

A CONSTRUCTION OF BRAIDED HOPF ALGEBRAS*

By

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Abstract. Under suitable assumption, we present a method to construct braided Hopf algebras (braided groups) \bar{B} and \underline{B} in Yetter-Drinfel'd category ${}^H\mathcal{YD}_1^H$ and ${}^H\mathcal{YD}_2^H$ respectively. As applications, we study some special cases in both module and comodule form for H quasitriangular and for H coquasitriangular respectively. Finally, some examples are given.

§1. Introduction and Preliminaries

There has been some interest in the theory of braided Hopf algebras (braided groups) or Hopf algebras in braided categories [AS1–2, D, Maj1–3]. Applications in physics include the spectrum generating quantum groups and the constructions of inhomogeneous quantum groups. Applications in pure mathematics include the proof of Schur's double centralizer theorems in [CFW], [FM], [XSW] and the complete classification of all pointed Hopf algebras of dimension p^2 or p^3 [AS1–2], and also linearly recursive sequences [NT].

Majid has introduced a procedure termed Bosonisation by which one can construct a braided Hopf algebras in the four categories denoted by ${}^H\mathcal{M}$, \mathcal{M}_H , ${}^H\mathcal{M}$ and \mathcal{M}^H for left, right modules and left, right comodules respectively. Bosonisation generalise the Jordan-Wigner bosonisation transform for Z_2 -graded systems in physics, and connectes to every Hopf algebra B in the braided category of representations of H an equivalent ordinary Hopf algebra $B \bowtie H$ (left handed cases) or $H \bowtie B$ (right handed cases).

We find that Bosonisation and Radford's biproducts [R] associated to Yetter-Drinfeld categories are not the same. This means that general braided Hopf algebras in the Yetter-Drinfeld category ${}^H\mathcal{YD}^H$ are much more general

*This research is Supported by a grant of NSF of China and also a grant awarded to him by NSF of Henan Province, China.

Received December 11, 2000.

Revised July 30, 2001.

than the braided Hopf algebras in ${}_H\mathcal{M}$ for H quasitriangular or in ${}^H\mathcal{M}$ for H coquasitriangular. Thus, a natural and further question is: Does there exist a more general method to construct a braided Hopf algebra in the Yetter-Drinfeld categories?

We give a description about this in this paper, this is one motivation of this paper. Another motivation is due to [AS1–2] and [D] in which the authors investigate braided Hopf algebras of order p in the category ${}_{H^{\mathcal{YD}}}\mathcal{D}^H$ of left-right Yetter-Drinfeld module over a more general Hopf algebra H and the trace formular for braided Hopf algebras respectively.

The present paper begins in Section 2 where we consider two braided monoidal category ${}_{H^{\mathcal{YD}}}\mathcal{D}_1^H$ and ${}_{H^{\mathcal{YD}}}\mathcal{D}_2^H$, and define one twisted algebra \bar{B} for a bialgebra B which is both in ${}_{H^{\mathcal{YD}}}\mathcal{D}_1^H$ and ${}_{H^{\mathcal{YD}}}\mathcal{D}_2^H$, and then under suitable assumption, we show that \bar{B} is a braided Hopf algebra in ${}_{H^{\mathcal{YD}}}\mathcal{D}_2^H$. Similarly, it is proved that there exists another braided Hopf algebra \underline{B} in ${}_{H^{\mathcal{YD}}}\mathcal{D}_1^H$.

Section 3 is concerned with the conditions under which \bar{B} and \underline{B} above respectively become braided Hopf algebras.

The main results of the paper are some detailed calculation of the braided Hopf algebra \bar{B} living in the category of modules of a quasitriangular Hopf algebra (H, R) associated to a dual pairing Hopf algebra (B, H, τ) and of comodules of a coquasitriangular Hopf algebra $(H, \langle | \rangle)$ associated to a dual R -Hopf algebra pair (B, H, R) , these results are investigated in Section 4.

Throughout this paper, k denotes a fixed field and $(H, m_H, 1_H, \Delta_H, \varepsilon_H)$ a Hopf algebra over k with multiplication m_H , unit 1_H , comultiplication $\Delta : H \rightarrow H \otimes H$ and counit $\varepsilon : H \rightarrow k$. We use Sweedler’s Hopf algebra notation [Mont, Sw] and use the “sigma” notation for $\Delta : \Delta(h) = h_1 \otimes h_2$, for all $h \in H$, where we omit parentheses on subscripts and the sum notation. S_H denotes the antipode of H . S_H^{-1} denotes its composition inverse if S_H is bijective. All maps are k -linear, \otimes means \otimes_k unless otherwise specified, etc. Denote by ${}_H\mathcal{M}$ the category of left H -modules and by \mathcal{M}^H the category of right H -comodules. For $(V, \rho) \in \mathcal{M}^H$, we use notation: $\rho(v) = v_{(0)} \otimes v_{(1)} \in V \otimes H$, for any $v \in V$. Also, for $(W, \delta) \in \mathcal{M}^H$, we use notation: $\delta(w) = w^{(0)} \otimes w^{(1)} \in W \otimes H$, for any $w \in W$.

