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# ANALYTIC REPRESENTATION OF GENERALIZED TEMPERED DISTRIBUTIONS OF EXPONENTIAL GROWTH BY WAVELETS

By

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**Abstract.** The analytic representation of the generalized tempered distributions of $e^{M(kx)}$-growth with restricted order, $\mathscr{K}_M^{r\prime}(R)$, is given in terms of series of analytic wavelets. These series converge uniformly on compact subsets of the upper and lower half planes.

## 1. Introduction

The analytic representation of functions or distributions on the real line $R$ is usually given by a Cauchy type formula, but in some cases may also be given by an orthogonal series. It is well-known that trigonometric series may be used for the analytic representation of periodic functions and distributions. Also, Hermite series and Legendre polynomials can be used for the representation of non-periodic functions and functions with compact support, respectively. Recently a new category of orthogonal systems has been introduced in [1]. These systems are composed of wavelets, i.e., orthogonal functions on $R$ consisting of dilations and translations of a fixed function. G. G. Walter has found an expansion in orthogonal wavelets and pointwise convergence of that expansion from $L^2(R)$ to the tempered distributions with restricted order of derivative, $\mathscr{S}_r'(R)$, in [6] and [8] and has showed an analytic representation of $\mathscr{S}_r'(R)$ in terms of series of analytic wavelets in [7]. These two results were extended by us to the case of the tempered distributions of exponential growth with restricted order in [3], [5]. Also, we have found the wavelet expansion of the tempered distributions of $e^{M(kx)}$-growth

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with restricted order and the pointwise convergence of the wavelet expansion of $\mathcal{K}_M^{r\prime}(R)$ in [4].

In this paper, we will present an analytic representation of $\mathcal{K}_M^{r\prime}(R)$ in terms of series of analytic wavelets. These series converge uniformly on compact subsets of the upper and lower half planes.

## 2. The Generalized Tempered Distributions Space $\mathcal{K}_M'(R)$

Let $\mu(\xi)$ $(0 \le \xi \le \infty)$ denote a continuous increasing function such that $\mu(0) = 0$, $\mu(\infty) = \infty$. For $x \ge 0$, we define

$$M(x) = \int_0^x \mu(\xi)\, d\xi.$$

The function $M(x)$ is an increasing, convex and continuous function with $M(0) = 0$, $M(\infty) = \infty$ and satisfies the fundamental convexity inequality $M(x_1) + M(x_2) \le M(x_1 + x_2)$. Further we define $M(x)$ for negative $x$ by means of the equality $M(-x) = M(x)$. Note that since the derivative $\mu(x)$ of $M(x)$ is unbounded in $R$, the function $M(x)$ will grow faster than any linear function as $|x| \to \infty$. Now we list some properties of $M(x)$ which will be frequently used in this paper.

$$M(x) + M(y) \le M(x + y) \quad \text{for all } x, y \ge 0. \tag{1}$$

$$M(x + y) \le M(2x) + M(2y) \quad \text{for all } x, y \ge 0. \tag{2}$$

Using the function $M(x)$ we define the space $\mathcal{K}_M(R)$ as the space of all functions $\phi \in C^\infty(R)$ such that

$$\nu_k(\phi) = \sup_{x \in R, \alpha \le k} e^{M(kx)} |D^\alpha \phi(x)| < \infty, \quad k = 1, 2, \ldots, \tag{3}$$

where $D^\alpha = d^\alpha/dx^\alpha$. The topology in $\mathcal{K}_M(R)$ is defined by the family of the semi-norms $\nu_k$. Then $\mathcal{K}_M(R)$ becomes a Fréchet space and the embeddings $\mathcal{D} \hookrightarrow \mathcal{K}_M \hookrightarrow \mathcal{S} \hookrightarrow \mathcal{E}$ are continuous; here $\mathcal{E}$ denotes the space of all $C^\infty$-functions, $\mathcal{S}$ the space of the tempered distributions of polynomial growth and $\mathcal{D}$ the space of $C^\infty$-functions with compact supports. By $\mathcal{K}_M'(R)$, we mean the space of continuous linear functionals on $\mathcal{K}_M(R)$. Pahk characterized the distributions in $\mathcal{K}_M'(R)$ by the growth at infinity [2, Theorem 2.3]; a distribution $T \in \mathcal{D}'$ is in $\mathcal{K}_M'(R)$ if and only if there exist positive integers $\alpha$, $k_0$ and a bounded continuous function $f(x)$ on $R$ such that

