper subword of $\mathsf{f}_n(\rho_{\nu-1})$ and w_2 is a proper subword of $\mathsf{f}_n(\rho_{\nu+h+1})$. If $|\mathsf{w}_1| = |\mathsf{w}_2| = 0$, then $|\mathsf{w}| = |\overline{\mathsf{w}}| = nt$. If $|\mathsf{w}_\mathsf{a}| \ne 0$ and $|\mathsf{w}_\mathsf{b}| = 0$ $(\mathsf{a}, \mathsf{b} \in \{1, 2\})$, then $|\overline{\mathsf{w}}| = nt$ and $1 \le |\mathsf{w}_\mathsf{a}| = r < n$. If $|\mathsf{w}_1| \ne 0$ and $|\mathsf{w}_2| \ne 0$, then $2 \le |\mathsf{w}_1| + |\mathsf{w}_2| \le 2n - 2$. Thus we have

$$nt + r - 2 \le |\overline{\mathsf{w}}| \le nt + r - 2n + 2.$$

Since $0 \le r < n$, we obtain

$$nt - 2 \le nt + r - 2 \le |\overline{\mathsf{w}}| \le nt + r - 2n + 2 < nt - n + 2 = n(t-1) + 2.$$

Namely

$$n(t-1) \le nt-2 \le |\overline{\mathbf{w}}| < n(t-1)+2.$$

Therefore $|\overline{\mathbf{w}}| = n(t-1)$. First, we consider the case $|\overline{\mathbf{w}}| = nt$. Then $|\mathbf{w}_1| + |\mathbf{w}_2| = r$ and $|\overline{\mathbf{w}}|_{x_\alpha} = t$, $0 \le |\mathbf{w}_1|_{x_\alpha} + |\mathbf{w}_2|_{x_\alpha} \le 2$. Since $|\mathbf{w}|_{x_\alpha} = |\mathbf{w}_1|_{x_\alpha} + |\overline{\mathbf{w}}|_{x_\alpha} + |\mathbf{w}_2|_{x_\alpha}$, we have

$$t \le |\mathbf{w}|_{x_\alpha} \le t+2. \tag{5.10}$$

Next, we consider the case $|\overline{\mathbf{w}}| = n(t-1)$. Then $|\mathbf{w}_1| + |\mathbf{w}_2| = n+r$ and $0 \le |\mathbf{w}_1|_{x_\alpha} + |\mathbf{w}_2|_{x_\alpha} \le 2$, and $|\overline{\mathbf{w}}|_{x_\alpha} = t-1$. Thus we have

$$t-1 \le |\mathbf{w}|_{x_\alpha} \le t+1. \tag{5.11}$$

By inequalities (5.10) and (5.11), we obtain $t-1 \le |\mathbf{w}|_{x_\alpha} \le t+2$. Therefore $\mathbf{S}$ is at most four-distance. Furthermore, if $n \ge 4$, it is easy to create a four-distance sequence. Next, we consider the following case: $n \le 3$.

Case 1: When $n = 2$, an arbitrary l is written as $l = 2t$ or $l = 2t+1$.

First, we assume $l = |\mathbf{w}| = 2t$. If $|\overline{\mathbf{w}}| = 2t$, then $|\mathbf{w}|_{x_\alpha} = |\overline{\mathbf{w}}|_{x_\alpha} = t$. If $|\overline{\mathbf{w}}| = 2(t-1)$, then $t-1 \le |\mathbf{w}|_{x_\alpha} \le t+1$. Hence, we obtain $t-1 \le |\mathbf{w}|_{x_\alpha} \le t+1$.

