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AN EXTENSION OF RAUCH COMPARISON THEOREM TO GLUED RIEMANNIAN SPACES

By

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Abstract. A glued Riemannian space is obtained from Riemannian manifolds M_1 and M_2 by identifying their isometric submanifolds B_1 and B_2. A curve on a glued Riemannian space which is a geodesic on each Riemannian manifold and satisfies certain passage law on the identified submanifold $B := B_1 \cong B_2$ is called a B-geodesic. Considering the variational problem with respect to arclength L of piecewise smooth curves through B, a critical point of L is a B-geodesic. A B-Jacobi field is a Jacobi field on each Riemannian manifold and satisfies certain passage condition on B. In this paper, we extend Rauch's theorem which gives a comparison of the lengths of Jacobi fields along geodesics in different Riemannian manifolds to B-Jacobi fields along B-geodesics in different glued Riemannian spaces.

0. Introduction

In Riemannian manifolds, various results have been given on geodesics by many authors. Recently, N. Innami studied a geodesic reflecting at a boundary point of a Riemannian manifold with boundary in [I1]. Let M be a Riemannian manifold with boundary B which is a union of smooth hypersurfaces. A curve on M is said to be a reflecting geodesic if it is a geodesic except at reflecting points and satisfies the reflection law. He dealt with the index form, conjugate points and so on, as in the case of a usual geodesic. Moreover, in [I2], he generalized these to the case of a glued Riemannian manifold which is a space obtained from Riemannian manifolds with boundary by identifying their isometric boundary hypersurfaces. Some collapsing Riemannian manifolds are considered to be a kind

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of glued Riemannian manifolds. In [T] the author gave the definition of a glued Riemannian space which is obtained from Riemannian manifolds by identifying their isometric submanifolds B_1 and B_2 and is a generalization of a glued Riemannian manifold. A curve on a glued Riemannian space which is a geodesic on each Riemannian manifold and satisfies certain passage law on the identified submanifold $B := B_1 \cong B_2$ was called a B-geodesic. Considering the variational problem with respect to arclength L of piecewise smooth curves through B, a critical point of L is a B-geodesic. Also, the definitions of the index form of B-geodesics, B-Jacobi fields and B-conjugate points were given. A B-Jacobi field is a Jacobi field on each Riemannian manifold and satisfies certain passage condition on B. The purpose of this paper is to generalize the Rauch comparison theorem to the case of glued Riemannian spaces. The Rauch comparison theorem yields a comparison of the lengths of Jacobi fields along geodesics in different Riemannian manifolds under suitable initial conditions and suitable hypotheses on the curvatures and on the nonexistence of conjugate points. In this paper, we show how Rauch's theorem extend to B-Jacobi fields satisfying the passage condition, which involves the passage endomorphism defined by using the shape operators of B in M_1 and M_2. So in the comparison theorem, we need an additional hypothesis comparing the passage endomorphisms. In Section 1, we review fundamental definitions and results ([T]) on a glued Riemannian space. In Section 2, we give a precise statement of a Rauch comparison theorem for B-Jacobi fields. Section 3 is devoted to the proof of this comparison theorem.

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1. Preliminaries

Let N_μ and M_λ be manifolds (possibly with boundary) for $\mu = 1, \dots, k$ and $\lambda = 1, \dots, l$. We allow the case where $\dim N_\mu \neq \dim N_\nu$ and $\dim M_\kappa \neq \dim M_\lambda$ for $\mu \neq \nu$ and $\kappa \neq \lambda$. A map $\bar{\varphi} : \overline{N} \to \overline{M}$ from the topological direct sum $\overline{N} := N_1 \coprod \cdots \coprod N_k$ to $\overline{M} := M_1 \coprod \cdots \coprod M_l$ is *smooth* if $\bar{\varphi}|N_\mu$ is smooth. A *tangent bundle* $T\overline{M}$ *of* $\overline{M}$ is the direct sum $T\overline{M} = TM_1 \coprod \cdots \coprod TM_l$, where TM_λ denotes the tangent bundle of M_λ. We note that a tangent bundle $T\overline{M}$ on $\overline{M}$ is not constant rank vector bundle on $\overline{M}$. We put $T_p\overline{M} := T_pM_\lambda$ for $p \in M_\lambda$. We define a map $\pi_{\overline{M}} : T\overline{M} \to \overline{M}$ by

$$\pi_{\overline{M}}(v_p) := p \quad \text{for } v_p \in T_pM_\lambda.$$

A *vector field* $\bar{V}$ *on* $\bar{M}$ is a map $\bar{V} : \bar{M} \to T\bar{M}$ such that $\pi_{\bar{M}} \circ \bar{V} = \mathrm{id}_{\bar{M}}$, where $\mathrm{id}_{\bar{M}}$ is the identity map on $\bar{M}$. If $\bar{V}|M_\lambda : M_\lambda \to TM_\lambda$ is smooth vector field on each M_λ, then $\bar{V}$ is smooth. Let I_μ be a closed interval in $\boldsymbol{R}$ which is a manifold with boundary, for $\mu = 1, \dots, k$. A map $\bar{\alpha} : \bar{I} := I_1 \coprod \cdots \coprod I_k \to \bar{M}$ is called a *curve on* $\bar{M}$ if $\bar{\alpha}$ is smooth.

Let M_λ be a manifold (possibly with boundary) with a submanifold B_λ for $\lambda = 1, 2$ and ψ a diffeomorphism from B_1 to B_2. A *glued space* $M = M_1 \cup_\psi M_2$ is defined as follows: M is the quotient topological space obtained from the topological direct sum $\bar{M} = M_1 \coprod M_2$ of M_1 and M_2 by identifying $p \in B_1$ with $\psi(p) \in B_2$. We allow the case where $B_1 = B_2 = \varnothing$, $M_1 = \varnothing$ or $M_2 = \varnothing$, where ψ is the empty map. Let $\pi : \bar{M} \to M$ be the natural projection which is defined by $\pi(p) = [p]$, where $[p]$ is the equivalence class of p. Let N_λ be a manifold with a submanifold C_λ $(\lambda = 1, 2)$, $\tau : C_1 \to C_2$ a diffeomorphism and $N = N_1 \cup_\tau N_2$ a glued space. A *glued smooth map* $\varphi : \bar{N} \to M$ *on* $\bar{N}$ *derived from a smooth map* $\bar{\varphi} : \bar{N} \to \bar{M}$ or, simply, *a smooth map on* N is defined by $\varphi = \pi \circ \bar{\varphi}$. We note that a glued smooth map on $\bar{N}$ is considered as a map on N which, possibly, take two values at $[p]$ $(p \in C_\lambda)$. A glued smooth map φ is *continuous* if $\varphi(p) = \varphi(\tau(p))$ holds for any $p \in C_1$.

A *glued tangent bundle* TM *of* M is the glued space $TM_1 \cup_{\psi_*} TM_2$, where $\psi_* : TB_1 \to TB_2$ is the differential map of ψ. Let $\hat{\pi} : T\bar{M} \to TM$ be the natural projection which is defined by $\hat{\pi}(v) = [v]$, where $[v]$ is the equivalence class of v. For $p \in \bar{M}$, we set $T_pM := \hat{\pi}(T_p\bar{M}) = \{[v] \in TM \mid v \in T_p\bar{M}\}$. We define a map $\pi_M : TM \to M$ by

$$\pi_M([v_p]) := [p] \quad \text{for } v_p \in T_p\bar{M}.$$

We note that $\pi \circ \pi_{\bar{M}} = \pi_M \circ \hat{\pi}$ holds. A *glued vector field* $V : \bar{M} \to TM$ *on* $\bar{M}$ *derived from a vector field* $\bar{V}$ *on* $\bar{M}$ or, simply, *a vector field on* M is defined by $V = \hat{\pi} \circ \bar{V}$. A glued vector field V is called a *smooth glued vector field* provided V is glued smooth. If a glued vector field V on $\bar{M}$ is continuous, then we can regard it as a cross section of TM over M; that is $\pi_M \circ V = \mathrm{id}_M$. Similarly, we can define a *glued vector field* (or *vector field*) *along a curve* $\bar{\alpha} : \bar{I} := I_1 \coprod I_2 \to \bar{M}$.

Let $T_p^*\bar{M}$ be the dual vector space of $T_p\bar{M}$. We put $T^*\bar{M} = T^*M_1 \coprod T^*M_2$, where T^*M_λ is the cotangent bundle of M_λ. For $\bar{\theta}_p$ $(\in T_p^*\bar{M})$, $\bar{\omega}_q$ $(\in T_q^*\bar{M}) \in T^*\bar{M}$, we define an equivalence relation $\sim$ as follows: $\bar{\theta}_p \sim \bar{\omega}_q$ if and only if $\bar{\theta}_p = \bar{\omega}_q$ $(p = q)$ or $\bar{\theta}_p|_{T_pB_1} = \psi^*(\bar{\omega}_q)$ $(p \in B_1, q = \psi(p))$ or $\bar{\omega}_q|_{T_qB_1} = \psi^*(\bar{\theta}_p)$ $(q \in B_1, p = \psi(q))$, where ψ^* is the dual map of ψ_*. The quotient space obtained from $T^*\bar{M}$ by this equivalence relation is denoted by T^*M. Let $\hat{\pi} : T^*\bar{M} \to T^*M$ be the natural projection, that is, $\hat{\pi}(\bar{\theta}) := [\bar{\theta}]$, where $[\bar{\theta}]$ is the equivalence class of $\bar{\theta}$. For $p \in \overline{M}$, we set $T_p^*M := \hat{\pi}(T_p^*\overline{M})$ and define a map $[\bar{\theta}] : T_pM \to \boldsymbol{R}$ by $[\bar{\theta}]([\bar{v}]) := \bar{\theta}(\bar{v})$ for $\bar{\theta} \in T_p^*\overline{M}$ and $\bar{v} \in T_p\overline{M}$. Then we can regard T_p^*M as the dual of T_pM. We put $T^{r,s}(\overline{M}) := T^{r,s}(M_1) \coprod T^{r,s}(M_2)$, where $T^{r,s}(M_\lambda)$ is the (r,s)-tensor bundle of M_λ. An *(r,s)-tensor field on $\overline{M}$* is a cross section of $T^{r,s}(\overline{M})$. The definition of the *smoothness* of a tensor field on $\overline{M}$ is similar to that of a vector field on $\overline{M}$. Similarly, we can define the equivalence relation on $T^{r,s}(\overline{M})$ induced from those on $T\overline{M}$ and $T^*\overline{M}$, and denote the quotient space by $T^{r,s}(M)$. Let $\hat{\pi} : T^{r,s}(\overline{M}) \to T^{r,s}(M)$ be the natural projection. A *glued tensor field T derived from a tensor field $\overline{T}$ on $\overline{M}$* is defined by $T = \hat{\pi} \circ \overline{T}$. A glued tensor field T derived from a tensor field $\overline{T}$ on $\overline{M}$ is (*glued*) *smooth* if $\overline{T}$ is smooth.

DEFINITION 1.1. Let (M_λ, g_λ) be a Riemannian manifold with a Riemannian submanifold B_λ for $\lambda = 1, 2$ and ψ an isometry from B_1 to B_2. Let $\bar{g}$ be the *metric on $\overline{M}$* which is defined to be $\bar{g}_p = (g_\lambda)_p$ for $p \in M_\lambda$. A *glued Riemannian space* $(M, g) = (M_1, g_1) \cup_\psi (M_2, g_2)$ is a pair of a glued space $M = M_1 \cup_\psi M_2$ and a *glued metric g on M* derived from $\bar{g}$ which is a glued tensor field derived from the $(0,2)$-tensor field $\bar{g}$.

We note that, for any glued smooth vector fields V and W on $\overline{M}$ derived from smooth vector fields $\overline{V}$ and $\overline{W}$ on $\overline{M}$, respectively, a map $g(V, W) : \overline{M} \to \boldsymbol{R}$ defined by

$$g(V, W)(p) := \bar{g}(\overline{V}_p, \overline{W}_p)$$

is glued smooth on $\overline{M}$ derived from a smooth map $\bar{g}(\overline{V}, \overline{W}) : \overline{M} \to \boldsymbol{R}$.