$$T = D^\alpha[e^{M(k_0 x)} f(x)].$$

Definition 1. *For a natural number r, we denote by* $\mathscr{K}_M^r(R)$ *the space of all functions* $\phi \in C^r(R)$ *such that*

$$v_k^r(\phi) = \sup_{x \in R, \alpha \le r} e^{M(kx)} |D^\alpha \phi(x)| < \infty, \quad k = 1, 2, 3, \ldots.$$

The topology of $\mathscr{K}_M^r(R)$ is defined by the family of semi-norms $\{v_k^r\}_{k=1,2,\ldots}$. By $\mathscr{K}_M^{r\prime}(R)$, we mean the space of continuous linear functionals on $\mathscr{K}_M^r(R)$. Each $S \in \mathscr{K}_M^{r\prime}(R)$ is characterized by

$$S = D^r[e^{M(k_0 x)} f(x)], \tag{4}$$

where $f(x)$ is a bounded continuous function on $R$ and $r$, $k_0 \in N$, the set of natural numbers, by the same method of the above $\mathscr{K}_M'$-case in [2, Theorem 2.3]. Similarly, we can define

$$\mathscr{S}_r(R) = \{\theta(t) \in C^r(R); |D^k \theta(t)| \le C_{pk}(1 + |t|)^{-p}, p \in N, k = 0, 1, \ldots, r\}$$

and its dual $\mathscr{S}_r'(R)$. For further details, we refer to [2].

## 3. Multiresolution Analysis of $L^2(R)$ Associated with $\phi \in \mathscr{K}_M^r(R)$

Let $\phi \in \mathscr{K}_M^r(R)$. In order for it to qualify as a scaling function, there must be associated with $\phi$ a multiresolution analysis of $L^2(R)$, i.e., a nested sequence of closed subspaces $\{V_m\}_{m \in Z}$ for the set of integers $Z$ such that

(i) $\{\phi(\cdot - n)\}$ is an orthonormal basis of $V_0$,
(ii) $\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots \subset L^2(R)$,
(iii) $f(\cdot) \in V_m \Leftrightarrow f(2\cdot) \in V_{m+1}$,
(iv) $\bigcap_m V_m = \{0\}$, $\overline{\bigcap_m V_m} = L^2(R)$.

Then $\phi$ has an expansion

$$\phi(t) = \sum_n c_n \sqrt{2} \phi(2t - n), \quad \{c_n\} \in l^2, t \in R, \tag{5}$$

where $l^2 = \{\{c_n\}; \sum_n |c_n|^2 < \infty\}$. Once we have the scaling function $\phi \in \mathscr{K}_M^r(R)$, we can obtain a mother wavelet $\psi$ such that $\{\psi(t - n)\}$ is an orthogonal basis of the space $W_0$, given by the orthogonal complement of $V_0$ in $V_1$. Also, $\psi$ has an expansion

$$\psi(t) = \sum_n d_n \sqrt{2} \phi(2t - n), \quad \{d_n\} \in l^2, \tag{6}$$

for $d_n$ corresponding to $c_n$ in (5). We will adopt the construction of a mother wavelet defined by $d_n = (-1)^n \overline{c_{1-n}}$. If such a $\psi(t)$ can be found, then $\psi_{mn}(t) = 2^{m/2}\psi(2^m t - n)$ is an orthogonal basis of $W_m$ which is the orthogonal complement of $V_m$ in $V_{m+1}$.

EXAMPLE. In [1], Corollary 5.5.3 states that it is impossible that $\psi$ has exponential decay and that $\psi \in C^\infty$, with all derivatives bounded, unless $\psi = 0$. Hence there is no mother wavelet $\psi \in \mathcal{K}_M(R)$. So we will restrict our attention to $\mathcal{K}_M^r(R)$. Daubechies' compactly supported wavelets are examples of $\mathcal{K}_M^r(R)$, but Battle-Lemarié's wavelets (in the page 152 of [1]) are not $\mathcal{K}_M^r(R)$ wavelets even if they have exponential decay and smoothness.