Next, we assume $l = |\mathbf{w}| = 2t+1$. If $|\overline{\mathbf{w}}| = 2t$, then $t \le |\mathbf{w}|_{x_\alpha} \le t+1$. We note that $|\overline{\mathbf{w}}| = 2(t-1)$ does not hold in this case. Because, if $|\overline{\mathbf{w}}| = 2(t-1)$, then we obtain $|\mathbf{w}_1| + |\mathbf{w}_2| = 3$. Hence $|\mathbf{w}_1| = 1$ and $|\mathbf{w}_2| = 2$, or $|\mathbf{w}_1| = 2$ and $|\mathbf{w}_2| = 1$. This is contrary to our assumption that $\mathbf{w}_1$ and $\mathbf{w}_2$ are proper subwords of $\mathbf{f}_n(\rho_{v-1})$ and $\mathbf{f}_n(\rho_{v+h+1})$, respectively.

Therefore, if $n = 2$, then $\mathbf{S}$ is three-distance.

Case 2: When $n = 3$, an arbitrary l is written as $l = 3t$ or $l = 3t+1$ or $l = 3t+2$.

First, we assume $l = |\mathbf{w}| = 3t$. If $|\overline{\mathbf{w}}| = 3t$, then $|\mathbf{w}|_{x_\alpha} = |\overline{\mathbf{w}}|_{x_\alpha} = t$. If $|\overline{\mathbf{w}}| = 3(t-1)$, then $t-1 \le |\mathbf{w}|_{x_\alpha} \le t+1$. Hence, we obtain $t-1 \le |\mathbf{w}|_{x_\alpha} \le t+1$.

Next, we assume $l = |\mathbf{w}| = 3t+1$. If $|\overline{\mathbf{w}}| = 3t$, then $t \le |\mathbf{w}|_{x_\alpha} \le t+1$. If $|\overline{\mathbf{w}}| = 3(t-1)$, then $t-1 \le |\mathbf{w}|_{x_\alpha} \le t+1$. Hence, we have $t-1 \le |\mathbf{w}|_{x_\alpha} \le t+1$.

Assume $l = |\mathbf{w}| = 3t+2$. If $|\overline{\mathbf{w}}| = 3t$, then $t \le |\mathbf{w}|_{x_\alpha} \le t+2$. We note that $|\overline{\mathbf{w}}| = 3(t-1)$ does not hold in this case. Because, if $|\overline{\mathbf{w}}| = 3(t-1)$, then we obtain $|\mathbf{w}_1| + |\mathbf{w}_2| = 5$. Hence $|\mathbf{w}_1| = 1$ and $|\mathbf{w}_2| = 4$, or $|\mathbf{w}_1| = 4$ and $|\mathbf{w}_2| = 1$, or $|\mathbf{w}_1| = 2$ and $|\mathbf{w}_2| = 3$, or $|\mathbf{w}_1| = 3$ and $|\mathbf{w}_2| = 2$. This is contrary to our assumption that $\mathbf{w}_1$ and $\mathbf{w}_2$ are proper subwords of $\mathbf{f}_n(\rho_{v-1})$ and $\mathbf{f}_n(\rho_{v+h+1})$, respectively.

Therefore, if $n=3$, then S is three-distance.

This completes the proof.

EXAMPLE 5.1. When $n=3$, $\#\{\mathbb{S}_3\}=6$. We put $\{x_1,x_2,x_3\}=\{\mathsf{A},\mathsf{B},\mathsf{C}\}$. Let $\mathsf{f}_3:\{1,2,\dots,6\}\to\mathbb{S}_3$ be a bijection given by:

$$1\mapsto \mathsf{ABC},\quad 2\mapsto \mathsf{ACB},\quad 3\mapsto \mathsf{BAC},\quad 4\mapsto \mathsf{BCA},\quad 5\mapsto \mathsf{CAB},\quad 6\mapsto \mathsf{CBA}.$$

By Proposition 5.1, an infinite sequence

$$R_3=\cdots 5243536456431143225352245135362462662531624334133462246624323\!5\ 543456625426166216231525522166544\cdots,$$

produces a three-distance sequence S $(\in\Sigma_{\mathsf{f}_3})$,

$$\mathsf{S}=\cdots\mathsf{CABACBBCABACCABBACCBABCACABCBABCABACABCA}\ \mathsf{BCBCA}\cdots.$$