By [RT] a left-right Yetter-Drinfeld category ${}_{H^{\mathcal{YD}}}\mathcal{D}^H$, whose morphisms are simultaneously module and comodule maps, is the category of objects (M, \cdot, δ) such that (M, \cdot) is in ${}_H\mathcal{M}$ and (M, δ) is in \mathcal{M}^H satisfying the compatibility condition:

$$h_1 \cdot m^{(0)} \otimes h_2 m^{(1)} = (h_2 \cdot m)^{(0)} \otimes (h_2 \cdot m)^{(1)} h_1 \tag{1.1}$$

for all $h \in H$ and $m \in M$.

If S_H is bijective, then the equation (1.1) is equivalent to:

$$\rho(h \cdot m) = h_2 \cdot m^{(0)} \otimes h_3 m^{(1)} S_H^{-1}(h_1).$$

If H is a finite-dimensional Hopf algebra, then the Yetter-Drinfeld ${}^H\mathcal{YD}^H$ can be identified with the category ${}_{D(H)}\mathcal{M}$ of left modules over the Drinfeld double $D(H)$, see [Mont, p. 214].

There exist two prebraiding monoidal structures on ${}^H\mathcal{YD}^H$ [RT] as follows. Let $V, W \in {}^H\mathcal{YD}^H$. For $v \otimes w \in V \otimes W$ and $h \in H$, one structure is defined by the following structure (1.2)–(1.4):

$$h \rightarrow (v \otimes w) = h_2 \cdot v \otimes h_1 \cdot w \tag{1.2}$$

$$\delta(v \otimes w) = v^{(0)} \otimes w^{(0)} \otimes v^{(1)} w^{(1)} \tag{1.3}$$

$$\tau'_{V,W}(v \otimes w) = v^{(1)} \cdot w \otimes v^{(0)} \tag{1.4}$$

and then the category ${}^H\mathcal{YD}_1^H$ denotes the category ${}^H\mathcal{YD}^H$ which is equipped with the above prebraiding monoidal structures. Then $(V \otimes W, \rightarrow, \delta)$ is in the ${}^H\mathcal{YD}_1^H$.

Another structure is given by the following structure (1.5)–(1.7):

$$h \rightarrow (v \otimes w) = h_1 \cdot v \otimes h_2 \cdot w \tag{1.5}$$

$$\rho(v \otimes w) = v_{(0)} \otimes w_{(0)} \otimes w_{(1)} v_{(1)} \tag{1.6}$$

$$\tau''_{V,W}(v \otimes w) = w_{(0)} \otimes w_{(1)} \cdot v \tag{1.7}$$

and by the category ${}^H\mathcal{YD}_2^H$ we denotes the category ${}^H\mathcal{YD}^H$ with the prebraiding monoidal structures (1.5), (1.6) and (1.7). Then $(V \otimes W, \rightarrow, \rho)$ is in the ${}^H\mathcal{YD}_2^H$.

Let A, H be two Hopf algebras. In [Milit], the author study a relationship between a Long dimodule category and some of D -equations. By a generalized Long dimodule category ${}^H\mathcal{L}^A$ we mean a triple (M, \rightarrow, δ) such that $(M, \rightarrow) \in {}_H\mathcal{M}$ and $(M, \delta) \in \mathcal{M}^A$ satisfying the compatibility condition:

$$\delta(h \rightarrow b) = h \rightarrow b^{(0)} \otimes b^{(1)} \tag{1.8}$$

whose morphisms are simultaneously module and comodule maps. Especially, when $A = H$ we get a Long dimodule category ${}^H\mathcal{L}^H$.

A quasitriangular Hopf algebra is a pair (H, R) , where H is a Hopf algebra over k and $R = R^{(1)} \otimes R^{(2)} \in H \otimes H$ is invertible such that the following ($r = R$):

$$(QT1) \quad \Delta R^{(1)} \otimes R^{(2)} = R^{(1)} \otimes r^{(1)} \otimes R^{(2)} r^{(2)}$$

$$(QT2) \quad R^{(1)} \otimes \Delta R^{(2)} = R^{(1)} r^{(1)} \otimes r^{(2)} \otimes R^{(2)}$$

$$(QT3) \quad \Delta^{cop}(h) = R \Delta(h) R^{-1}$$

is satisfied, where $\Delta^{cop}(h) = h_2 \otimes h_1$ for all $h \in H$. If $R^{-1} = R^{(2)} \otimes R^{(1)}$, then (H, R) is called a triangular Hopf algebra.

As a dual concept (see [DT, Sect 3]), a coquasitriangular Hopf algebra is a pair $(H, \langle | \rangle)$ where H is a Hopf algebra over k and $\langle | \rangle : H \otimes H \rightarrow k$ is a k -linear form which is convolution invertible in $Hom_k(H \otimes H, k)$ such that the following hold:

$$(BR1) \quad \langle h | gl \rangle = \langle h_1 | l \rangle \langle h_2 | g \rangle$$

$$(BR2) \quad \langle hg | l \rangle = \langle h | l_1 \rangle \langle g | l_2 \rangle$$

$$(BR3) \quad \langle h_1 | g_1 \rangle h_2 g_2 = g_1 h_1 \langle h_2 | g_2 \rangle.$$

If $\langle h_1 | g_1 \rangle \langle g_2 | h_2 \rangle = \varepsilon(g)\varepsilon(h)$ then $(H, \langle | \rangle)$ is called a cotriangular Hopf algebra.