The reproducing kernel of $V_0$ is given by

$$q(x,t) = \sum\nolimits_n \overline{\phi(x-n)}\phi(t-n),$$

where $\phi(x)$ is the scaling function. The series and its derivatives with respect to $t$ of order $\le r$ converge uniformly on $x \in R$ because of the regularity of $\phi \in \mathcal{K}_M^r(R)$, i.e.,

$$|\phi^{(\alpha)}(x)| \le C_{\alpha k} e^{-M(kx)}, \quad \alpha = 0, 1, \dots, r; \; k = 1, 2, \dots. \tag{7}$$

The reproducing kernel for $V_m$ is given by

$$q_m(x,t) = 2^m q(2^m x, 2^m t).$$

Similarly, we can define the reproducing kernel $r_m(x,t)$ for $W_m$ by

$$r_m(x,t) = 2^m \sum\nolimits_n \overline{\psi(2^m x - n)}\psi(2^m t - n),$$

where $\psi(t)$ is the mother wavelet.

The sequence $\{q_m(x,t)\}$ is a delta sequence in $\mathcal{S}_r'(R) \subset \mathcal{K}_M^{r\prime}(R)$, i.e., $q_m(x,t) \to \delta(x-t)$. This follows from the fact that

$$\int_{-\infty}^{\infty} q_m(x,t)\theta(t)\,dt \to \theta(x) \quad \text{as } m \to \infty,$$

for each $\theta \in \mathcal{K}_M^r(R) \subset \mathcal{S}_r(R)$, where the convergence is in the $L^2$-sense. These kernels have a number of interesting properties, some of which come out of the wavelet moment theorem. Since $\mathcal{K}_M^r(R) \subset \mathcal{S}_r(R)$, we have by [1], LEMMA 2. *Let* $\psi \in \mathcal{K}_M^r(R)$ *with* $\psi_{mn}(x) = 2^{m/2}\psi(2^m x - n)$ *an orthogonal system in* $L^2(R)$. *Then*

$$\int_{-\infty}^{\infty} x^k \psi(x)\, dx = 0, \quad k = 0, 1, \ldots, r.$$

DEFINITION 3. *We define the spaces* $T_0$ *and* $U_0$ *by* $T_0 = \{f; f(t) = \sum_n a_n \phi(t-n)$ *for some sequence of complex numbers with* $a_n = \mathcal{O}(e^{M(k_1 n)})$ *for some* $k_1 \in N\}$ *and* $U_0 = \{g; g(t) = \sum_n a_n \psi(t-n)$ *for some sequence of complex numbers with* $a_n = \mathcal{O}(e^{M(k_1 n)})$ *for some* $k_1 \in N\}$. *We denote by* $T_m$ *and* $U_m$ *their corresponding dilation spaces, i.e.,* $f \in T_0 \Leftrightarrow f(2^m t) \in T_m$ *and* $g \in U_0 \Leftrightarrow g(2^m t) \in U_m$.

We may expect that a multiresolution analysis of $\mathcal{K}_M^{r\prime}(R)$ exists, namely,

$$\cdots \subset T_{-m} \cdots \subset T_{-1} \subset T_0 \subset T_1 \cdots \subset T_m \subset \cdots \subset \mathcal{K}_M^{r\prime}(R) \tag{8}$$

and

$$\overline{\bigcup_m T_m} = \mathcal{K}_M^{r\prime}(R),$$

where the closure is in the topology of $\mathcal{K}_M^{r\prime}(R)$.

Now in [3], we have found the expansion in orthogonal wavelets from $L^2(R)$ to $\mathcal{K}_M^{r\prime}(R)$.