From now on, identifying B_1 with B_2 by ψ, we put $B := B_1 \cong B_2$ and $T_pB := T_pB_1 \cong T_pB_2$ for $p \in B$ and omit the symbol $[\cdot]$ of the equivalence class. In particular, $[M_\lambda] := \pi(M_\lambda)$ will be denoted by M_λ. We call a map $\alpha : [a, t_0] \coprod [t_0, b] \to M$ a *glued curve derived from a curve* $\bar{\alpha} : [a, t_0] \coprod [t_0, b] \to \overline{M}$ or, simply, *a curve on M* if $\alpha : [a, t_0] \coprod [t_0, b] \to M$ is a continuous glued smooth map derived from $\bar{\alpha}$. Let $\alpha : [a, t_0] \coprod [t_0, b] \to M$ be a glued curve derived from a curve $\bar{\alpha} : [a, t_0] \coprod [t_0, b] \to \overline{M}$. The (*glued*) *velocity vector field of* α is $\alpha' := \hat{\pi} \circ \bar{\alpha}'$. We put $\alpha'(t_0 - 0) := \hat{\pi} \circ \bar{\alpha}_1'(t_0)$ and $\alpha'(t_0 + 0) := \hat{\pi} \circ \bar{\alpha}_2'(t_0)$, where $\bar{\alpha}_1 := \bar{\alpha} \,|\, [a, t_0] : [a, t_0] \to \overline{M}$ and $\bar{\alpha}_2 := \bar{\alpha} \,|\, [t_0, b] : [t_0, b] \to \overline{M}$. We note that a glued velocity vector field is considered as a glued vector field along $\bar{\alpha}$ and not generally continuous. We call $\alpha : [a, b] \to M$ a *piecewise smooth curve on M* provided there is a partition $a = a_0 < a_1 < \cdots < a_k < a_{k+1} = b$ of $[a, b]$ such that $\alpha \,|\, [a_{i-1}, a_{i+1}] : [a_{i-1}, a_i] \coprod [a_i, a_{i+1}] \to M$ is a glued curve. We call a_j $(j = 1, \dots, k)$ the *break*. A function $\lambda : [a, t_0] \coprod [t_0, b] \to \{1, 2\}$ is defined by

$$\lambda(t) := \begin{cases} 1 & \text{on } [a, t_0] \\ 2 & \text{on } [t_0, b] \end{cases}.$$

For simplicity, we put $\lambda := \lambda(t)$.

If M is a glued Riemannian space such that $(M, g) = (M_1, g_1) \cup_\psi (M_2, g_2)$, then let $\Omega_{t_0}(M_1, M_2; B) =: \Omega_{t_0}$ $(t_0 \in (a, b))$ be the set of all piecewise smooth curves $\alpha : [a, b] \to M$ such that $\alpha(t_0) \in B$, $\alpha([a, t_0]) \subset M_1$ and $\alpha([t_0, b]) \subset M_2$. The projection from $T_p M_\lambda$ to $T_p B$ is denoted by tan. Let D^λ be Levi-Civita connection of Riemannian manifold M_λ $(\lambda = 1, 2)$. A curve $\alpha \in \Omega_{t_0}$ is a *B-geodesic* if α satisfies the following conditions:

(1.1) $\alpha \,|\, [a, t_0]$ and $\alpha \,|\, [t_0, b]$ are geodesics, that is $D^\lambda_{\alpha'} \alpha' = 0$,

on M_1 and M_2, respectively,

(1.2) $$\tan \alpha'(t_0 - 0) = \tan \alpha'(t_0 + 0),$$

(1.3) $$g_1(\alpha'(t_0 - 0), \alpha'(t_0 - 0)) = g_2(\alpha'(t_0 + 0), \alpha'(t_0 + 0)).$$

We assume that geodesics and B-geodesics are parametrized by arclength.

Let $q \in B$, $u \in T_q M_1$ and $v \in T_q M_2$ with $\|u\|_1 = \|v\|_2$, $\tan u = \tan v$ and $v \notin T_q B$. We define a linear map $Q_{u,v} : T_q B \oplus \mathrm{Span}\{\mathrm{nor}_1\, u\} \to T_q B \oplus \mathrm{Span}\{\mathrm{nor}_2\, v\}$ as

$$Q_{u,v}(w) = \left\{ w - \frac{g_1(w, \mathrm{nor}_1\, u)}{g_1(u, \mathrm{nor}_1\, u)} \mathrm{nor}_1\, u \right\} + \frac{g_1(w, \mathrm{nor}_1\, u)}{g_1(u, \mathrm{nor}_1\, u)} \mathrm{nor}_2\, v$$

for any $w \in T_q B \oplus \mathrm{Span}\{\mathrm{nor}_1\, u\}$, where $\mathrm{nor}_\lambda : T_q M_\lambda \to T_q B^\perp$ is the projection. The following hold:

$$Q_{u,v}(x) = x \quad \text{for any } x \in T_q B.$$

$$Q_{u,v}(\mathrm{nor}_1\, u) = \mathrm{nor}_2\, v.$$

$$g_2(Q_{u,v}(w), x) = g_1(w, x)$$

for any $x \in T_q B$ and $w \in T_q B \oplus \mathrm{Span}\{\mathrm{nor}_1\, u\}$.

$$g_2(Q_{u,v}(w), Q_{u,v}(w)) = g_1(w, w)$$

for any $w \in T_q B \oplus \mathrm{Span}\{\mathrm{nor}_1\, u\}$. Let $\gamma \in \Omega_{t_0}$ be a B-geodesic with $\gamma'(t_0 + 0) \notin T_{\gamma(t_0)} B$. Then we have

$$Q_{\gamma'(t_0 - 0), \gamma'(t_0 + 0)}(\gamma'(t_0 - 0)) = \gamma'(t_0 + 0).$$

REMARK. Let $q \in B$, $u \in T_qM_1$ and $v \in T_qM_2$ with $\|u\|_1 = \|v\|_2$, $\tan u = \tan v$ and $v \notin T_qB$. If we define a linear map $Q_{v,u} : T_qB \oplus \mathrm{Span}\{\mathrm{nor}_2\, v\} \to T_qB \oplus \mathrm{Span}\{\mathrm{nor}_1\, u\}$ as

$$Q_{v,u}(z) = \left\{ z - \frac{g_2(z, \mathrm{nor}_2\, v)}{g_2(v, \mathrm{nor}_2\, v)} \mathrm{nor}_2\, v \right\} + \frac{g_2(z, \mathrm{nor}_2\, v)}{g_2(v, \mathrm{nor}_2\, v)} \mathrm{nor}_1\, u$$

for any $z \in T_qB \oplus \mathrm{Span}\{\mathrm{nor}_2\, v\}$. The following hold:

$$Q_{u,v} \circ Q_{v,u} = \mathrm{id}, \quad Q_{v,u} \circ Q_{u,v} = \mathrm{id},$$

$$g_2(Q_{u,v}(w), z) = g_1(w, Q_{v,u}(z))$$

for $w \in T_qB \oplus \mathrm{Span}\{\mathrm{nor}_1\, u\}$ and $z \in T_qB \oplus \mathrm{Span}\{\mathrm{nor}_2\, v\}$.

If $\gamma \in \Omega_{t_0}$ is a B-geodesic with $\gamma'(t_0 + 0) \notin T_{\gamma(t_0)}B$, the set $T_\gamma\Omega_{t_0}$ consists of all vector fields Y along γ which satisfy the following condition:

$$Q_{\gamma'(t_0-0),\gamma'(t_0+0)}(Y(t_0 - 0)) = Y(t_0 + 0). \tag{1.4}$$

Let p and q be points of M_1 and M_2 such that $\gamma(a) = p$ and $\gamma(b) = q$. A subspace $T_\gamma\Omega_{t_0}(p,q)$ in $T_\gamma\Omega_{t_0}$ is defined by

$$T_\gamma\Omega_{t_0}(p,q) := \{Y \in T_\gamma\Omega_{t_0} \mid Y(a) = 0, Y(b) = 0\}.$$

For $\lambda = 1, 2$, let R^λ be the Riemannian curvature tensor of a Riemannian manifold M_λ defined as

$$R^\lambda(X, Y)W := D^\lambda_X D^\lambda_Y W - D^\lambda_Y D^\lambda_X W - D^\lambda_{[X,Y]} W,$$

for any vector field X, Y and W on M_λ, and S^λ_Z the shape operator of $B \subset M_\lambda$ defined as

$$S^\lambda_Z(V) := -\tan D^\lambda_V Z,$$

for any vector field V tangent to B and Z normal to B. Especially, if $B = \{p\}$, we have that $S^\lambda_Z = 0$ for $Z \in T_pM_\lambda$. A vector field Y along a piecewise smooth curve $\alpha : [a,b] \to M$ is a *tangent to* α if $Y = f\alpha'$ for some function f on $[a,b]$ and *perpendicular to* α if $g_\lambda(Y, \alpha') = 0$. If $\|\alpha'\|_\lambda \neq 0$, then each tangent space $T_{\alpha(t)}M_\lambda$ has a direct sum decomposition $\mathbf{R}\alpha' + \{\alpha'\}^\perp$. Hence each vector field Y along α has a unique expression $Y = Y^T + Y^\perp$, where Y^T is tangent to α and $Y^\perp$ is perpendicular to α, that is,

$$Y^{\perp} = Y - \frac{g_\lambda(Y, \alpha')}{g_\lambda(\alpha', \alpha')}\alpha'.$$

If α is a B-geodesic, then $(Y^T)' = (Y')^T$ and $(Y^{\perp})' = (Y')^{\perp}$.

Let $q \in B$ and $v \in T_qM_\lambda$ $(\lambda = 1, 2)$ is not tangent to B. A linear operator $P_\lambda^v : T_qB \oplus \mathrm{Span}\{\mathrm{nor}_\lambda\, v\} \to T_qB$ is defined by

$$P_\lambda^v(w) := w - \frac{g_\lambda(w, \mathrm{nor}_\lambda\, v)}{g_\lambda(v, \mathrm{nor}_\lambda\, v)} v$$

for any $w \in T_qB \oplus \mathrm{Span}\{\mathrm{nor}_\lambda\, v\}$ $(\subset T_qM_\lambda)$. We note that P_λ^v is surjective and $P_\lambda^v(v) = 0$.

Let $q \in B$, $u \in T_qM_1$ and $v \in T_qM_2$ with $\|u\|_1 = \|v\|_2$, $\tan u = \tan v$ and $v \notin T_qB$. We define a symmetric linear map $A_{u,v} : T_qB \oplus \mathrm{Span}\{\mathrm{nor}_2\, v\} \to T_qB \oplus \mathrm{Span}\{\mathrm{nor}_2\, v\}$ as

$$A_{u,v}(w) = (S^1_{\mathrm{nor}_1 u} - S^2_{\mathrm{nor}_2 v})(P_2^v(w)) - \frac{g_2((S^1_{\mathrm{nor}_1 u} - S^2_{\mathrm{nor}_2 v})(P_2^v(w)), v)}{g_2(v, \mathrm{nor}_2\, v)} \mathrm{nor}_2\, v$$

for any $w \in T_qB \oplus \mathrm{Span}\{\mathrm{nor}_2\, v\}$. We call this map $A_{u,v}$ a *passage endomorphism*. The following hold:

$$A_{u,v}(w) \perp v \quad \text{and} \quad A_{u,v}(v) = 0.$$

The *index form* $I_\gamma : T_\gamma\Omega_{t_0} \times T_\gamma\Omega_{t_0} \to \boldsymbol{R}$ of a B-geodesic $\gamma \in \Omega_{t_0}$ is the symmetric bilinear form defined as

$$\begin{aligned} I_\gamma(Y, W) = &\int_a^{t_0} \{g_1(Y^{\perp\prime}, W^{\perp\prime}) - g_1(R^1(Y, \gamma')\gamma', W)\}\, dt \\ &+ \int_{t_0}^b \{g_2(Y^{\perp\prime}, W^{\perp\prime}) - g_2(R^2(Y, \gamma')\gamma', W)\}\, dt \\ &+ g_2(A_{\gamma'(t_0-0), \gamma'(t_0+0)}(Y(t_0+0)), W(t_0+0)), \end{aligned}$$

for all $Y, W \in T_\gamma\Omega_{t_0}$. It follows that

$$I_\gamma(Y, W) = I_\gamma(Y^{\perp}, W^{\perp}) \quad \text{for all } Y, W \in T_\gamma\Omega_{t_0}.$$

Thus there is no loss of information in restricting the index form I_γ to

$$T_\gamma^{\perp}\Omega_{t_0} := \{Y \in T_\gamma\Omega_{t_0} \mid Y \perp \gamma'\}.$$

We write $I_\gamma^{\perp}$ for this restriction. We put

$$T_\gamma^\perp \Omega_{t_0}(\gamma(a), \gamma(b)) := \{ Y \in T_\gamma \Omega_{t_0}(\gamma(a), \gamma(b)) \mid Y \perp \gamma' \}$$

and write $I_\gamma^{0,\perp}$ for the restriction of the index form I_γ to this.