A Hopf pairing (B, H, τ) means a triple (B, H, τ) , where B, H are Hopf algebras and the $\tau : B \times H \rightarrow k$ is a (convolution) invertible bilinear form satisfying:

$$(DP1) \quad \tau(ab, h) = \tau(a, h_1)\tau(b, h_2)$$

$$(DP2) \quad \tau(a, hl) = \tau(a_1, h)\tau(a_2, l)$$

$$(DP3) \quad \tau(1, h) = \varepsilon(h)1$$

$$(DP4) \quad \tau(a, 1) = \varepsilon(a)1$$

It is easy to see that (DP1) and (DP2) yield

$$(DP1)' \quad \tau^{-1}(ab, h) = \tau^{-1}(a, h_2)\tau^{-1}(b, h_1)$$

$$(DP2)' \quad \tau^{-1}(a, hl) = \tau^{-1}(a_1, l)\tau^{-1}(a_2, h)$$

for $a, b \in B, h, l \in H$.

Let C be a coalgebra. The opposite coalgebra C^{cop} is C as a k -module with comultiplication given by $\Delta^{cop}(c) = c_2 \otimes c_1$ for $c \in C$, where we write $\Delta(c) = c_1 \otimes c_2$. Let H be a Hopf algebra. Suppose the antipode S of H is bijective (this holds true if H is quasitriangular or coquasitriangular). Then, H^{op} and H^{cop} are both Hopf algebras with antipode S^{-1} .

EXAMPLE 1.1. Let (B, H, τ) be skew-pairing Hopf algebra ([DT]). Then (B, H^{cop}, τ) and (B^{op}, H, τ) are Hopf pairings.

EXAMPLE 1.2. Let $(H, \langle | \rangle)$ be coquasitriangular Hopf algebra. Then $(H, H^{cop}, \langle | \rangle)$ and $(H^{op}, H, \langle | \rangle)$ are Hopf pairings.

EXAMPLE 1.3. Let H be a finite-dimensional Hopf algebra. Then $(H^*, H, \langle, \rangle)$ is Hopf pairing, where H^* is the dual Hopf algebra and \langle, \rangle is the evaluation map.

Dually, we define a dual R -Hopf algebra is a triple (B, H, R) , where B, H are two Hopf algebras and the $R = R^{(1)} \otimes R^{(2)} \in B \otimes H$ is an invertible element such that the following ($r = R$):

$$(QT1) \quad \Delta R^{(1)} \otimes R^{(2)} = R^{(1)} \otimes r^{(1)} \otimes R^{(2)} r^{(2)}$$

$$(QT2) \quad R^{(1)} \otimes \Delta R^{(2)} = R^{(1)} r^{(1)} \otimes R^{(2)} \otimes r^{(2)}$$

is satisfied, for all $h \in H$. It is not hard to show $R^{(1)} \otimes R^{(2)} = SR^{(1)} \otimes SR^{(2)} = S^2 R^{(1)} \otimes R^{(2)} = R^{(1)} \otimes S^2 R^{(2)}$ and $R^{-1} = SR^{(1)} \otimes R^{(2)} = R^{(1)} \otimes SR^{(2)}$.

EXAMPLE 1.4. Let (B, H, R) be a R -Hopf algebras ([WZJ]). Then (B^{op}, H, R) and (B, H^{cop}, R) are dual R -Hopf algebras.

EXAMPLE 1.5. Let (H, R) be quasitriangular Hopf algebra. Then (H^{op}, H, R) and (H, H^{cop}, R) are dual R -Hopf algebras.

EXAMPLE 1.6. Let H be a finite-dimensional Hopf algebra. Then (H, H^*, R) is dual R -Hopf algebras, where H^* is the dual Hopf algebra and $R = \sum_{i=1}^n h_i \otimes h_i^*$ here $\{h_i\}$ and $\{h_i^*\}$ are dual basis of H and H^* .

§2. Braided Bialgebras in ${}^H\mathcal{YD}^H$

Let (A, \cdot, δ_A) be an algebra in ${}^H\mathcal{YD}_1^H$ where \cdot and δ_A is a left H -module structure and a right H -comodule structure on A respectively. We define $A^* = A$ as linear space, with a twisted multiplication given by:

$$a * b = (b^{(1)} \cdot a) b^{(0)}$$

PROPOSITION 2.1. $(A^*, *)$ is an associative algebra.