However,

$$\mathcal{D}_x(\mathsf{S})=\cdots\mathsf{CBCBBCBCCBBCCBB}\cdots$$

and

$$\mathcal{D}_y(\mathsf{S})=\cdots\mathsf{CAACCAACCAACCACAC}\cdots,$$

$$\mathcal{D}_z(\mathsf{S})=\cdots\mathsf{ABABBABAABBABABAABBABAB}\cdots$$

are not two-distances with two symbols BC, CA, AB respectively. Namely, there does not exist a line in $\mathbb{R}^2$ which has $\mathcal{D}_{\mathsf{u}}(\mathsf{S})$ as its 2D cutting sequence. Therefore S is a three-distance sequence which is not a 3D cutting sequence. From the above construction, it is easy to see that there are infinitely many such sequences.

The set of the elements of Σ_{f_3} which are not 3D cutting sequences is denoted by $\Sigma^*_{\mathsf{f}_3}$.

COROLLARY 5.2. $card\,\Sigma^*_{\mathsf{f}_3}=card\,\Sigma_{\mathsf{f}_3}=card\,\mathbb{R}$.

PROOF. The set of bi-infinite sequences with symbols $1,2,\dots,6$ is denoted by $\mathcal{R}_3$. For a sequence $R_3=\cdots r_{-1}r_0r_1r_2\cdots\in\mathcal{R}_3$ with $r_\nu\in\{1,2,\dots,6\}$ $(\nu\in\mathbb{Z})$, we define the infinite sequence $R_3^*=\cdots r_{-1}135r_0r_1r_2\cdots$. We put

$$\mathcal{R}_3^*=\{R_3^*:R_3\in\mathcal{R}_3\}.$$

Then we have $card\ \mathcal{R}_3^* = card\ \mathcal{R}_3 = card\ \mathbb{R}$. Note that

$$\mathcal{D}_z \circ \mathsf{f}_3(135) = \mathcal{D}_z(\mathsf{f}_3(1)\mathsf{f}_3(3)\mathsf{f}_3(5)) = \mathcal{D}_z(\mathsf{ABCBACCAB}) = \mathsf{ABBAAB}.$$

Hence, for any element R_3^* of $\mathcal{R}_3^*$, $\mathcal{D}_z \circ \mathsf{f}_3(R_3^*)$ is not two-distance with two symbols A, B. Thus $\mathcal{D}_z \circ \mathsf{f}_3(R_3^*)$ cannot be a 2D cutting sequence. From Remark 2.1, we see $\mathsf{f}_3(R_3^*) \in \Sigma_{\mathsf{f}_3}^*$. We put

$$\Sigma_{\mathsf{f}_3}^*(135) = \{\mathsf{f}_3(R_3^*) : R_3^* \in \mathcal{R}_3^*\}.$$

Note that $\Sigma_{\mathsf{f}_3}^*(135) \subset \Sigma_{\mathsf{f}_3}^*$. Since there exists an injection:

$$\mathcal{R}_3^* \to \Sigma_{\mathsf{f}_3}^*(135), \quad R_3^* \mapsto \mathsf{f}_3(R_3^*),$$

we have $card\ \mathbb{R} \leq card\ \Sigma_{\mathsf{f}_3}^*(135)$. Hence $card\ \mathbb{R} \leq card\ \Sigma_{\mathsf{f}_3}^*$. Therefore we obtain

$$card\ \mathbb{R} \leq card\ \Sigma_{\mathsf{f}_3}^* \leq card\ \Sigma_{\mathsf{f}_3} \leq card\ \mathbb{R},$$

and

$$card\ \Sigma_{\mathsf{f}_3}^* = card\ \Sigma_{\mathsf{f}_3} = card\ \mathbb{R}. \qquad \text{Q.E.D}$$

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