Let $\mathrm{pr}^1 : T_{\gamma(t_0)}M_1 \to T_{\gamma(t_0)}B \oplus \mathrm{Span}\{\mathrm{nor}_1\, \gamma'(t_0 - 0)\}$ and $\mathrm{pr}^2 : T_{\gamma(t_0)}M_2 \to T_{\gamma(t_0)}B \oplus \mathrm{Span}\{\mathrm{nor}_2\, \gamma'(t_0 + 0)\}$ be orthogonal projections. The following holds:

LEMMA 1.2 ([T1]). *Let $\gamma \in \Omega_{t_0}(p, q)$ be a B-geodesic with $\gamma'(t_0 + 0) \notin T_{\gamma(t_0)}B$. If Y and $W \in T_\gamma\Omega_{t_0}(p, q)$ have breaks $a_1 < \cdots < t_0 = a_j < \cdots < a_k$, then we have that*

$$\begin{aligned} I_\gamma(Y, W) = &-\left\{ \int_a^{t_0} g_1(Y^{\perp\prime\prime} + R^1(Y, \gamma')\gamma', W^\perp)\, dt \right. \\ &\left. + \int_{t_0}^b g_2(Y^{\perp\prime\prime} + R^2(Y, \gamma')\gamma', W^\perp)\, dt \right\} \\ &+ g_2(A_{\gamma'(t_0-0), \gamma'(t_0+0)}(Y(t_0 + 0)), W(t_0 + 0)) \\ &+ g_1(\mathrm{pr}^1(Y^{\perp\prime}(t_0 - 0)), W^\perp(t_0 - 0)) \\ &- g_2(\mathrm{pr}^2(Y^{\perp\prime}(t_0 + 0)), W^\perp(t_0 + 0)) \\ &+ \sum_{i=1}^{j-1} g_1(\Delta_{a_i} Y^{\perp\prime}, W^\perp(a_i)) + \sum_{i=j+1}^{k} g_2(\Delta_{a_i} Y^{\perp\prime}, W^\perp(a_i)) \\ &+ g_2(Y^{\perp\prime}(b), W^\perp(b)) - g_1(Y^{\perp\prime}(a), W^\perp(a)). \end{aligned}$$

Let $\gamma \in \Omega_{t_0}$ be a B-geodesic with $\gamma'(t_0 + 0) \notin T_{\gamma(t_0)}B$. If $Y \in T_\gamma\Omega_{t_0}$ satisfies

$$Y'' + R^\lambda(Y, \gamma')\gamma' = 0 \quad \text{on } M_\lambda \ (\lambda = 1, 2), \tag{1.5}$$

$$\begin{aligned} &-A_{\gamma'(t_0-0), \gamma'(t_0+0)}(Y(t_0 + 0)) \\ &\quad = Q_{\gamma'(t_0-0), \gamma'(t_0+0)}(\mathrm{pr}^1(Y'(t_0 - 0))) - \mathrm{pr}^2(Y'(t_0 + 0)), \end{aligned} \tag{1.6}$$

and

$$g_1(Y'(t_0 - 0), \gamma'(t_0 - 0)) = g_2(Y'(t_0 + 0), \gamma'(t_0 + 0)), \tag{1.7}$$

then Y is called a *B-Jacobi field along* γ. Let $\mathscr{J}_\gamma$ be the set of all B-Jacobi fields along γ. A B-Jacobi field Y along γ is *perpendicular* if Y is perpendicular to γ. Let $\mathscr{J}_\gamma^\perp$ be the set of all perpendicular B-Jacobi fields along γ. Let $\mathscr{J}_\gamma^0$ be the set of all B-Jacobi field such that $Y(a) = 0$.

If Y is a B-Jacobi field along γ, then we have that

$$I_\gamma(Y, Y) = g_2(Y^{\perp\prime}(b), Y^{\perp}(b)) - g_1(Y^{\perp\prime}(a), Y^{\perp}(a)). \tag{1.8}$$

LEMMA 1.3 ([T]). *Let $\gamma \in \Omega_{t_0}$ be a B-geodesic with $\gamma'(t_0 + 0) \notin T_{\gamma(t_0)}B$. Then $Y \in T_\gamma^\perp \Omega_{t_0}(\gamma(a), \gamma(b))$ is an element of the nullspace of $I_\gamma^{0,\perp}$ if and only if Y is a B-Jacobi field along γ.*

Let $\gamma \in \Omega_{t_0}$ be a B-geodesic with $\gamma'(t_0 + 0) \notin T_{\gamma(t_0)}B$, and Y a B-Jacobi field. We say that Y is *strong* if it holds

$$\mathrm{nor}_1\, Y'(t_0 - 0) = \frac{g_1(Y'(t_0 - 0), \mathrm{nor}_1\, \gamma'(t_0 - 0))}{g_1(\gamma'(t_0 - 0), \mathrm{nor}_1\, \gamma'(t_0 - 0))}\, \mathrm{nor}_1\, \gamma'(t_0 - 0),$$

and

$$\mathrm{nor}_2\, Y'(t_0 + 0) = \frac{g_2(Y'(t_0 + 0), \mathrm{nor}_2\, \gamma'(t_0 + 0))}{g_2(\gamma'(t_0 + 0), \mathrm{nor}_2\, \gamma'(t_0 + 0))}\, \mathrm{nor}_2\, \gamma'(t_0 + 0).$$

Let $\mathscr{J}_\gamma^{st}$ be the set of all strong B-Jacobi fields. $\mathscr{J}_\gamma^{st}$ forms a real vector space. We note that if $\dim M_1 = \dim M_2 = \dim B + 1$, then all B-Jacobi fields are strong.

LEMMA 1.4 ([T]). *Let $\gamma \in \Omega_{t_0}$ be a B-geodesic with $\gamma'(t_0 + 0) \notin T_{\gamma(t_0)}B$, and Y a B-Jacobi field. Then there exist a strong B-Jacobi field W with $W(t_0 - 0) = Y(t_0 - 0)$, $\tan W'(t_0 - 0) = \tan Y'(t_0 - 0)$ and $g_1(W'(t_0 - 0), \mathrm{nor}_1\, \gamma'(t_0 - 0)) = g_1(Y'(t_0 - 0), \mathrm{nor}_1\, \gamma'(t_0 - 0))$, and a B-Jacobi field V with $V(t_0 - 0) = 0$ such that $Y(t) = W(t) + V(t)$. And this decomposition is unique.*

Lemma 1.4 gives the direct sum decomposition

$$\mathscr{J}_\gamma = \mathscr{J}_\gamma^{st} + \mathscr{J}_\gamma^{M_1, M_2}.$$

Elements of $\mathscr{J}_\gamma^{M_1, M_2}$ are called (M_1, M_2)-*Jacobi fields*. Then we have that

$$\mathscr{J}_\gamma^{M_1, M_2} = \mathscr{J}_\gamma^{M_1} + \mathscr{J}_\gamma^{M_2},$$

where $\mathscr{J}_\gamma^{M_\lambda}$ is the set of all (M_1, M_2)-Jacobi fields which is identically zero on M_μ $(\lambda \neq \mu)$. The resulting projections $\mathrm{pr}_{st} : \mathscr{J}_\gamma \to \mathscr{J}_\gamma^{st}$ and $\mathrm{pr}_{(M_1, M_2)} : \mathscr{J}_\gamma \to \mathscr{J}_\gamma^{M_1, M_2}$ are obviously $\boldsymbol{R}$-linear. For $Y \in \mathscr{J}_\gamma$, we put $\mathrm{pr}_{st}(Y) =: Y^1$ and $\mathrm{pr}_{(M_1, M_2)}(Y) =: Y^2$.

Let $\gamma \in \Omega_{t_0}$ be a B-geodesic with $\gamma'(t_0 + 0) \notin T_{\gamma(t_0)}B$. We say that $\gamma(t_2)$ $(t_2 \in (a, b])$ is a *B-conjugate point to* $\gamma(t_1)$ $(t_1 \in [a, b), t_1 < t_2)$ *along* γ if there exists a B-Jacobi field Y along γ such that $Y(t_1) = 0$, $Y(t_2) = 0$ and $Y \mid [t_1, t_2]$ is nontrivial.

LEMMA 1.5 ([T]). *Let $\gamma \in \Omega_{t_0}$ be a B-geodesic with $\gamma'(t_0 + 0) \notin T_{\gamma(t_0)}B$. We assume that $\gamma(t_0)$ and $\gamma(b)$ are not B-conjugate points to $\gamma(a)$. Then, for any $v_1 \in T_{\gamma(a)}M_1$ and $v_2 \in T_{\gamma(b)}M_2$, there is a unique $Y \in \mathscr{J}_\gamma$ with $Y(a) = v_1$ and $Y(b) = v_2$.*

LEMMA 1.6 ([T]). *Let $\gamma \in \Omega_{t_0}$ be a B-geodesic with $\gamma'(t_0 + 0) \notin T_{\gamma(t_0)}B$. If $\gamma(t_1)$ $(t_1 \in (t_0, b])$ is not a B-conjugate point to $\gamma(a)$ and also $\gamma(t_1)$ $(t_1 \in (a, t_0])$ is not a conjugate point to $\gamma(a)$, then, for any $Y \in T_\gamma\Omega_{t_0}$ with $Y(a) = 0$, there exist a unique B-Jacobi field $J \in \mathscr{J}_\gamma^0$ such that $J(b) = Y(b)$ and*

$$I_\gamma(J, J) \leq I_\gamma(Y, Y).$$

In particular, the equality holds if and only if $J^\perp = Y^\perp$.

LEMMA 1.7 ([T]). *Let $\gamma \in \Omega_{t_0}$ be a B-geodesic with $\gamma'(t_0 + 0) \notin T_{\gamma(t_0)}B$. The following are equivalent:*

(1) *$\gamma(t_1)$ $(t_1 \in (t_0, b])$ is not a B-conjugate point to $\gamma(a)$ and also $\gamma(t_1)$ $(t_1 \in (a, t_0])$ is not a conjugate point to $\gamma(a)$.*

(2) *$I_\gamma^{0,\perp}$ is positive definite.*

We define the function $\rho_K : [a, b] \to \boldsymbol{R}$ and $f_K : [a, b] \to \boldsymbol{R}$ by

$$\rho_K(t) = \begin{cases} t & \text{if } K = 0 \\ \dfrac{1}{\sqrt{K}} \tan \sqrt{K}t & \text{if } K > 0 \\ \dfrac{1}{\sqrt{-K}} \tanh \sqrt{-K}t & \text{if } K < 0 \end{cases}$$

and

$$f_K(t) = \begin{cases} t & \text{if } K = 0 \\ \dfrac{1}{\sqrt{K}} \sin \sqrt{K}t & \text{if } K > 0 \\ \dfrac{1}{\sqrt{-K}} \sinh \sqrt{-K}t & \text{if } K < 0 \end{cases},$$

respectively. We put $\Gamma_2(\gamma') := T_{\gamma(t_0)}B \oplus \mathrm{Span}\{\mathrm{nor}_2\, \gamma'(t_0 + 0)\}$, $\Gamma_2^\perp(\gamma') := \{v \in \Gamma_2(\gamma') \mid g_2(v, \gamma'(t_0 + 0)) = 0\}$ and $A := A_{\gamma'(t_0 - 0), \gamma'(t_0 + 0)} \mid \Gamma_2^\perp(\gamma')$.

LEMMA 1.8 (The passage equation [T]). *Let M_1 and M_2 be Riemannian manifolds of constant curvature K_1 and K_2, respectively. Let $\gamma : [a,b] \to M$ be a B-geodesic with $\gamma(t_0) \in B$ and $\gamma'(t_0+0) \notin T_{\gamma'(t_0)}B$. If $\gamma(b)$ is the first B-conjugate point to $\gamma(a)$, then we get that $f_{K_1}(t-a) > 0$ for $t \in (a, t_0]$ and*

$$\lambda_A = \frac{-1}{\rho_{K_1}(t_0 - a)} + \frac{-1}{\rho_{K_2}(b - t_0)} \quad \textit{when } f_{K_2}(b - t_0) > 0,$$

where λ_A is a minimal eigenvalue of A.