PROOF. It is easy to see that 1_A is a unit of A^* . As to associativity of $*$, one has:

$$\begin{aligned} (a * b) * c &= (c^{(1)} \cdot [(b^{(1)} \cdot a) b^{(0)}]) c^{(0)} \\ &\stackrel{(1,2)}{=} (c^{(1)}_2 b^{(1)} \cdot a) (c^{(1)}_1 \cdot b^{(0)}) c^{(0)} \quad (\text{since } A \text{ is } H\text{-module algebra}) \end{aligned}$$

$$\begin{aligned} &\stackrel{(1.1)}{=} [(c^{(1)}_2 \cdot b)^{(1)} c^{(1)}_1 \cdot a] (c^{(1)}_2 \cdot b)^{(0)} c^{(0)} \\ &= [(c^{(1)} \cdot b)^{(1)} c^{(0)(1)} \cdot a] (c^{(1)} \cdot b)^{(0)} c^{(0)(0)} \\ &\stackrel{(1.3)}{=} [((c^{(1)} \cdot b) c^{(0)})^{(1)} \cdot a] ((c^{(1)} \cdot b) c^{(0)})^{(0)} = a * (c^{(1)} \cdot b) c^{(0)} = a * (b * c). \end{aligned}$$

This concludes the proof.

REMARK. That (A, \cdot, δ_A) is an algebra in ${}^H\mathcal{YD}_1^H$ is not a necessary condition for $(A^*, *)$ to be an associative algebra. This is seen in the following (2.10) and the proof of Theorem 2.3.

Similarly, for any $(A, \cdot, \rho) \in {}^H\mathcal{YD}_2^H$ we define $A^* = A$ as linear space with a twisted multiplication defined by:

$$a * b = a_{(0)}(a_{(1)} \cdot b),$$

and we have the following proposition:

PROPOSITION 2.2. (A^*, \star) is an associative algebra.

Let (B, \dashv, δ) be an algebra in ${}^H\mathcal{YD}_1^H$ and (B, \dashv, ρ) an algebra in ${}^H\mathcal{YD}_2^H$ such that (B, \dashv, ρ) and (B, \dashv, δ) are in ${}^H\mathcal{L}^H$.

Now, we assume that the following condition (A) are satisfied:

CONDITION (A).

$$h \dashv (l \dashv b) = l \dashv (h \dashv b) \tag{2.1}$$

$$(h_1 \dashv b_1) \otimes (h_2 \dashv b_2) = \varepsilon(h) b_1 \otimes b_2 \tag{2.2}$$

$$\Delta(h \dashv b) = (h \dashv b_1) \otimes b_2 \tag{2.3}$$

$$\Delta(h \dashv b) = b_1 \otimes (h \dashv b_2) \tag{2.4}$$

$$b^{(1)} \otimes b^{(0)}_{(1)} \otimes b^{(0)}_{(0)} = b_{(0)}^{(1)} \otimes b_{(1)} \otimes b_{(0)}^{(0)} \tag{2.5}$$

$$b_{1(0)} \otimes b_2^{(0)} \otimes b_2^{(1)} b_{1(1)} = b_1 \otimes b_2 \otimes 1 \tag{2.6}$$

$$b_1^{(0)} \otimes b_2 \otimes b_1^{(1)} = b^{(0)}_1 \otimes b^{(0)}_2 \otimes b^{(1)} \tag{2.7}$$

$$b_1 \otimes b_{2(0)} \otimes b_{2(1)} = b_{(0)1} \otimes b_{(0)2} \otimes b_{(1)}, \tag{2.8}$$

for any $b \in B$ and $h, l \in H$.

Then, we define:

$$h \rightarrow b = h_1 \rightarrow (h_2 \rightarrow b) \tag{2.9}$$

$$a \star b = (b^{(1)} \rightarrow a)b^{(0)} \tag{2.10}$$

$$\chi_B(b) = b_{(0)}^{(0)} \otimes b_{(1)}b_{(0)}^{(1)}. \tag{2.11}$$

It is not hard to verify that (B, \rightarrow) is a left H -module; that (B, χ_B) is a right H -comodule; and that (B, \rightarrow, χ_B) is an object in ${}^H\mathcal{YD}^H$. But the (B, m_B, \rightarrow) is not an algebra in ${}^H\mathcal{YD}_2^H$. In fact, we have

$$\begin{aligned} h \rightarrow (ab) &= h_1 \rightarrow (h_2 \rightarrow (ab)) \stackrel{(1.5)}{=} h_1 \rightarrow [(h_2 \rightarrow a)(h_3 \rightarrow b)] \\ &\stackrel{(1.2)}{=} (h_2 \rightarrow (h_3 \rightarrow a))(h_1 \rightarrow (h_4 \rightarrow b)) = (h_2 \rightarrow a)(h_1 \rightarrow (h_3 \rightarrow b)) \\ &\neq (h_1 \rightarrow a)(h_2 \rightarrow b), \end{aligned}$$

and this proves that $(B, \rightarrow, \delta'_B)$ is not an H -module algebra. Thus we cannot apply Proposition 2.1 to (B, m_B, \rightarrow) . However, one can calculate:

$$\begin{aligned} a \star (b \star c) &\stackrel{(2.10)}{=} a \star (c^{(1)} \rightarrow b)c^{(0)} \\ &\stackrel{(1.3)(2.10)}{=} [(c^{(1)} \rightarrow b)^{(1)}c^{(0)(1)} \rightarrow a](c^{(1)} \rightarrow b)^{(0)}c^{(0)(0)} \\ &\stackrel{(2.9)}{=} [(c^{(1)}_2 \rightarrow (c^{(1)}_3 \rightarrow b))^{(1)}c^{(1)}_1 \rightarrow a](c^{(1)}_2 \rightarrow (c^{(1)}_3 \rightarrow b))^{(0)}c^{(0)} \\ &\stackrel{(1.1)}{=} [c^{(1)}_2(c^{(1)}_3 \rightarrow b)^{(1)} \rightarrow a](c^{(1)}_1 \rightarrow (c^{(1)}_3 \rightarrow b)^{(0)})c^{(0)} \\ &\stackrel{(1.8)}{=} (c^{(1)}_2b^{(1)} \rightarrow a)(c^{(1)}_1 \rightarrow (c^{(1)}_3 \rightarrow b^{(0)}))c^{(0)} \\ &\stackrel{(2.9)}{=} [c^{(1)}_2 \rightarrow (c^{(1)}_3 \rightarrow (b^{(1)} \rightarrow a))](c^{(1)}_1 \rightarrow (c^{(1)}_4 \rightarrow b^{(0)}))c^{(0)} \\ &\stackrel{(1.2)}{=} (c^{(1)}_1 \rightarrow [(c^{(1)}_2 \rightarrow (b^{(1)} \rightarrow a))(c^{(1)}_3 \rightarrow b^{(0)})])c^{(0)} \\ &= (c^{(1)}_1 \rightarrow (c^{(1)}_2 \rightarrow [(b^{(1)} \rightarrow a)b^{(0)}]))c^{(0)} \\ &= (c^{(1)} \rightarrow [(b^{(1)} \rightarrow a)b^{(0)}])c^{(0)} = (a \star b) \star c, \end{aligned}$$

and this proves that (B, \star) is an associative algebra with the unit 1_B .

THEOREM 2.3. *Let H be any Hopf algebra, and B a bialgebra. Let (B, \rightarrow, δ) be an algebra in ${}^H\mathcal{YD}_1^H$ and (B, \rightarrow, ρ) an algebra in ${}^H\mathcal{YD}_2^H$ such that (B, \rightarrow, ρ) and (B, \rightarrow, δ) are in ${}^H\mathcal{L}^H$. If the condition (A) hold, then there exists a bialgebra \bar{B} in ${}^H\mathcal{YD}_2^H$, $\bar{B} = B$ as a linear space and as an object in ${}^H\mathcal{YD}_2^H$ with a H -module*

structure defined by (2.9), and H -comodule structure defined by (2.11) respectively. The coalgebra structure and unit in \bar{B} coincide with those of B . The multiplication is given by (2.10).

PROOF. We show that (B, \rightarrow, \star) is a left H -module algebra in ${}^H\mathcal{YD}_2^H$ as follows. Thus

$$\begin{aligned}
h \rightarrow (a \star b) &\stackrel{(2.10)(1.5)}{=} h_1 \rightarrow [(h_2 \rightarrow (b^{(1)} \rightarrow a))(h_3 \rightarrow b^{(0)})] \\
&\stackrel{(1.2)}{=} (h_2 \rightarrow (h_3 \rightarrow (b^{(1)} \rightarrow a)))(h_1 \rightarrow (h_4 \rightarrow b^{(0)})) \\
&\stackrel{(2.1)}{=} (h_3 \rightarrow (h_2 \rightarrow (b^{(1)} \rightarrow a)))(h_4 \rightarrow (h_1 \rightarrow b^{(0)})) \\
&\stackrel{(2.9)}{=} (h_3 \rightarrow (h_2 b^{(0)(1)} \rightarrow (b^{(1)} \rightarrow a)))(h_4 \rightarrow (h_1 \rightarrow b^{(0)(0)})) \\
&\stackrel{(1.1)}{=} (h_3 \rightarrow ((h_2 \rightarrow b^{(0)})^{(1)} h_1 \rightarrow (b^{(1)} \rightarrow a)))(h_4 \rightarrow (h_2 \rightarrow b^{(0)})^{(0)}) \\
&\stackrel{(2.1)}{=} ((h_2 \rightarrow b^{(0)})^{(1)} h_1 \rightarrow (h_3 b^{(1)} \rightarrow a))(h_4 \rightarrow (h_2 \rightarrow b^{(0)})^{(0)}) \\
&\stackrel{(1.1)}{=} ((h_3 \rightarrow b)^{(0)(1)} h_1 \rightarrow (h_3 \rightarrow b)^{(1)} h_2 \rightarrow a)(h_4 \rightarrow (h_3 \rightarrow b)^{(0)(0)}) \\
&\stackrel{(2.9)}{=} ((h_2 \rightarrow b)^{(1)} h_1 \rightarrow a)(h_3 \rightarrow (h_2 \rightarrow b)^{(0)}) \\
&\stackrel{(1.8)}{=} ((h_3 \rightarrow (h_2 \rightarrow b)^{(1)} h_1 \rightarrow a)(h_3 \rightarrow (h_2 \rightarrow b))^{(0)}) \\
&\stackrel{(2.1)(2.9)}{=} ((h_2 \rightarrow b)^{(1)} h_1 \rightarrow a)(h_2 \rightarrow b)^{(0)} = (h_1 \rightarrow a) \star (h_2 \rightarrow b),
\end{aligned}$$

which is as required.