THEOREM 4. *Let the scaling function* $\phi \in \mathcal{K}_M^r(R)$ *satisfy the dilation equation* (5) *with* $c_k = \mathcal{O}(e^{-M(lk)})$ *for all* $l \in N$, *and have an associated multiresolution analysis in* $L^2(R)$; *let* $\psi \in \mathcal{K}_M^r(R)$ *be the mother wavelet given in* (6). *Then there exists a multiresolution analysis* (8) *of closed dilation subspaces* $\{T_m\}$ *whose union is dense in* $\mathcal{K}_M^{r\prime}(R)$; *the closed subspace* $U_m$ *in Definition* 3 *is a complementary subspace of* $T_m$ *in* $T_{m+1}$ *and*

$$T_m = U_0 \oplus U_1 \oplus \cdots \oplus U_m \oplus T_0,$$

*where* $\oplus$ *denotes the nonorthogonal direct sum.*

## 4. Analytic Representation of Distributions of $\mathcal{K}_M^{r\prime}$ by Wavelets

A *quasi-positive delta sequence* is a sequence $\{\delta_m(\cdot, y)\}$ of functions in $L^1(R)$ with a parameter $y \in R$ which satisfies the following:

(a) there is a $C > 0$ such that

$$\int_{-\infty}^{\infty} |\delta_m(x, y)|\, dx \le C, \quad y \in R, m \in N;$$

(b) there is a $c > 0$ such that

$$\int_{y-c}^{y+c} \delta_m(x, y)\, dx \to 1$$

uniformly on compact subsets of $R$, as $m \to \infty$;

(c) for each $\gamma > 0$,

$$\sup_{|x-y| \le \gamma} |\delta_m(x, y)| \to 0 \quad \text{as } m \to \infty.$$

Then since $\mathscr{K}_M^r(R) \subset \mathscr{S}_r(R)$, we have the following important lemmas as in [8]:

LEMMA 5. *Let $\{\delta_m(x, y)\}$ be a quasi-positive delta sequence and let $f \in L^1(R)$ be continuous on $(a, b)$. Then*

$$f_m(y) = \int_{-\infty}^{\infty} \delta_m(x, y) f(x)\, dx \to f(y) \quad \textit{as } m \to \infty$$

*uniformly on compact subsets of $(a, b)$.*

LEMMA 6. *If the scaling function $\phi \in \mathscr{K}_M^r(R)$, then the reproducing kernel $q_m(x, y)$ and $K_m(x, t) = \frac{(x-t)}{\alpha!} \frac{\partial^\alpha}{\partial t^\alpha} q_m(x, t)$ for $\alpha \in N$, $0 \le \alpha \le r$, are quasi-positive delta sequences on $R$.*

In order to represent an element of $\mathscr{K}_M^{r\prime}(R)$ by series of analytic wavelets, we impose conditions on the scaling function $\phi$ again. Since $\mathscr{K}_M^r(R) \subset L^2(R)$, an analytic representation of $\phi$ is given by

$$\phi^{\pm}(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\phi(x)}{x - z}\, dx, \quad Im\, z \gtrless 0,$$

where $\phi^{\pm}$ are analytic in the upper half-plane and the lower half-plane, respectively. An analytic representation of the mother wavelet is also given by

$$\psi^{\pm}(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\psi(x)}{x - z}\, dx, \quad Im\, z \gtrless 0,$$

and the analytic wavelets $\psi_{mn}^{\pm}$ are obtained by dilation and translation of $\psi^{\pm}$. Now, we define $T_0^{\pm} = \{f(z) = \sum_n a_n \phi^{\pm}(z - n); a_n = \mathcal{O}(c^{l_0 M(n)})$ for some $l_0 \in N\}$ and we denote by the subspaces $T_m^{\pm}$ of $T_0^{\pm}$ the corresponding dilation spaces. Then the spaces $T_m^+$ and $T_m^-$ are composed of analytic functions in the upper and the lower half-planes, respectively, whose boundary functions are continuous functions of $e^{M(x)}$-growth. Since $\overline{\bigcup T_m} = \mathscr{K}_M^{r\prime}(R)$, we might expect to obtain an analytic representation of $f \in \mathscr{K}_M^{r\prime}(R)$ in terms of wavelets,

$$f^+(z) = \sum_{n=-\infty}^{\infty} a_n \phi^+(z-n) + \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} b_{mn} 2^{m/2} \psi^+(2^m z - n),$$

where the first series may not converge. Since an analytic representation is a continuous map from $\mathscr{K}_M^{r\prime}(R)$ to a corresponding space of analytic functions and $f_m(x) = (f, q_m(x,t)) \to f(x) = D^r F(x)$ in $\mathscr{K}_M^{r\prime}(R)$ for a continuous function of $e^{M(x)}$-growth $F(x)$ [cf. (4)] by Lemmas 5 and 6, $f_m^+(z) \to f^+(z)$ uniformly on bounded subsets of the upper half-plane. Moreover, $f^+(z) = D_z^r F^+(z)$, where $F^+(z)$ is an analytic representation of $F(z)$, and is given by