2. Comparison Theorems on Glued Riemannian Spaces

Let (M_λ, g_λ) (resp. $(\overline{M}_\lambda, \bar{g}_\lambda)$) be Riemannian manifold with Riemannian submanifold B_λ (resp. $\bar{B}_\lambda$) for $\lambda = 1, 2$, and ψ (resp. $\bar{\psi}$) isometry from B_1 to B_2 (resp. $\bar{B}_1$ to $\bar{B}_2$). Let $(M, g) = (M_1, g_1) \cup_\psi (M_2, g_2)$ and $(\overline{M}, \bar{g}) = (\overline{M}_1, \bar{g}_1) \cup_{\bar{\psi}} (\overline{M}_2, \bar{g}_2)$ be glued Riemannian spaces. We put $B := B_1 \cong B_2$ and $\bar{B} := \bar{B}_1 \cong \bar{B}_2$ and assume that $\dim \bar{B} > 0$ if $\dim B > 0$. Let $\gamma : [a,b] \to M$ (resp. $\bar{\gamma} : [a,b] \to \overline{M}$) be a B-geodesic (resp. $\bar{B}$-geodesic) such that $\gamma(t_0) \in B$ (resp. $\bar{\gamma}(t_0) \in \bar{B}$) and $\gamma'(t_0+0) \notin T_{\gamma(t_0)}B$ (resp. $\bar{\gamma}'(t_0+0) \notin T_{\bar{\gamma}(t_0)}\bar{B}$). For $\lambda = 1, 2$, let R^λ (resp. $\bar{R}^\lambda$) be the Riemannian curvature tensor of Riemannian manifold M_λ (resp. $\overline{M}_\lambda$). We define operators $R_t^\lambda : \{\gamma'(t)\}^\perp \to \{\gamma'(t)\}^\perp$ and $\bar{R}_t^\lambda : \{\bar{\gamma}'(t)\}^\perp \to \{\bar{\gamma}'(t)\}^\perp$ by

$$R_t^\lambda v = R^\lambda(v, \gamma'(t))\gamma'(t) \quad \text{for } v \in \{\gamma'(t)\}^\perp$$

and

$$\bar{R}_t^\lambda \bar{v} = \bar{R}^\lambda(\bar{v}, \bar{\gamma}'(t))\bar{\gamma}'(t) \quad \text{for } \bar{v} \in \{\bar{\gamma}'(t)\}^\perp,$$

where $\{\gamma'(t)\}^\perp := \{v \in T_{\gamma(t)}M_\lambda \mid g_\lambda(v, \gamma'(t)) = 0\}$ and $\{\bar{\gamma}'(t)\}^\perp := \{\bar{v} \in T_{\bar{\gamma}(t)}\overline{M}_\lambda \mid \bar{g}_\lambda(\bar{v}, \bar{\gamma}'(t)) = 0\}$. Similarly, a bar is used to distinguish objects in $\overline{M}$ from the corresponding objects in M.

We assume that $\dim M_\lambda \geq 2$ and $\dim \overline{M}_\lambda \geq 2$. Then, the following assertion holds and is proved in Section 3:

THEOREM 2.1. *We assume that the following conditions hold:*
(1) *For any $t \in [a,b]$,*

$$(\text{the maximal eigenvalue of } R_t^\lambda) \leq (\text{the minimal eigenvalue of } \bar{R}_t^\lambda)$$

(2) *If* $\dim B > 0$, *then*

$$(\text{the minimal eigenvalue of } A) \geq (\text{the maximal eigenvalue of } \bar{A}),$$

where $A := A_{\gamma'(t_0-0), \gamma'(t_0+0)} \mid \Gamma_2^\perp(\gamma')$ and $\bar{A} := \bar{A}_{\bar{\gamma}'(t_0-0), \bar{\gamma}'(t_0+0)} \mid \bar{\Gamma}_2^\perp(\bar{\gamma}')$.

(3) $\bar{\gamma}(t)$ $(t \in (a, t_0])$ *is not a conjugate point to* $\bar{\gamma}(a)$ and also $\bar{\gamma}(t)$ $(t \in (t_0, b])$ *is not a* $\bar{B}$*-conjugate point to* $\bar{\gamma}(a)$.

Then $\gamma(t)$ $(t \in (a, t_0])$ *is not a conjugate point to* $\gamma(a)$ *and also* $\gamma(t)$ $(t \in (t_0, b])$ *is not a B-conjugate point to* $\gamma(a)$. *Moreover, if a perpendicular B-Jacobi field J with* $J(a) = 0$ *and a perpendicular strong* $\bar{B}$*-Jacobi field* $\bar{J}$ *with* $\bar{J}(a) = 0$ *satisfy* $\|J'(a)\|_1 = \|\bar{J}'(a)\|_1$, *then*

$$\|J(t)\|_\lambda \geq \|\bar{J}(t)\|_\lambda \quad on \ [a, b].$$

In particular, if there is $d \in (a, b]$ *such that* $\|J(d)\|_\lambda = \|\bar{J}(d)\|_\lambda$, *then*

$$\|J(t)\|_\lambda = \|\bar{J}(t)\|_\lambda \quad on \ [a, d].$$

The condition that dim $\bar{B} > 0$ if dim $B > 0$ is necessary. We give an example which shows this:

EXAMPLE 1. Let $M = M_1 \cup_{id} M_2$ be a glued Riemannian space which consists of the following two surfaces in the Euclidean space E^3 and B a boundary (submanifold) of M_λ $(\lambda = 1, 2)$:

$$M_1 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1, y \geq 0\},$$

$$M_2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1, y \leq 0\},$$

$$B = \{(x, 0, z) \mid x^2 + z^2 = 1\},$$

and g_λ, $\lambda = 1, 2$, are Riemannian metrics induced from the natural Euclidean metric of E^3. We defined a B-geodesic $\gamma : [0, \pi] \to M$ by

$$\gamma(t) = (0, \cos t, \sin t).$$

Then $Y(t) = (\sin t) U_1$ is a B-Jacobi field along γ, where $U_1 := \partial/\partial x$, $U_2 := \partial/\partial y$ and $U_3 := \partial/\partial z$ is a natural frame field on E^3. Hence $\gamma(\pi)$ is a B-conjugate point to $\gamma(0)$.

Let $\bar{M} = \bar{M}_1 \cup_{id} \bar{M}_2$ be a glued Riemannian space which consists of the following two surfaces in the Euclidean space E^3 and $\bar{B}$ a submanifold of $\bar{M}_\lambda$ $(\lambda = 1, 2)$:

$$\bar{M}_1 = S^2(1) := \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\},$$

$$\bar{M}_2 = \{(x, y, z) \mid x^2 + (y + 2)^2 + z^2 = 1\},$$

$$\bar{B} = \{(0, -1, 0)\},$$

and $\bar{g}_\lambda$, $\lambda = 1, 2$, are Riemannian metrics induced from the natural Euclidean metric of $\boldsymbol{E}^3$. We defined a $\bar{B}$-geodesic $\bar{\gamma} : [0, \pi] \to M$ by

$$\bar{\gamma}(t) = \begin{cases} (0, \cos(t + \pi/2), \sin(t + \pi/2)) & \text{on } [0, \pi/2] \\ (0, \cos(t - \pi/2) - 2, \sin(t - \pi/2)) & \text{on } [\pi/2, \pi] \end{cases}.$$

Then, for any $t \in (0, \pi]$, $\bar{\gamma}(t)$ are not $\bar{B}$-conjugate points to $\bar{\gamma}(0)$.

THEOREM 2.2. *We assume that the conditions* (1), (2) *and* (3) *in Theorem* 2.1 *hold. If* $\dim \overline{M}_\lambda = \dim \bar{B} + 1$ $(\lambda = 1, 2)$ *and a perpendicular B-Jacobi field J with* $J(a) = 0$ *and a perpendicular $\bar{B}$-Jacobi field $\bar{J}$ with* $\bar{J}(a) = 0$ *satisfy* $\|J'(a)\|_1 = \|\bar{J}'(a)\|_1$, *then*

$$\|J(t)\|_\lambda \geq \|\bar{J}(t)\|_\lambda \quad \textit{on } [a, b].$$

In particular, if there is $d \in (a, b]$ *such that* $\|J(d)\|_\lambda = \|\bar{J}(d)\|_\lambda$, *then*

$$\|J(t)\|_\lambda = \|\bar{J}(t)\|_\lambda \quad \textit{on } [a, d].$$

PROOF. If $\dim \overline{M}_\lambda = \dim \bar{B} + 1$, then any $\bar{B}$-Jacobi fields are strong. Hence, by Theorem 2.1, the assertion holds. □

The condition that $\bar{J}$ is strong in Theorem 2.1 and $\dim \overline{M}_\lambda = \dim \bar{B} + 1$ $(\lambda = 1, 2)$ in Theorem 2.2 is necessary. We give an example which shows this:

EXAMPLE 2. Let $S^3(1)$ be the 3-sphere of constant curvature 1 and γ a geodesic on $S^3(1)$. Let $(e_1(t), e_2(t), \gamma'(t))$ be a parallel orthonormal frame along γ. Let τ be the geodesic through $\gamma(0)$ with $\tau'(0) = e_1(0)$. We put $M_\lambda := S^3(1)$ $(\lambda = 1, 2)$, $B := \{\tau(t) \mid t \in \boldsymbol{R}\}$, $\psi = \mathrm{id}_B$ and $M = M_1 \cup_\psi M_2$. Then $\gamma : [-\pi/2, \pi/2] \to M$ is a B-geodesic. Let $J(t) = (\cos t) e_1(t)$ and

$$\bar{J}(t) = \begin{cases} (\cos t) e_1(t) & \text{on } [-\pi/2, 0] \\ (\cos t) e_1(t) + (\sin t) e_2(t) & \text{on } [0, \pi/2] \end{cases}.$$

Then J and $\bar{J}$ are both perpendicular B-Jacobi fields along γ such that $J(-\pi/2) = 0$ and $\bar{J}(-\pi/2) = 0$. We set $a := -\pi/2$, $t_0 := 0$ and $b \in (t_0, \pi/2)$. $\|J'(-\pi/2)\| = \|\bar{J}'(-\pi/2)\|$, and (1), (2) and (3) in Theorem 2.1 hold trivially, but $\|J(t)\| < \|\bar{J}(t)\|$ for $t_0 < t < b$.

We say *the Jacobi equation splits along γ relative to B* if the curvature transformation $R_t^1 := R^1(\cdot, \gamma'(t))\gamma'(t)$ preserve the parallel translates of $T_{\gamma(t_0)}B \oplus \mathrm{Span}\{\mathrm{nor}_1\, \gamma'(t_0 - 0)\}$ along $\gamma \mid [a, t_0]$, and the curvature transformation $R_t^2 := R^2(\cdot, \gamma'(t))\gamma'(t)$ preserve the parallel translates of $T_{\gamma(t_0)}B \oplus \mathrm{Span}\{\mathrm{nor}_2\, \gamma'(t_0 + 0)\}$

along $\gamma \mid [t_0, b]$. We note that if $\dim M_1 = \dim M_2 = \dim B + 1$ or M_λ $(\lambda = 1, 2)$ has constant curvature, then the Jacobi equation always splits along γ relative to B. We say that $\gamma(t_2)$ $(t_2 \in [a, b])$ is a *strong B-conjugate point to* $\gamma(t_1)$ $(t_1 \in [a, b], t_1 \neq t_2)$ *along* γ provided there exists a nontrivial strong B-Jacobi field along γ which vanishes at t_1 and t_2.

The following assertions hold and are proved in Section 3:

LEMMA 2.3. *Suppose the Jacobi equation splits along γ relative to B. Let J be a B-Jacobi field and let $J = J^1 + J^2$ be the decomposition of Lemma* 1.4. *This decomposition is orthogonal, that is, $g_\lambda(J^1(t), J^2(t)) = 0$.*

THEOREM 2.4. *We assume that the Jacobi equation splits along $\bar{\gamma}$ relative to $\bar{B}$, the conditions* (1), (2) *in Theorem* 2.1 *hold and any $\bar{\gamma}(t)$ $(t \in (a, b])$ are not strong $\bar{B}$-conjugate points to $\bar{\gamma}(a)$. If a perpendicular B-Jacobi field J with $J(a) = 0$ and a perpendicular strong $\bar{B}$-Jacobi field $\bar{J}$ with $\bar{J}(a) = 0$ satisfy $\|J'(a)\|_1 = \|\bar{J}'(a)\|_1$, then*

$$\|J(t)\|_\lambda \geq \|\bar{J}(t)\|_\lambda \quad on \ [a, b].$$

In particular, if there is $d \in (a, b]$ such that $\|J(d)\|_\lambda = \|\bar{J}(d)\|_\lambda$, then

$$\|J(t)\|_\lambda = \|\bar{J}(t)\|_\lambda \quad on \ [a, d].$$

COROLLARY 2.5. *We assume that the Jacobi equation splits along $\bar{\gamma}$ relative to $\bar{B}$, the conditions* (1), (2) *in Theorem* 2.1 *hold and any $\bar{\gamma}(t)$ $(t \in (a, b])$ are not strong $\bar{B}$-conjugate points to $\bar{\gamma}(a)$. If a perpendicular B-Jacobi field J with $J(a) = 0$ and a perpendicular $\bar{B}$-Jacobi field $\bar{J}$ with $\bar{J}(a) = 0$ satisfy $\|J'(a)\|_1 = \|\bar{J}'(a)\|_1$, then*

$$\|J(t)\|_1 \geq \|\bar{J}(t)\|_1 \quad on \ [a, t_0]$$

and

$$\|J(t)\|_2 \geq \|\bar{J}(t)\|_2 - \|\bar{J}^2(t)\|_2 \quad on \ [t_0, b].$$

In particular, if equality occurs for some d, then equality holds on $[a, d]$.