And then, we check that (B, \star, χ_B) is a right H -comodule algebra in ${}^H\mathcal{YD}_2^H$ according to the equation (1.6). In fact, one has

$$\begin{aligned}
\chi_B(a) \chi_B(b) &= a_{(0)}^{(0)} \star b_{(0)}^{(0)} \otimes b_{(1)} b_{(0)}^{(1)} a_{(1)} a_{(0)}^{(1)} \\
&= (b_{(0)}^{(0)(1)} \rightarrow a_{(0)}^{(0)}) b_{(0)}^{(0)(0)} \otimes b_{(1)} b_{(0)}^{(1)} a_{(1)} a_{(0)}^{(1)},
\end{aligned}$$

on the other hand,

$$\begin{aligned}
\chi_B(a \star b) &\stackrel{(2.10)}{=} \chi_B((b^{(1)} \rightarrow a) b^{(0)}) \\
&\stackrel{(2.11)}{=} ((b^{(1)} \rightarrow a) b^{(0)})_{(0)}^{(0)} \otimes ((b^{(1)} \rightarrow a) b^{(0)})_{(1)} ((b^{(1)} \rightarrow a) b^{(0)})_{(0)}^{(1)} \\
&\stackrel{(1.6)(1.3)}{=} (b^{(1)} \rightarrow a)_{(0)}^{(0)} b_{(0)}^{(0)(0)} \otimes b_{(1)}^{(0)} (b^{(1)} \rightarrow a)_{(1)} (b^{(1)} \rightarrow a)_{(0)}^{(1)} b_{(0)}^{(0)(1)}
\end{aligned}$$

$$\begin{aligned}
 &\stackrel{(2.9)}{=} (b^{(1)}_1 \rightarrow (b^{(1)}_2 \rightarrow a))_{(0)} b^{(0)}_{(0)} \otimes b^{(0)}_{(1)} (b^{(1)}_1 \rightarrow (b^{(1)}_2 \rightarrow a))_{(1)} \\
 &\quad (b^{(1)}_1 \rightarrow (b^{(1)}_2 \rightarrow a))_{(0)} b^{(0)}_{(0)} \\
 &\stackrel{(1.8)}{=} (b^{(1)}_1 \rightarrow (b^{(1)}_2 \rightarrow a))_{(0)} b^{(0)}_{(0)} \otimes b^{(0)}_{(1)} (b^{(1)}_2 \rightarrow a)_{(1)} \\
 &\quad (b^{(1)}_1 \rightarrow (b^{(1)}_2 \rightarrow a))_{(0)} b^{(0)}_{(0)} \\
 &\stackrel{(2.5)}{=} (b_{(0)}^{(1)}_1 \rightarrow (b_{(0)}^{(1)}_2 \rightarrow a)_{(0)}) b_{(0)}^{(0)} \otimes b_{(1)} (b_{(0)}^{(1)}_2 \rightarrow a)_{(1)} \\
 &\quad (b_{(0)}^{(1)}_1 \rightarrow (b_{(0)}^{(1)}_2 \rightarrow a)_{(0)}) b_{(0)}^{(0)} \\
 &\stackrel{(1.1)}{=} (b_{(0)}^{(1)}_1 \rightarrow (b_{(0)}^{(1)}_3 \rightarrow a)_{(0)}) b_{(0)}^{(0)} \otimes b_{(1)} (b_{(0)}^{(1)}_3 \rightarrow a)_{(1)} \\
 &\quad b_{(0)}^{(1)}_2 (b_{(0)}^{(1)}_3 \rightarrow a)_{(0)} \\
 &\stackrel{(2.5)}{=} (b^{(1)}_1 \rightarrow (b^{(1)}_3 \rightarrow a)_{(0)}) b^{(0)}_{(0)} \otimes b^{(0)}_{(1)} (b^{(1)}_3 \rightarrow a)_{(1)} \\
 &\quad \times b^{(1)}_2 (b^{(1)}_3 \rightarrow a)_{(0)} \\
 &\stackrel{(1.1)}{=} (b^{(1)}_1 \rightarrow (b^{(1)}_2 \rightarrow a_{(0)})_{(0)}) b^{(0)}_{(0)} \otimes b^{(0)}_{(1)} b^{(1)}_3 a_{(1)} (b^{(1)}_2 \rightarrow a_{(0)})_{(1)} \\
 &\stackrel{(1.8)}{=} (b^{(1)}_1 \rightarrow (b^{(1)}_2 \rightarrow a_{(0)})_{(0)}) b^{(0)}_{(0)} \otimes b^{(0)}_{(1)} b^{(1)}_3 a_{(1)} a_{(0)} \\
 &\stackrel{(2.9)}{=} (b^{(1)}_1 \rightarrow a_{(0)}) b^{(0)}_{(0)} \otimes b^{(0)}_{(1)} b^{(1)}_2 a_{(1)} a_{(0)} \\
 &\stackrel{(2.5)}{=} (b_{(0)}^{(0)} \rightarrow a_{(0)}) b_{(0)}^{(0)} \otimes b_{(1)} b_{(0)}^{(1)} a_{(1)} a_{(0)},
 \end{aligned}$$

again as required.