$$F^+(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{F(x)}{x-z} e^{-M(kx)} e^{M(kz)}\, dx,$$

for a sufficiently large $k$ such that $F(x)e^{-M(kx)} \in L^2(R)$. Here for $z \in C$, we define $e^{M(z)}$ as $e^{M(|z|)}$.

We may express $f_m$ as

$$f_m = f_0 + f_m - f_0 = f_0 + \sum_{k=0}^{m-1} \sum_{n=-\infty}^{\infty} b_{kn} \psi_{kn},$$

and if the inner sum converges,

$$f_m^+(z) - f_0^+(z) = \sum_{k=0}^{m-1} \sum_{n=-\infty}^{\infty} b_{kn} \psi_{kn}^+(z) + g_m(z), \tag{9}$$

where $g_m(z)$ is an entire function.

LEMMA 7. *Let* $\psi \in \mathscr{K}_M^r(R)$ *and* $b_n = \mathcal{O}(e^{M(kn)-\varepsilon})$ *for any* $k \in N$ *and some* $\varepsilon > 0$. *Then*

$$\sum_{n=-\infty}^{\infty} b_n \psi^+(z-n)$$

*converges uniformly on compact subsets of the upper half-plane.*

PROOF. The proof is based on the moment property, Lemma 2,

$$\int_{-\infty}^{\infty} x^l \psi(x)\, dx = 0, \quad l = 0, 1, \dots, r.$$

Hence, for any $k \in N$ and a natural number $p \le r+1$,

$$\begin{aligned}
e^{M(kz)}\psi^{+}(z) &= \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{M(kz)}}{z^{p}}\cdot\frac{z^{p}}{x-z}\psi(x)\,dx \\
&= \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{M(kz)}}{z^{p}}\cdot\frac{z^{p}-x^{p}}{x-z}\psi(x)\,dx \\
&\quad+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{M(kz)}}{z^{p}}\cdot\frac{x^{p}}{x-z}\psi(x)\,dx \\
&= -\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{M(kz)}}{z^{p}}\cdot(x^{p-1}+zx^{p-2}+\cdots+z^{p-2}x+z^{p-1})\psi(x)\,dx \\
&\quad+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{M(kz)}}{z^{p}}\cdot\frac{x^{p}}{x-z}\psi(x)\,dx \\
&= \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{M(kz)}}{z^{p}}\cdot\frac{x^{p}}{x-z}\psi(x)\,dx
\end{aligned}$$

holds. By the growth condition of $\psi \in \mathscr{K}_M^r(R)$, $|e^{M(kz)}\psi^{+}(z)|$ is uniformly bounded on compact subsets of the half-plane $Im\, z \geq \varepsilon > 0$ for any $k \in N$ and a natural number $p \leq r+1$. Hence, the preceding fact holds for any $k \in N$ and any $p \leq r+1$. Thus the conclusion follows.

THEOREM 8. *For natural numbers $s$, $r$ with $s < r$, let $f \in \mathscr{K}'^{s}_M(R)$, $\phi, \psi \in K_M^r(R)$ and let $b_{mn} = \langle f, \psi_{mn}\rangle$, $m = 0, 1, 2, \ldots$; $n = 0, \pm 1, \pm 2, \ldots$ be the wavelet coefficients of $f$. Then an analytic representation of $f$ is given by*

$$f^{+}(z) = f_0^{+}(z) + \sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty} b_{mn}\psi_{mn}^{+}(z),$$

*where the series converges uniformly on compact subsets of the half-plane $Im\, z \geq 1$ and $f_0^{+}(z)$ is an analytic representation of $f_0$, the projection of $f$ on $T_0$.*