PROOF. By Theorem 2.4, $\|J(t)\|_\lambda \geq \|\bar{J}^1(t)\|_\lambda$ for $t \in [a, b]$. But $\|\bar{J}^1(t)\|_\lambda = \|\bar{J}(t)\|_\lambda - \|\bar{J}^2(t)\|_\lambda$, since $\bar{g}_\lambda(\bar{J}^1(t), \bar{J}^2(t)) = 0$ along $\bar{\gamma}$. Moreover, since any $\bar{\gamma}(t)$ $(t \in (a, t_0])$ are not conjugate points, it holds that $\bar{J}^2(t) = 0$ for $t \in [a, t_0]$. □

Some of the most useful comparison spaces are the manifolds of constant curvature. By using Lemma 1.8 (passage equation) and Theorem 2.1, the following corollary holds:

COROLLARY 2.6. *Let a, t_0 and K_1 be any real numbers such that $a < t_0$ and $f_{K_1}(t-a) > 0$ for any $t \in (a, t_0]$ (f_{K_1} is defined in Section 1). Let δ and K_2 be any real numbers such that $K_1 = K_2$ if $\delta = 0$. Let b_1 ($>t_0$) be the smallest solution of*

$$\delta = \frac{-1}{\rho_{K_1}(t_0 - a)} + \frac{-1}{\rho_{K_2}(t - t_0)},$$

b_2 ($>t_0$) the smallest value which satisfies $f_{K_2}(t - t_0) = 0$ and $b := \min\{b_1, b_2\}$. Assume that $\dim B > 0$,

$$(\text{the maximal eigenvalue of } R_t^\lambda) \leq K_\lambda \quad \textit{for any } t \in [a, b],$$

and

$$(\text{the minimal eigenvalue of } A) \geq \delta,$$

where $A := A_{\gamma'(t_0-0),\gamma'(t_0+0)} \,|\, \Gamma_2^\perp(\gamma')$. Then there are no conjugate points along $\gamma \,|\, [a, t_0]$ and no B-conjugate points along $\gamma \,|\, [a, b)$ to $\gamma(a)$.

PROOF. If $\delta \neq 0$, we put $\delta_1 := (K_1 - K_2 - \delta^2)/(-2\delta)$ and $\delta_2 := (K_1 - K_2 + \delta^2)/(-2\delta)$. If $\delta = 0$, we put $\delta_1 = \delta_2$. Choose a complete Riemannian manifold $\overline{M}_\lambda$ ($\lambda = 1, 2$) of $\dim \geq 2$ and with constant curvature K_λ, $p_\lambda \in \overline{M}_\lambda$, a unit vector v_λ in $T_{p_\lambda}\overline{M}_\lambda$ and totally umbilic hypersurface $\tilde{B}_\lambda$ in $\overline{M}_\lambda$ through p_λ such that $\dim \overline{M}_1 = \dim \overline{M}_2$, $v_\lambda \in T_{p_\lambda}\tilde{B}_\lambda^\perp$ and all the eigenvalues of the shape operator $\bar{S}_{v_1}^1$ (resp. $\bar{S}_{v_2}^2$) are equal to δ_1 (resp. δ_2). Then $\tilde{B}_\lambda$ has constant curvature $K_\lambda + \delta_\lambda^2$ and $K_1 + \delta_1^2 = K_2 + \delta_2^2$. Hence there exists a neighborhood $\bar{B}_\lambda$ ($\subset \tilde{B}_\lambda$) of p_λ such that $\bar{B}_1$ is isometric to $\bar{B}_2$. Let $\bar{\psi} : \bar{B}_1 \to \bar{B}_2$ be the isometry such that $\bar{\psi}(p_1) = p_2$. Let $\bar{\gamma}$ be a $\bar{B}$-geodesic such that $\bar{\gamma}(t_0) = p_2$, $\bar{\gamma}'(t_0 - 0) = v_1$ and $\bar{\gamma}'(t_0 + 0) = v_2$. Since $\delta_1 - \delta_2 = \delta$, $\bar{A}_{v_1, v_2}(w) = \delta w$ for $w \in \bar{\Gamma}_2^\perp(\gamma')$. Comparing M with $\overline{M}$, assumptions (1) and (2) of Theorem 2.1 are hold. Also, by Lemma 1.8, (3) holds. □

3. Proofs of Comparison Theorems

Let V be a m-dimensional vector space ($m > 0$) with an inner product $\langle \cdot, \cdot \rangle$, $e_1, \ldots, e_m$ a basis of V, $\|w\|$ a norm of $w \in V$ and $R_t : V \to V$ ($t \in [a, b]$) a self-adjoint linear transformation such that $t \mapsto R_t$ is continuous. Let $\Theta(V)$

be the set of all piecewise smooth curves in V defined on $[a,b]$. For $Y(t) = \sum_{i=1}^{m} y^i(t)e_i \in \Theta(V)$, we put $Y'(t) = \sum_{i=1}^{m}(dy^i/dt)(t)e_i$. A curve $Y \in \Theta(V)$ that satisfies $Y''(t) + R_t Y(t) = 0$ is called a *(V, R_t)-Jacobi field.* A point t_2 of $[a,b]$ is a *(V, R_t)-conjugate point to t_1* $(\neq t_2)$, $t_1 \in [a,b]$, provided there is a nontrivial (V, R_t)-Jacobi field J with $J(t_1) = 0$ and $J(t_2) = 0$. We put $v \in V$ and assume that $R_t v = 0$ and $R_t w \in \{v\}^\perp$ for any $w \in V$, where $\{v\}^\perp := \{w \in V \mid \langle w, v\rangle = 0\}$. A (V, R_t)-Jacobi field Y is a *perpendicular (V, R_t)-Jacobi field* if Y is perpendicular to v, that is, $\langle Y(t), v\rangle = 0$. In general, when V is a vector space with an inner product and W is a subspace of V, let $\mathrm{pr}_{V,W} : V \to W$ be an orthogonal projection from V to W. We put $X^\perp := \mathrm{pr}_{V,\{v\}^\perp}(X)$ for $X \in \Theta(V)$. The *index form* $I : \Theta(V) \times \Theta(V) \to \boldsymbol{R}$ is defined by

$$I(X, Y) := I_a^b(X, Y) := \int_a^b \{\langle X^{\perp\prime}, Y^{\perp\prime}\rangle - \langle R_t X, Y\rangle\}\, dt,$$

for any $X, Y \in \Theta(V)$. For two triplex $\mathscr{V} := (V, v, R_t)$ and $\bar{\mathscr{V}} := (\bar{V}, \bar{v}, \bar{R}_t)$, we assume that if $\dim V = 1$, then $v = 0$, and if $\dim \bar{V} = 1$, then $\bar{v} = 0$. Then, the following comparison theorem is shown as usual:

Comparison theorem. We assume that the following conditions (1) and (2) hold:

(1) For any $t \in [a,b]$,

(the maximal eigenvalue of $R_t|\{v\}^\perp$) $\leq$ (the minimal eigenvalue of $\bar{R}_t|\{\bar{v}\}^\perp$).

(2) Any $t \in (a,b]$ are not $\bar{\mathscr{V}}$-conjugate points to a.

Then any $t \in (a,b]$ are not $\mathscr{V}$-conjugate points to a. Moreover, if a perpendicular $\mathscr{V}$-Jacobi field J with $J(a) = 0$ and a perpendicular $\bar{\mathscr{V}}$-Jacobi field $\bar{J}$ with $\bar{J}(a) = 0$ satisfy $\|J'(a)\| = \|\bar{J}'(a)\|$, then

$$\|J(t)\| \geq \|\bar{J}(t)\| \quad \text{on } [a,b].$$

In particular, if there is $d \in (a,b]$ such that $\|J(d)\| = \|\bar{J}(d)\|$, then

$$\|J(t)\| = \|\bar{J}(t)\| \quad \text{on } [a,d].$$

Let V_λ $(\lambda = 1,2)$ be a m_λ-dimensional vector space $(m_\lambda > 0)$ with an inner product $\langle\cdot,\cdot\rangle_\lambda$ and W_λ a subspace of V_λ such that there is a linear isometry $\psi : W_1 \to W_2$. Let $\|v\|_\lambda$ be the norm of $v \in V_\lambda$. We put $v_\lambda \in V_\lambda$ such that $\|v_1\|_1 = \|v_2\|_2$ and $\psi(\mathrm{pr}_{V_1,W_1}(v_1)) = \mathrm{pr}_{V_2,W_2}(v_2)$. Let $R_t^1 : V_1 \to V_1$ (resp. $R_t^2 : V_2 \to V_2$) be a self-adjoint linear transformation for any $t \in [a, t_0]$ (resp. $t \in [t_0, b]$). We assume that $t \mapsto R_t^\lambda$ is continuous, $R_t^\lambda v_\lambda = 0$ and $R_t^\lambda w \in \{v_\lambda\}^\perp$, where $\{v_\lambda\}^\perp :=$

$\{w \in V_\lambda \mid \langle w, v_\lambda \rangle_\lambda = 0\}$. We set $\Gamma(v_\lambda) := W_\lambda + \mathrm{Span}\{v_\lambda\} = W_\lambda \oplus Z_\lambda$, where Z_λ is an orthogonal complement of W_λ in $\Gamma(v_\lambda)$. Let $A : \Gamma(v_2) \to \Gamma(v_2)$ be a self-adjoint linear map such that $A(v_2) = 0$ and $A(w) \in \Gamma^\perp(v_2)$ for all $w \in \Gamma(v_2)$, where $\Gamma^\perp(v_2) := \{w \in \Gamma(v_2) \mid \langle w, v_2 \rangle_2 = 0\}$. For a ten-tuple $(V_1, V_2; W_1, W_2; \psi; v_1, v_2; R_t^1, R_t^2; A)$, we denote it by $\mathscr{V} = (V_1, V_2; W_1, W_2; \psi; v_1, v_2; R_t^1, R_t^2; A)$. Fix a ten-tuple $\mathscr{V} = (V_1, V_2; W_1, W_2; \psi; v_1, v_2; R_t^1, R_t^2; A)$. Let $Q := Q_{v_1, v_2} : \Gamma(v_1) \to \Gamma(v_2)$ be a linear map such that $Q|W_1 = \psi$ and $Q(v_1) = v_2$. Let $\Theta(V_1, V_2; W_1, W_2; \psi; v_1, v_2)$ be the set of all pair $Y := (Y_1, Y_2)$ such that $Y_1 : [a, t_0] \to V_1$ and $Y_2 : [t_0, b] \to V_2$ are piecewise smooth curves, $Y_1(t_0) \in \Gamma(v_1)$ and $Y_2(t_0) = Q(Y_1(t_0))$. A element Y of $\Theta(V_1, V_2; W_1, W_2; \psi; v_1, v_2)$ is called a *$\mathscr{V}$-Jacobi field* if it satisfies that

(3.1) $$Y_\lambda''(t) + R_t^\lambda Y_\lambda(t) = 0 \quad \text{for } t \in [a, b]$$

(3.2) $$A(Y_2(t_0)) = \mathrm{pr}_{V_2, \Gamma(v_2)}(Y_2'(t_0)) - Q(\mathrm{pr}_{V_1, \Gamma(v_1)}(Y_1'(t_0))),$$

and

(3.3) $$\langle Y_1'(t_0), v_1 \rangle_1 = \langle Y_2'(t_0), v_2 \rangle_2.$$

Let $Y = (Y_1, Y_2)$ be a $\mathscr{V}$-Jacobi field. We say that Y is *strong* if it holds

$$\mathrm{pr}_{V_\lambda, W_\lambda^\perp}(Y_\lambda'(t_0)) \in Z_\lambda \quad \text{for } \lambda = 1, 2.$$

Similarly to Lemma 1.4, we have the following. Let $Y = (Y_1, Y_2)$ be a $\mathscr{V}$-Jacobi field. Then there exist a strong $\mathscr{V}$-Jacobi field $X = (X_1, X_2)$ with $X_1(t_0) = Y_1(t_0)$, $\mathrm{pr}_{V_1, W_1}(X_1'(t_0)) = \mathrm{pr}_{V_1, W_1}(Y_1'(t_0))$ and $\mathrm{pr}_{V_1, Z_1}(X_1'(t_0)) = \mathrm{pr}_{V_1, Z_1}(Y_1'(t_0))$, and a $\mathscr{V}$-Jacobi field $J = (J_1, J_2)$ with $J_1(t_0) = 0$ such that $Y_\lambda(t) = X_\lambda(t) + J_\lambda(t)$. And this decomposition is unique.

For points t_1 and t_2 on $[a, b]$ with $t_1 < t_2$, t_2 is a *$\mathscr{V}$-conjugate point to t_1* provided there is a $\mathscr{V}$-Jacobi field J which satisfies one of the following conditions.

(1) If $t_1 \in [a, t_0)$ and $t_2 \in (t_0, b]$, then $J_1(t_1) = 0$, $J_2(t_2) = 0$ and, $J_1 \mid [t_1, t_0]$ or $J_2 \mid [t_0, t_2]$ are nontrivial.

(2) If $t_2 \in (a, t_0]$, then $J_1(t_1) = 0$, $J_1(t_2) = 0$ and $J_1 \mid [t_1, t_2]$ is nontrivial.