It is easy to see that $(B, \Delta_B, \rightarrow)$ is a module coalgebra in ${}^H\mathcal{YD}_2^H$ by the conditions (2.2), (2.3) and (2.4). It follows from the formulae (2.6)–(2.8) that (B, Δ_B, χ_B) is a left H -comodule coalgebra in ${}^H\mathcal{YD}_2^H$.

In final, using the braiding τ'' in ${}^H\mathcal{YD}_2^H$ (see (1.7)), we can form a braided tensor product $\bar{B} \otimes \bar{B} : (a \otimes b)(c \otimes d) = ac_{(0)} \otimes (c_{(1)} \rightarrow b)d$ for any $a, b, c, d \in B$. We are prepared to show that Δ_B is an algebra map from \bar{B} to $\bar{B} \otimes \bar{B}$. Thus, one does a calculation as follows:

$$\begin{aligned}
 \Delta_B(a \star b) &= (b^{(1)} \rightarrow a)_1 b^{(0)}_1 \otimes (b^{(1)} \rightarrow a)_2 b^{(0)}_2 \\
 &= (b^{(1)}_1 \rightarrow a_1) b^{(0)}_1 \otimes (b^{(1)}_2 \rightarrow a_2) b^{(0)}_2 \\
 &\quad (\text{since } (B, \Delta_B, \rightarrow) \in {}^H\mathcal{YD}_2^H \text{ is a module coalgebra})
 \end{aligned}$$

$$\stackrel{(2.9)}{=} (b^{(1)}_1 \rightarrow (b^{(1)}_2 \rightarrow a_1)b^{(0)}_1 \otimes (b^{(1)}_3 \rightarrow b^{(1)}_4 \rightarrow a_2)b^{(0)}_2$$

$$\stackrel{(2.1)(2.2)}{=} (b^{(1)}_1 \rightarrow a_1)b^{(0)}_1 \otimes (b^{(1)}_2 \rightarrow a_2)b^{(0)}_2,$$

and on the other hand, we have

$$a_1 \star b_{1(0)}^{(0)} \otimes (b_{1(1)}b_{1(0)}^{(1)} \rightarrow a_2) \star b_2$$

$$= (b_{1(0)}^{(0)(1)} \rightarrow a_1)b_{1(0)}^{(0)(0)} \otimes (b_2^{(1)}b_{1(1)}b_{1(0)}^{(1)} \rightarrow a_2)b_2^{(0)}$$

$$\stackrel{(2.5)}{=} (b_1^{(0)(1)} \rightarrow a_1)b_1^{(0)(0)}_{(0)} \otimes (b_2^{(1)}b_1^{(0)(0)}_{(1)}b_1^{(1)} \rightarrow a_2)b_2^{(0)}$$

$$= (b_1^{(1)}_1 \rightarrow a_1)b_1^{(0)}_{(0)} \otimes (b_2^{(1)}b_1^{(0)}_{(1)}b_1^{(1)}_2 \rightarrow a_2)b_2^{(0)}$$

$$\stackrel{(2.9)}{=} (b_1^{(1)}_1 \rightarrow b_1^{(1)}_2 \rightarrow a_1)b_1^{(0)}_{(0)} \otimes (b_2^{(1)}_1b_1^{(0)}_{(1)1}b_1^{(1)}_3$$

$$\rightarrow b_2^{(1)}_2b_1^{(0)}_{(1)2}b_1^{(1)}_4 \rightarrow a_2)b_2^{(0)}$$

$$\stackrel{(2.1)(2.2)}{=} (b_1^{(1)}_1 \rightarrow a_1)b_1^{(0)}_{(0)} \otimes (b_2^{(1)}_1b_1^{(0)}_{(1)1} \rightarrow b_2^{(1)}_2b_1^{(0)}_{(1)2}b_1^{(1)}_2 \rightarrow a_2)b_2^{(0)}$$

$$\stackrel{(2.7)}{=} (b^{(1)}_1 \rightarrow a_1)b^{(0)}_{1(0)} \otimes (b^{(0)}_2^{(1)}b^{(0)}_{1(1)1} \rightarrow b^{(0)}_2^{(1)}b^{(0)}_{1(1)2}b^{(1)}_2 \rightarrow a_2)b^{(0)}_2^{(0)}$$

$$= (b^{(1)}_1 \rightarrow a_1)b^{(0)}_{1(0)} \otimes ((b^{(0)}_2^{(1)}b^{(0)}_{1(1)1} \rightarrow (b^{(0)}_2^{(1)}b^{(0)}_{1(0)2}b^{(1)}_2 \rightarrow a_2)b^{(0)}_2^{(0)})$$

$$\stackrel{(2.9)}{=} (b^{(1)}_1 \rightarrow a_1)b^{(0)}_{1(0)} \otimes (b^{(0)}_2^{(1)}b^{(0)}_{1(1)} \rightarrow (b^{(1)}_2 \rightarrow a_2))b^{(0)}_2^{(0)}$$

$$\stackrel{(2.6)}{=} (b^{(1)}_1 \rightarrow a_1)b^{(0)}_1 \otimes (b^{(1)}_2 \rightarrow a_2)b^{(0)}_2.$$

Thus, (B, Δ, \star) is a bialgebra in ${}^H\mathcal{YD}^H$ concluding the proof.