PROOF. First, we will estimate $|b_{mn}|$. Each $f \in \mathscr{K}'^{s}_M(R)$ is characterized by

$$f = D^{s}[e^{M(k_0 x)}\mu]$$

for some integer $k_0$ and finite measure $\mu$ on $R$. Each $\psi \in \mathscr{K}_M^r(R)$ satisfies

$$|\psi^{(l)}(x)| \leq C_j e^{-M(jx)}, \quad l = 1, 2, \ldots, r; j \geq 0.$$

If we use integration by parts $s$-times, we have, for $m > 1$,

$$|b_{mn}| \le \int_{-\infty}^{\infty} |D^s[e^{M(k_0x)}]\psi_{mn}(x)|\, d|\mu| \le \int_{-\infty}^{\infty} e^{M(k_0x)}|\psi_{mn}^{(s)}(x)|\, d|\mu|$$

$$\le \int_{-\infty}^{\infty} e^{M(k_0x)} c_{k_0} 2^{m/2+sm} e^{-M(k_0(2^m x-n))}\, d|\mu|$$

$$\le \int_{-\infty}^{\infty} e^{M(2k_0(x-n2^{-m}))} e^{M(2k_0n2^{-m})} c_{k_0} 2^{m/2+sm} e^{-M(2^m k_0(x-n2^{-m}))}\, d|\mu|$$

$$\le c'_{k_0} 2^{m/2+sm} e^{M(2k_0n2^{-m})}.$$

By the fact in the proof of Lemma 7, on every compact subset $K$ of the half-plane $Im\, z \ge 1$, there exists a constant $c$ such as $|\psi^+(z)| \le ce^{-M(kz)}$ for any $k \in N$.

Hence if we take $k$ sufficiently large with $k > \sup\{k_0, \frac{3}{2}+s\}$, then for $z \in K$,

$$\sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty} |b_{mn}\psi_{mn}^+(z)|$$

$$\le \sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty} c'_{k_0} e^{(1/2+s)m} e^{M(2k_0n2^{-m})} c2^{m/2} e^{-M(k(2^m z-n))}$$

$$\le \sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty} cc'_{k_0} e^{(3/2+s)m} e^{M(2k_0n2^{-m})} e^{-M(k2^m(n2^{-m}-Re\, z-i\, Im\, z))} 2^{-m/2}$$

$$\le \sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty} cc'_{k_0} e^{(3/2+s)m} e^{M(2k_0n2^{-m})}$$

$$\times\, e^{-M(k2^m(n2^{-m}-Re\, z))} e^{-M(2^{m-1}k)} 2^{-m/2}$$

$$\le \left\{\sum_{m=0}^{2}\sum_{n=-\infty}^{\infty} + \sum_{m=3}^{\infty}\sum_{n=-\infty}^{\infty}\right\} cc'_{k_0} e^{(3/2+s)m} e^{M(2k_0n2^{-m})}$$

$$\times\, e^{-M(k2^m(n2^{-m}-Re\, z))} e^{-M(2^{m-1}k)} 2^{-m/2}$$

$$\le \sum_{m=3}^{\infty} C_{k_0,z} 2^{-m/2} < \infty,$$

where we use the properties (1), (2) and the inequality $\sqrt{2}|z| \ge |Re\, z| + |Im\, z| \ge |Re\, z| + 1$ for $|Im\, z| \ge 1$. Hence the series $\sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty} b_{mn}\psi_{mn}^+(z)$ converges uniformly on compact subsets of the half-plane $Im\, z \ge 1$.

Now, by taking the limit in (9) as $m \to \infty$, we have

$$f^+(z) = f_0^+(z) + \sum_{k=0}^{\infty}\sum_{n=-\infty}^{\infty} b_{kn}\psi_{kn}^+(z) + g_\infty(z),$$

where $g_\infty(z) = \lim_{m\to\infty} g_m(z)$ is an entire function. Since an analytic representation plus an entire function is an analytic representation, we can drop $g_\infty$ in (9).

Remark. We have only worked out the convergence for $f^+$ but proof for $f^-$ is parallel. Then by the same method as in the proof of Theorem 8, an analytic representation of $f$ is given by

$$f^-(z) = f_0^-(z) + \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} b_{mn}\psi_{mn}^-(z),$$

where the series converges uniformly on compact subsets of the half-plane $Im\, z \leq -1$ and $f_0^-(z)$ is an analytic representation of $f_0$, the projection of $f$ on $T_0$.

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