(3) If $t_1 \in [t_0, b]$, then $J_2(t_1) = 0$, $J_2(t_2) = 0$ and $J_2 \mid [t_1, t_2]$ is nontrivial.

For $w \in V_\lambda$, let $w^\perp$ be the perpendicular component to v_λ, that is,

$$w^\perp := \mathrm{pr}_{V_\lambda, \{v_\lambda\}^\perp}(w).$$

Jacobi fields are *perpendicular* provided they are equal to their perpendicular component to v_λ. The *index form* $I : \Theta(V_1, V_2; W_1, W_2; \psi; v_1, v_2) \times \Theta(V_1, V_2; W_1, W_2; \psi; v_1, v_2) \to \boldsymbol{R}$ is defined to be

$$I(X,Y) := I_a^b(X,Y) := I_a^{t_0}(X_1,Y_1) + I_{t_0}^b(X_2,Y_2) + \langle A(X_2(t_0)), Y_2(t_0)\rangle_2$$
$$= \int_a^{t_0} \{\langle X_1^{\perp\prime}, Y_1^{\perp\prime}\rangle_1 - \langle R_t^1 X_1, Y_1\rangle_1\}\, dt$$
$$+ \int_{t_0}^b \{\langle X_2^{\perp\prime}, Y_2^{\perp\prime}\rangle_2 - \langle R_t^2 X_2, Y_2\rangle_2\}\, dt + \langle A(X_2(t_0)), Y_2(t_0)\rangle_2,$$

for $X = (X_1, X_2)$, $Y = (Y_1, Y_2) \in \Theta(V_1, V_2; W_1, W_2; \psi; v_1, v_2)$. We put $Y^\perp := (Y_1^\perp, Y_2^\perp)$ for $Y = (Y_1, Y_2) \in \Theta(V_1, V_2; W_1, W_2; \psi; v_1, v_2)$ and $\Theta^\perp(V_1, V_2; W_1, W_2; \psi; v_1, v_2) := \{Y = (Y_1, Y_2) \in \Theta(V_1, V_2; W_1, W_2; \psi; v_1, v_2) \mid \langle Y_\lambda, v_\lambda\rangle_\lambda = 0\}$. It follows immediately that $I(X, Y) = I(X^\perp, Y^\perp)$ for all $X, Y \in \Theta(V_1, V_2; W_1, W_2; \psi; v_1, v_2)$. We set $I^\perp := I \mid \Theta^\perp(V_1, V_2; W_1, W_2; \psi; v_1, v_2)$. If $X = (X_1, X_2)$, $Y = (Y_1, Y_2) \in \Theta(V_1, V_2; W_1, W_2; \psi; v_1, v_2)$ and X is a $\mathscr{V}$-Jacobi field, then we have that

$$I(X,Y) = \langle X_2^{\perp\prime}(b), Y_2^\perp(b)\rangle_2 - \langle X_1^{\perp\prime}(a), Y_1^\perp(a)\rangle_1,$$

and, in particular,

$$I(X,X) = \langle X_2^{\perp\prime}(b), X_2^\perp(b)\rangle_2 - \langle X_1^{\perp\prime}(a), X_1^\perp(a)\rangle_1 \tag{3.4}$$
$$= \frac{1}{2}\{\langle X_2^\perp, X_2^\perp\rangle_2'(b) - \langle X_1^\perp, X_1^\perp\rangle_1'(a)\}.$$

We put

$$\Theta^0(V_1, V_2; W_1, W_2; \psi; v_1, v_2)$$
$$:= \{Y = (Y_1, Y_2) \in \Theta(V_1, V_2; W_1, W_2; \psi; v_1, v_2) \mid Y_1(a) = 0 \text{ and } Y_2(b) = 0\},$$
$$\Theta^{0,\perp}(V_1, V_2; W_1, W_2; \psi; v_1, v_2)$$
$$:= \Theta^\perp(V_1, V_2; W_1, W_2; \psi; v_1, v_2) \cap \Theta^0(V_1, V_2; W_1, W_2; \psi; v_1, v_2)$$

and

$$I^{0,\perp} := I \mid \Theta^{0,\perp}(V_1, V_2; W_1, W_2; \psi; v_1, v_2).$$

Then, similarly to Lemma 1.3, we have the following:

LEMMA 3.1. *$Y = (Y_1, Y_2) \in \Theta^{0,\perp}(V_1, V_2; W_1, W_2; \psi; v_1, v_2)$ is an element of the nullspace of $I^{0,\perp}$ if and only if Y is a $\mathscr{V}$-Jacobi field.*

Similarly to Lemma 1.6, we have the following:

LEMMA 3.2. *If any $t \in (a, t_0]$ are not (V_1, R_t^1)-conjugate points to a and also* *any* $t \in (t_0, b]$ *are not* $\mathscr{V}$*-conjugate points to* a*, then, for any* $Y = (Y_1, Y_2) \in \Theta(V_1, V_2; W_1, W_2; \psi; v_1, v_2)$ *with* $Y_1(a) = 0$*, there exist a unique* $\mathscr{V}$*-Jacobi field* $J = (J_1, J_2) \in \Theta(V_1, V_2; W_1, W_2; \psi; v_1, v_2)$ *such that* $J_1(a) = 0$*,* $J_2(b) = Y_2(b)$ *and*

$$I(J, J) \leq I(Y, Y).$$

In particular, the equality holds if and only if $J^\perp = Y^\perp$.

Similarly to Lemma 1.7, we have the following:

LEMMA 3.3. *The followings are equivalent.*

(1) *Any* $t \in (a, t_0]$ *are not* (V_1, R_t^1)*-conjugate points to* a *and also any* $t \in (t_0, b]$ *are not* $\mathscr{V}$*-conjugate points to* a.

(2) $I^{0,\perp}$ *is positive definite.*

For two ten-tuples $\mathscr{V} = (V_1, V_2; W_1, W_2; \psi; v_1, v_2; R_t^1, R_t^2; A)$ and $\bar{\mathscr{V}} = (\bar{V}_1, \bar{V}_2; \bar{W}_1, \bar{W}_2; \bar{\psi}; \bar{v}_1, \bar{v}_2; \bar{R}_t^1, \bar{R}_t^2; \bar{A})$, we assume that $\dim\{v_\lambda\}^\perp > 0$ and $\dim\{\bar{v}_\lambda\}^\perp > 0$ for $\lambda = 1, 2$. We put $m_\lambda := \dim V_\lambda$, $\bar{m}_\lambda := \dim \bar{V}_\lambda$, $n := \dim \Gamma^\perp(v_1)$ and $\bar{n} := \dim \bar{\Gamma}^\perp(\bar{v}_1)$. From now on, we shall assume that $\bar{n} > 0$ if $n > 0$, and, $\dim \bar{V}_\lambda \geq 2$ for $\lambda = 1, 2$. Then, by using Lemma 3.3, the following assertion holds:

LEMMA 3.4. *We assume that the following conditions hold.*

(1) *For any* $t \in [a, b]$,

$$\begin{aligned} &(\text{the maximal eigenvalue of } R_t^\lambda|\{v_\lambda\}^\perp) \\ &\qquad \leq (\text{the minimal eigenvalue of } \bar{R}_t^\lambda|\{\bar{v}_\lambda\}^\perp). \end{aligned}$$

(2) *If* $n > 0$*, then*

$$\begin{aligned} &(\text{the minimal eigenvalue of } A \mid \Gamma^\perp(v_2)) \\ &\qquad \geq (\text{the maximal eigenvalue of } \bar{A} \mid \bar{\Gamma}^\perp(\bar{v}_2)). \end{aligned}$$

(3) *Any* $t \in (a, t_0]$ *are not* $(\bar{V}_1, \bar{R}_t^1)$*-conjugate points to* a *and also any* $t \in (t_0, b]$ *are not* $\bar{\mathscr{V}}$*-conjugate points to* a.

Then any $t \in (a, t_0]$ *is not* (V_1, R_t^1)*-conjugate points to* a*. and also any* $t \in (t_0, b]$ *are not* $\mathscr{V}$*-conjugate points to* a.

PROOF. We will show that any $t \in (t_0, b]$ are not $\mathscr{V}$-conjugate points to a if any $t \in (a, t_0]$ is not $(\bar{V}_1, \bar{R}_t^1)$-conjugate points to a and also any $t \in (t_0, b]$ are not $\bar{\mathscr{V}}$-conjugate points to a.

1. *Case where $m_1 + 1 \leq \bar{m}_1$ and $m_2 + 1 \leq \bar{m}_2$ and $n \leq \bar{n}$.* We assume that a point $d \in (t_0, b]$ is a $\mathscr{V}$-conjugate point to a. Then there is a nontrivial $\mathscr{V}$-Jacobi field $\tilde{Y} := (\tilde{Y}_1, \tilde{Y}_2) \in \Theta^{\perp}(V_1, V_2; W_1, W_2; \psi; v_1, v_2)$ such that $\tilde{Y}_1(a) = 0$ and $\tilde{Y}_2(d) = 0$. We define $Y := (Y_1, Y_2) \in \Theta^{\perp}(V_1, V_2; W_1, W_2; \psi; v_1, v_2)$ by $Y_1(t) = \tilde{Y}_1(t)$ and

$$Y_2(t) = \begin{cases} \tilde{Y}_2(t), & t \in [t_0, d] \\ 0, & t \in [d, b] \end{cases}.$$

Since $I_a^d(\tilde{Y}, \tilde{Y}) = 0$ holds, we have

$$I(Y, Y) = I_a^d(Y, Y) + I_d^b(Y_2, Y_2) = 0.$$

Let $e_1^-, \ldots, e_{m_1}^-$ be an orthonormal basis of V_1 and $e_1^+, \ldots, e_{m_2}^+$ an orthonormal basis of V_2 such that $e_1^-, \ldots, e_n^-$ is an orthonormal basis of $\Gamma^{\perp}(v_1)$, $e_1^+, \ldots, e_n^+$ is an orthonormal basis of $\Gamma^{\perp}(v_2)$ and $Q(e_i^-) = e_i^+$ for $i = 1, \ldots, n$. (Such orthonormal basis $e_1^-, \ldots, e_{m_1}^-$ and $e_1^+, \ldots, e_{m_2}^+$ are called orthonormal basis of V_1 and V_2 *adapted to* Q, $\Gamma^{\perp}(v_1)$ and $\Gamma^{\perp}(v_2)$.) We can denote $Y(t)$ by

$$Y_1(t) = \sum_{i=1}^{m_1} y_-^i(t) e_i^-, \quad t \in [a, t_0]$$

and

$$Y_2(t) = \sum_{i=1}^{m_2} y_+^i(t) e_i^+, \quad t \in [t_0, b].$$

Since $Y \in \Theta^{\perp}(V_1, V_2; W_1, W_2; \psi; v_1, v_2)$, it holds that $y_-^i(t_0) = y_+^i(t_0)$ for $i = 1, \ldots, n$, and, $y_-^{i_1}(t_0) = 0$ and $y_+^{i_2}(t_0) = 0$ for $i_\lambda = n+1, \ldots, m_\lambda$ $(\lambda = 1, 2)$. Let $\bar{e}_1^-, \ldots, \bar{e}_{\bar{m}_1}^-$ and $\bar{e}_1^+, \ldots, \bar{e}_{\bar{m}_2}^+$ be orthonormal basis of $\bar{V}_1$ and $\bar{V}_2$ adapted to $\bar{\Gamma}^{\perp}(\bar{v}_1)$ and $\bar{\Gamma}^{\perp}(\bar{v}_2)$ such that $\bar{e}_{\bar{m}_1}^- = \bar{v}_1/\|\bar{v}_1\|_1$ and $\bar{e}_{\bar{m}_2}^+ = \bar{v}_2/\|\bar{v}_2\|_2$ if $\bar{v}_1 \neq 0$. If we put

$$\bar{Y}_1(t) = \sum_{i=1}^{m_1} y_-^i(t) \bar{e}_i^-, \quad t \in [a, t_0]$$

and

$$\bar{Y}_2(t) = \sum_{i=1}^{m_2} y_+^i(t) \bar{e}_i^+, \quad t \in [t_0, b],$$

then it holds that $\bar{Y} = (\bar{Y}_1, \bar{Y}_2) \in \Theta^{\perp}(\bar{V}_1, \bar{V}_2; \bar{W}_1, \bar{W}_2; \bar{\psi}; \bar{v}_1, \bar{v}_2)$, since

$$\bar{Y}_2(t_0) = \sum_{i=1}^{n} y_+^i(t_0) \bar{e}_i^+ = \sum_{i=1}^{n} y_-^i(t_0) \bar{Q}(\bar{e}_i^-) = \bar{Q}(\bar{Y}_1(t_0)).$$

Furthermore, by the definition, we have that $\|\bar{Y}_\lambda(t)\|_\lambda = \|Y_\lambda(t)\|_\lambda$ and $\|\bar{Y}'_\lambda(t)\|_\lambda = \|Y'_\lambda(t)\|_\lambda$. Hence, from the assumption (1) and (2), we get

$$\langle R_t^\lambda Y_\lambda(t), Y_\lambda(t)\rangle_\lambda \leq \langle \bar{R}_t^\lambda \bar{Y}_\lambda(t), \bar{Y}_\lambda(t)\rangle_\lambda$$

and

$$\langle A(Y_2(t_0)), Y_2(t_0)\rangle_2 \geq \langle \bar{A}(\bar{Y}_2(t_0)), \bar{Y}_2(t_0)\rangle_2.$$

Then we have that

$$I(Y, Y) \geq \bar{I}(\bar{Y}, \bar{Y}).$$

Since $I(Y, Y) = 0$, it holds that $\bar{I}(\bar{Y}, \bar{Y}) \leq 0$. But, by the definition of Y, $\bar{Y}$ is a nontrivial. Since any $t \in (a, t_0]$ are not $(\bar{V}_1, \bar{R}_t^1)$-conjugate points to a and any $t \in (t_0, b]$ are not $\bar{\mathscr{V}}$-conjugate points to a, from Lemma 3.3, we have that $\bar{I}(\bar{Y}, \bar{Y}) > 0$. Hence $t \in (t_0, b]$ is not a $\bar{\mathscr{V}}$-conjugate point to a.

2. *Case where $m_1 + 1 > \bar{m}_1$ or $m_2 + 1 > \bar{m}_2$ or $n > \bar{n}$.* By using the case 1, we can prove the assertion. By the assumption, $\bar{n} > 0$ if $n > 0$. Hence, we have that $\bar{n} \neq 0$. Let $\bar{\mu}^\lambda(t)$ be the minimal eigenvalue of $\bar{R}_t^\lambda|\{\bar{v}_\lambda\}^\perp$, $\bar{\eta}$ the maximal eigenvalue of $\bar{A}\,|\,\bar{\Gamma}^\perp(\bar{v}_2)$, $\hat{V}_\lambda := \bar{V}_\lambda \oplus \boldsymbol{R}^m$, $\hat{R}_t^\lambda := \bar{R}_t^\lambda \oplus \bar{\mu}^\lambda(t)\,\mathrm{id}_{\boldsymbol{R}^m}$, $\hat{W}_\lambda := \bar{W}_\lambda \oplus \boldsymbol{R}^m$, $\hat{\psi} := \bar{\psi} \oplus \mathrm{id}_{\boldsymbol{R}^m}$, $\hat{v}_\lambda := \bar{v}_\lambda \oplus 0$ and $\hat{A} := \bar{A} \oplus \bar{\eta}\,\mathrm{id}_{\boldsymbol{R}^m}$, where $m := \max\{m_1 + 1 - \bar{m}_1, m_2 + 1 - \bar{m}_2, n - \bar{n}\}$ and $\mathrm{id}_{\boldsymbol{R}^m} : \boldsymbol{R}^m \to \boldsymbol{R}^m$ is the identity map. Then $\hat{V}_\lambda$ is a vector space with an inner product, $\hat{\Gamma}(\hat{v}_\lambda) = \bar{\Gamma}(\bar{v}_\lambda) \oplus \boldsymbol{R}^m$, $\hat{\Gamma}^\perp(\hat{v}_\lambda) = \bar{\Gamma}^\perp(\bar{v}_\lambda) \oplus \boldsymbol{R}^m$, $\hat{Q} = \bar{Q} \oplus \mathrm{id}_{\boldsymbol{R}^m}$, $\hat{m}_\lambda := \dim \hat{V}_\lambda = \bar{m}_\lambda + m \geq \bar{m}_\lambda + (m_\lambda + 1 - \bar{m}_\lambda) = m_\lambda + 1$ and $\hat{n} := \dim \hat{\Gamma}^\perp(\hat{v}_1) = \bar{n} + m \geq \bar{n} + (n - \bar{n}) = n$. Comparing a ten-tuple $\mathscr{V}$ with a ten-tuple $\mathscr{W} := (\hat{V}_1, \hat{V}_2; \hat{W}_1, \hat{W}_2; \hat{\psi}; \hat{v}_1, \hat{v}_2; \hat{R}_t^1, \hat{R}_t^2; \hat{A})$, assumptions (1) and (2) of this lemma are hold. Furthermore if $\dim n > 0$, then $\dim \hat{n} > 0$. By case 1, if any $t \in (a, t_0]$ are not $(\hat{V}_1, \hat{R}_t^1)$-conjugate points to a and also any $t \in (t_0, b]$ are not $\mathscr{W}$-conjugate points to a, then any $t \in (t_0, b]$ are not $\mathscr{V}$-conjugate points to a. We will show that any $t \in (t_0, b]$ are not $\mathscr{W}$-conjugate points to a. Let $\hat{Y} = (\hat{Y}_1, \hat{Y}_2) \in \Theta(\hat{V}_1, \hat{V}_2; \hat{W}_1, \hat{W}_2; \hat{\psi}; \hat{v}_1, \hat{v}_2)$ be a $\mathscr{W}$-Jacobi field such that $\hat{Y}_1(a) = 0$ and $\hat{Y}_2(d) = 0$ ($d \in (t_0, b]$). Since $\hat{V}_\lambda$ is a direct sum, $\hat{Y}_\lambda$ have a form $\hat{Y}_\lambda = \bar{Y}_\lambda \oplus (\tilde{Y}_\lambda^1, \ldots, \tilde{Y}_\lambda^m)$, where $\bar{Y} := (\bar{Y}_1, \bar{Y}_2) \in \Theta(\bar{V}_1, \bar{V}_2; \bar{W}_1, \bar{W}_2; \bar{\psi}; \bar{v}_1, \bar{v}_2)$ and $\tilde{Y}^i := (\tilde{Y}_1^i, \tilde{Y}_2^i) \in \Theta(\boldsymbol{R}, \boldsymbol{R}; \boldsymbol{R}, \boldsymbol{R}; \mathrm{id}_{\boldsymbol{R}}; 0, 0)$. We put $\mathscr{R} := (\boldsymbol{R}, \boldsymbol{R}; \boldsymbol{R}, \boldsymbol{R}; \mathrm{id}_{\boldsymbol{R}}; 0, 0; \bar{\mu}^1(t)\,\mathrm{id}_{\boldsymbol{R}}, \bar{\mu}^2(t)\,\mathrm{id}_{\boldsymbol{R}}; \bar{\eta}\,\mathrm{id}_{\boldsymbol{R}})$. Then $\bar{Y}$ is a $\bar{\mathscr{V}}$-Jacobi field such that $\bar{Y}_1(a) = 0$ and $\bar{Y}_2(d) = 0$, and $\tilde{Y}^i$ is a $\mathscr{R}$-Jacobi field such that $\tilde{Y}_1^i(a) = 0$ and $\tilde{Y}_2^i(d) = 0$. Since, by the hypothesis, there is not a point $t \in (t_0, b]$ which is a $\bar{\mathscr{V}}$-conjugate point to a, $\bar{Y}$ is trivial. Furthermore, comparing a ten-tuple $\mathscr{R}$ with $\bar{\mathscr{V}}$, we see that any point $t \in (t_0, b]$ are not $\mathscr{R}$-conjugate points to a by the case 1. Hence $\tilde{Y}^i$ is trivial and $\hat{Y}$ is also. □

Using Lemma 3.2 and Lemma 3.4, we can prove the following:

LEMMA 3.5. *For two ten-tuple* $\mathscr{V}$ *and* $\overline{\mathscr{V}}$, *we assume that the conditions* (1), (2), (3) *in Lemma* 3.4. *If a perpendicular* $\mathscr{V}$*-Jacobi field* $J = (J_1, J_2)$ *with* $J_1(a) = 0$ *and a perpendicular strong* $\overline{\mathscr{V}}$*-Jacobi field* $\bar{J} = (J_1, J_2)$ *with* $\bar{J}_1(a) = 0$ *satisfy* $\|J_1'(a)\|_1 = \|\bar{J}_1'(a)\|_1$, *then*

$$\|J_\lambda(t)\|_\lambda \geq \|\bar{J}_\lambda(t)\|_\lambda \quad on \ [a,b].$$

In particular, if there is $d \in (a,b]$ *such that* $\|J_\lambda(d)\|_\lambda = \|\bar{J}_\lambda(d)\|_\lambda$, *then*

$$\|J_\lambda(t)\|_\lambda = \|\bar{J}_\lambda(t)\|_\lambda \quad on \ [a,d].$$

To show this lemma it is necessary to prove the following lemma:

LEMMA 3.6. *Let* $f : [a,b] \to \boldsymbol{R}$ *and* $\bar{f} : [a,b] \to \boldsymbol{R}$ *be piecewise smooth functions which are smooth except at* $t_0 \in (a,b)$ *and satisfies the following conditions*:

(1) $f(t),\ \bar{f}(t) > 0$ *for any* $t \in (a,b]$.

(2) $\lim_{t \to a}(f(t)/\bar{f}(t)) = 1$.

(3) $(f'(t)/f(t)) \geq (\bar{f}'(t)/\bar{f}(t))$ *except at* a *and* t_0.

Then, for any $t \in [a,b]$, $f(t) \geq \bar{f}(t)$. *In particular, if there is* $d \in (a,b]$ *with* $f(d) = \bar{f}(d)$, *then* $f(t) = \bar{f}(t)$ *for any* $t \in [a,d]$.

PROOF. We put $F(t) := f(t)/\bar{f}(t)$. By the assumption (1) and (3), we get

$$F'(t) = \frac{1}{\bar{f}(t)^2}\{f'(t)\bar{f}(t) - f(t)\bar{f}'(t)\} = F(t)\left\{\frac{f'(t)}{f(t)} - \frac{\bar{f}'(t)}{\bar{f}(t)}\right\} \geq 0,$$

except at a and t_0. Moreover, it is obtained that

$$\lim_{t \to a} F(t) = 1,$$

from the assumption (2). Since $F(t)$ is continuous, $F(t) \geq 1$ for any $t \in (a,b]$. If there is $d \in (a,b]$ with $F(t) = 1$, then $F(t) = 1$ for any $t \in [a,d]$. This completes the proof. □

PROOF OF LEMMA 3.5. For a perpendicular $\mathscr{V}$-Jacobi field $J = (J_1, J_2)$ with $J_1(a) = 0$ and a perpendicular strong $\overline{\mathscr{V}}$-Jacobi field $\bar{J} = (J_1, J_2)$ with $\bar{J}_1(a) = 0$, we set

$$f(t) := \langle J_\lambda(t), J_\lambda(t)\rangle_\lambda \quad \text{and} \quad \bar{f}(t) := \langle \bar{J}_\lambda(t), \bar{J}_\lambda(t)\rangle_\lambda.$$

We will show that f and $\bar{f}$ satisfy the assumption of Lemma 3.6. Then the proof will be complete.

By Lemma 3.4, there are no (V_1, R_t^1)-conjugate points to a on $(a, t_0]$ and $\mathscr{V}$-conjugate points to a on $(t_0, b]$. Since $\bar{J}$ is strong, if $J_1'(a) = 0$ and $\bar{J}_1'(a) = 0$, then $J_1 = 0$, $\bar{J}_1 = 0$ and $\bar{J}_2 = 0$. This shows that lemma is true. If $\|J_1'(a)\|_1 = \|\bar{J}_1'(a)\|_1 \neq 0$, then we get $J_\lambda(t) \neq 0$ and $\bar{J}_\lambda(t) \neq 0$ for any $t \in (a, b]$. Hence we have that f and $\bar{f}$ are piecewise smooth and smooth except at t_0. The assumption (1) of Lemma 3.6 holds. We can prove the assumption (2) of Lemma 3.6 as follows:

$$\lim_{t\to a} \frac{f(t)}{\bar{f}(t)} = \lim_{t\to a} \frac{f'(t)}{\bar{f}'(t)} = \lim_{t\to a} \frac{f''(t)}{\bar{f}''(t)}$$
$$= \lim_{t\to a} \frac{\langle J_1'(t), J_1'(t)\rangle_1 - \langle J_1(t), R_t^1 J_1\rangle_1}{\langle \bar{J}_1'(t), \bar{J}_1'(t)\rangle_1 - \langle \bar{J}_1(t), \bar{R}_t^1 \bar{J}_1\rangle_1} = \frac{\|J_1'(a)\|_1^2}{\|\bar{J}_1'(a)\|_1^2} = 1.$$

Finally, we shall show that the assumption (3) of Lemma 3.6.

1. *Case where* $m_1 + 1 \leq \bar{m}_1$, $m_2 + 1 \leq \bar{m}_2$ *and* $n \leq \bar{n}$. Fix a point $c \in (t_0, b]$. We define $Y = (Y_1, Y_2) \in \Theta^\perp(V_1, V_2; W_1, W_2; \psi; v_1, v_2)$ and $\bar{Y} = (\bar{Y}_1, \bar{Y}_2) \in \Theta^\perp(\bar{V}_1, \bar{V}_2; \bar{W}_1, \bar{W}_2; \bar{\psi}; \bar{v}_1, \bar{v}_2)$ by

$$Y_\lambda(t) := \frac{J_\lambda(t)}{\|J_2(c)\|_2} \quad \text{and} \quad \bar{Y}_\lambda(t) := \frac{\bar{J}_\lambda(t)}{\|\bar{J}_2(c)\|_2},$$

respectively. Then Y is a perpendicular $\mathscr{V}$-Jacobi field and $\bar{Y}$ is a perpendicular $\bar{\mathscr{V}}$-Jacobi field. Hence, from (3.4), we have that

$$I_a^c(Y, Y) = \frac{1}{2}\langle Y_2, Y_2\rangle_2'(c) = \frac{f_1'(c)}{2f_1(c)} \tag{3.5}$$

and

$$\bar{I}_a^c(\bar{Y}, \bar{Y}) = \frac{1}{2}\langle \bar{Y}_2, \bar{Y}_2\rangle_2'(c) = \frac{f_2'(c)}{2f_2(c)}. \tag{3.6}$$

Let $e_1^+ = Y_2(c), e_2^+, \ldots, e_{m_2}^+$ (resp. $\bar{e}_1^+ = \bar{Y}_2(c), \bar{e}_2^+, \ldots, \bar{e}_{\bar{m}_2}^+$) be an orthonormal basis of V_2 (resp. $\bar{V}_2$) such that $\bar{e}_{\bar{m}_2}^+ = \bar{v}_2/\|\bar{v}_2\|_2$ if $\bar{v}_2 \neq 0$, $e_{i_1}^+, \ldots, e_{i_n}^+$ $(1 \leq i_1 < \cdots < i_n \leq m_2)$ is an orthonormal basis of $\Gamma^\perp(v_2)$ and $\bar{e}_{i_1}^+, \ldots, \bar{e}_{i_n}^+$ are elements of $\bar{\Gamma}^\perp(\bar{v}_2)$. Let $e_1^-, \ldots, e_{m_1}^-$ (resp. $\bar{e}_1^-, \ldots, \bar{e}_{\bar{m}_1}^-$) be an orthonormal basis of V_1 (resp. $\bar{V}_1$) such that $\bar{e}_{\bar{m}_1}^- = \bar{v}_1/\|\bar{v}_1\|_1$ if $\bar{v}_2 \neq 0$, and, $Q(e_{i_\alpha}^-) = e_{i_\alpha}^+$ and $\bar{Q}(\bar{e}_{i_\alpha}^-) = \bar{e}_{i_\alpha}^+$ for $\alpha = 1, \ldots, n$. We can denote $Y(t)$ by

$$Y_1(t) = \sum_{i=1}^{m_1} y_-^i(t)e_i^-, \quad t \in [a, t_0]$$

and

$$Y_2(t) = \sum_{i=1}^{m_2} y_+^i(t)e_i^+, \quad t \in [t_0, b].$$

Since $Y = (Y_1, Y_2) \in \Theta^\perp(V_1, V_2; W_1, W_2; \psi; v_1, v_2)$, $y_-^{i_\alpha}(t_0) = y_+^{i_\alpha}(t_0)$ for $\alpha = 1, \dots, n$. We define $\bar{Z} = (\bar{Z}_1, \bar{Z}_2) \in \Theta^\perp(\bar{V}_1, \bar{V}_2; \bar{W}_1, \bar{W}_2; \bar{\psi}; \bar{v}_1, \bar{v}_2)$ by

$$\bar{Z}_1(t) = \sum_{i=1}^{m_1} y_-^i(t)\bar{e}_i^-, \quad t \in [a, t_0]$$

and

$$\bar{Z}_2(t) = \sum_{i=1}^{m_2} y_+^i(t)\bar{e}_i^+, \quad t \in [t_0, b].$$

In fact,

$$\bar{Z}_2(t_0) = \sum_{\alpha=1}^{n} y_+^{i_\alpha}(t_0)\bar{e}_{i_\alpha}^+ = \sum_{\alpha=1}^{n} y_-^{i_\alpha}(t_0)\bar{Q}(\bar{e}_{i_\alpha}^-) = \bar{Q}(\bar{Z}_1(t_0)).$$

Since $Y_2(c) = e_1^+$, $\bar{Z}_2(c) = \bar{e}_1^+ = \bar{Y}_2(c)$. Therefore, from Lemma 3.2, we obtain that

$$\bar{I}_a^c(\bar{Z}, \bar{Z}) \geq \bar{I}_a^c(\bar{Y}, \bar{Y}).$$

Moreover, for any $t \in (a, c]$, $\| Y_\lambda(t)\|_\lambda = \|\bar{Z}_\lambda(t)\|_\lambda$ and $\| Y_\lambda'(t)\|_\lambda = \|\bar{Z}_\lambda'(t)\|_\lambda$ hold. Hence we have that

$$I_a^c(Y, Y) \geq \bar{I}_a^c(\bar{Z}, \bar{Z})$$

from the condition (1) and (2). Then we get

$$I_a^c(Y, Y) \geq \bar{I}_a^c(\bar{Y}, \bar{Y}). \tag{3.7}$$

By (3.5), (3.6) and (3.7), f and $\bar{f}$ satisfy the assumption (3) of Lemma 3.6 for any $t \in (t_0, b]$. We can prove the case of $t \in (a, t_0]$ as usual.

2. *Case where $m_1 + 1 > \bar{m}_1$ or $m_2 + 1 > \bar{m}_2$ or $n > \bar{n}$.* Let $\mathscr{W}$ be as in proof of Lemma 3.4. We put $\mathscr{U} := (\boldsymbol{R}^m, \boldsymbol{R}^m; \boldsymbol{R}^m, \boldsymbol{R}^m; \mathrm{id}_{\boldsymbol{R}^m}; 0, 0; \bar{\mu}^1(t)\ \mathrm{id}_{\boldsymbol{R}^m}, \bar{\mu}^2(t)\ \mathrm{id}_{\boldsymbol{R}^m}; \bar{\eta}\ \mathrm{id}_{\boldsymbol{R}^m})$. A strong $\mathscr{W}$-Jacobi field $\hat{J} = (\hat{J}_1, \hat{J}_2)$ such that $\hat{J}_1(a) = 0$ and $\|\hat{J}_1'(a)\|_1 = \|J_1'(a)\|_1$ have a form $\hat{J}_\lambda = \bar{J} \oplus \tilde{J}$, where $\bar{J} = (\bar{J}_1, \bar{J}_2)$ is a strong $\bar{\mathscr{V}}$-Jacobi field

with $\bar{J}_1(a) = 0$ and $\tilde{J} = (\tilde{J}_1, \tilde{J}_2)$ is a $\mathscr{U}$-Jacobi field with $\tilde{J}_1(a) = 0$. If ten-tuples $\mathscr{V}$ and $\mathscr{W}$ satisfy the assumption (1), (2) and (3) of Lemma 3.4, then it is reduced to the case 1. Since (1) and (2) is true, we may prove (3). We assume that $c \in (t_0, b]$ is a $\mathscr{W}$-conjugate point to a. If $\hat{J} = (\hat{J}_1, \hat{J}_2)$ is a $\mathscr{W}$-Jacobi field with the decomposition $\hat{J}_\lambda = \bar{J}_\lambda \oplus \tilde{J}_\lambda$ such that $\hat{J}_1(a) = 0$ and $\hat{J}_2(c) = 0$, then we get $\bar{J}_1(a) = 0$, $\tilde{J}_1(a) = 0$, $\bar{J}_2(c) = 0$ and $\tilde{J}_2(c) = 0$. By the hypothesis, we have that $\bar{J}_\lambda$ is trivial on $[a, c]$. Moreover, since $\mathscr{U}$ and $\bar{\mathscr{V}}$ satisfy the assumption of Lemma 3.4, $\tilde{J}_\lambda$ is trivial on $[a, c]$. □

Lemma 3.4 and Lemma 3.5 show that Theorem 2.1 holds.

We say *the Jacobi equation splits relative to* $\Gamma(v_\lambda)$ if R_t^λ preserve $\Gamma(v_\lambda)$. We say that $t_2 \in [a, b]$ is a *strong* $\mathscr{V}$*-conjugate point to* $t_1 \in [a, b]$ $(t_1 \neq t_2)$ if there exists a nontrivial strong $\mathscr{V}$-Jacobi field which vanishes at t_1 and t_2.

Suppose the Jacobi equation splits relative to $\Gamma(v_\lambda)$. Then R_t^λ preserves $\Gamma(v_\lambda)$. Since R_t^λ is self-adjoint, it also preserves $\Gamma(v_\lambda)^\perp$. Let $R_t^{\lambda,1} := R_t^\lambda \mid \Gamma(v_\lambda)$ and $R_t^{\lambda,2} := R_t^\lambda \mid \Gamma(v_\lambda)^\perp$. Then $R_t^\lambda = R_t^{\lambda,1} \oplus R_t^{\lambda,2}$ and the following holds:

Lemma 3.7. *Suppose the Jacobi equation splits relative to* $\Gamma(v_\lambda)$. *Let* $Y = (Y_1, Y_2)$ *be a* $\mathscr{V}$*-Jacobi field and let* $Y_\lambda = Y_\lambda^1 + Y_\lambda^2$, *where* $Y_\lambda^1 \in \Gamma(v_\lambda)$ *and* $Y_\lambda^2 \in \Gamma(v_\lambda)^\perp$. *Then* $Y^1 := (Y_1^1, Y_2^1)$ *and* $Y^2 := (Y_1^2, Y_2^2)$ *are* $\mathscr{V}$*-Jacobi field and Jacobi equation becomes*

$$(Y_\lambda^1)'' + R_t^{\lambda,1}(Y_\lambda^1) = 0 \quad and \quad (Y_\lambda^2)'' + R_t^{\lambda,2}(Y_\lambda^2) = 0.$$

Moreover Y^1 *is a strong* $\mathscr{V}$*-Jacobi field and* Y^2 *is a* $\mathscr{V}$*-Jacobi field with* $Y_1^2(t_0) = 0$. *In particular, if* Y *is a strong* $\mathscr{V}$*-Jacobi field, then* $Y = Y^1$.

This lemma shows that Lemma 2.3 holds.

The following assertion holds:

Lemma 3.8. *We assume that the Jacobi equation splits relative to* $\bar{\Gamma}(\bar{v}_\lambda)$, *the conditions* (1), (2) *in Lemma* 3.4 *hold and any* $t \in (a, b]$ *are not strong* $\bar{\mathscr{V}}$*-conjugate point to* a. *If a perpendicular* $\mathscr{V}$*-Jacobi field* $J := (J_1, J_2)$ *with* $J_1(a) = 0$ *and a perpendicular strong* $\bar{\mathscr{V}}$*-Jacobi field* $\bar{J} := (\bar{J}_1, \bar{J}_2)$ *with* $\bar{J}_1(a) = 0$ *satisfy* $\|J_1'(a)\|_1 = \|\bar{J}_1'(a)\|_1$, *then*

$$\|J_\lambda(t)\|_\lambda \geq \|\bar{J}_\lambda(t)\|_\lambda \quad on \ [a, b].$$

In particular, if there is $d \in (a, b]$ *such that* $\|J_\lambda(d)\|_\lambda = \|\bar{J}_\lambda(d)\|_\lambda$, *then*

$$\|J_\lambda(t)\|_\lambda = \|\bar{J}_\lambda(t)\|_\lambda \quad on \ [a, d].$$

PROOF. Let $\hat{V}_\lambda := \bar{\Gamma}(\bar{v}_\lambda)$, $\hat{W}_\lambda := \overline{W}_\lambda$, $\hat{\psi} := \bar{\psi}$, $\hat{v}_\lambda := \bar{v}_\lambda$, $\hat{R}_t^\lambda := \bar{R}_t^\lambda \,|\, \bar{\Gamma}(v_\lambda)$ and $\hat{A} := \bar{A}$. Then $\hat{\Gamma}(\hat{v}_\lambda) = \bar{\Gamma}(\bar{v}_\lambda)$ and $\hat{Q} = \bar{Q}$. We put $\mathscr{W} := (\hat{V}_1, \hat{V}_2; \hat{W}_1, \hat{W}_2; \hat{\psi}; \hat{v}_1, \hat{v}_2; \hat{R}_t^1, \hat{R}_t^2; \hat{A})$. The assumption that there are no strong $\overline{\mathscr{V}}$-conjugate points to a on $(a, b]$ means that there are no $\mathscr{W}$-conjugate points to a on $(a, b]$. Since $\bar{J}$ is a strong $\overline{\mathscr{V}}$-Jacobi field, $\bar{J}$ is a $\mathscr{W}$-Jacobi field. By Lemma 3.5, we obtain the consequence. □

This lemma shows that Theorem 2.4 holds.

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