Similarly, we can make another definition as follows:

$$h \succ b = h_1 \rightarrow (h_2 \rightarrow b) \quad (2.12)$$

$$a \bar{\star} b = a_{(0)}(a_{(1)} \succ b) \quad (2.13)$$

$$\zeta_B(b) = b_{(0)}^{(0)} \otimes b_{(0)}^{(1)}b_{(1)} \quad (2.14)$$

In what follows, we postulate the following condition (2.15) and (2.16) respectively instead of the equation (2.2) and (2.6). In the condition (A).

$$(h_2 \rightarrow b_1) \otimes (h_1 \rightarrow b_2) = \varepsilon(h)b_1 \otimes b_2 \quad (2.15)$$

$$b_{1(0)} \otimes b_2^{(0)} \otimes b_{1(1)}b_2^{(1)} = b_1 \otimes b_2 \otimes 1. \quad (2.16)$$

Then we have another main theorem of this section:

THEOREM 2.4. *Let H be any Hopf algebra, and B a bialgebra. Let (B, \rightarrow, δ) be an algebra in ${}^H\mathcal{YD}_1^H$ and (B, \rightarrow, ρ) an algebra in ${}^H\mathcal{YD}_2^H$ such that (B, \rightarrow, ρ) and (B, \rightarrow, δ) are in ${}^H\mathcal{L}^H$. Assume that the conditions (2.1), (2.3)–(2.5), (2.7)–(2.8), and the conditions (2.15)–(2.16) hold. Then there exists a bialgebra \underline{B} in ${}^H\mathcal{YD}_1^H$, $\underline{B} = B$ as a linear space and as an object in ${}^H\mathcal{YD}_1^H$ with a H -module structure defined by (2.12), and H -comodule structure defined by (2.14) respectively. The coalgebra structure and unit in \bar{B} coincide with those of B . The multiplication is given by (2.13).*

PROOF. Similar to that of the theorem 2.3.

REMARK 2.5. The left Yetter-Drinfeld modules form the braided category ${}^H_H\mathcal{YD}$, see [Mont, p. 214]. Similarly, the right such modules form \mathcal{YD}_H^H . We have natural identification of braided categories,

$${}^H\mathcal{YD}_1^H = {}^H{}_{H^{cop}}\mathcal{YD}, \quad {}^H\mathcal{YD}_2^H = \mathcal{YD}_{H^{op}}^H.$$

Replace H by H^{op} , and identify $H^{op,cop}$ with H via $S : H \xrightarrow{\cong} H^{op,cop}$. Thus, if M is an object in ${}^H{}_{H^{op}}\mathcal{YD}_1^H$ with structures $(h^{op}, m) \mapsto h^{op}m$, $H^{op} \otimes M \rightarrow M$ and $m \mapsto m^{(0)} \otimes m^{(1)}$, $M \rightarrow M \otimes H$, then it is identified with an object in ${}^H_H\mathcal{YD}$ with the structures given by

$$hm = S(h)^{op}m, \quad \lambda(m) = S^{-1}(m^{(1)}) \otimes m^{(0)} \in H \otimes M.$$

Theorem 2.3 is translated as follows.

Let B be a bialgebra. Suppose the algebra B is further an algebra object in ${}^H_H\mathcal{YD}$ and also in \mathcal{YD}_H^H . Suppose that each pair of structures indicated by

$${}^HB_H, \quad {}^HB^H, \quad {}_HB^H, \quad {}_HB_H$$

commutes with each other (the commutativity in ${}^HB^H$, for example, means that B is an H -bicomodule). Denote the left and the right H -comodule structures on B by

$$\lambda(b) = b^{(-1)} \otimes b^{(0)}, \quad \rho(b) = b^{(0)} \otimes b^{(1)} \quad (b \in B),$$

respectively, and suppose further that

$$\begin{aligned} b_1h \otimes b_2 &= b_1 \otimes hb_2, & \rho(b_1) \otimes b_2 &= b_1 \otimes \lambda(b_2); \\ \Delta(hb) &= hb_1 \otimes b_2, & \Delta(bh) &= b_1 \otimes b_2h; \\ \lambda(b_1) \otimes b_2 &= b^{(-1)} \otimes \Delta(b^{(0)}), & b_1 \otimes \rho(b_2) &= \Delta(b^{(0)}) \otimes b^{(1)}, \end{aligned}$$

where $b \in B$, $h \in H$. Then the coalgebra B forms a bialgebra in \mathcal{YD}_H^H , given the following new structures:

$$\begin{aligned} b \leftarrow h &:= S^{-1}(h_1)bh_2; \\ a \star b &:= (a \leftarrow S(b^{(-1)}))b^{(0)} = (b^{(-1)}aS(b^{(-2)}))b^{(0)}; \\ b \mapsto b^{(0)} \otimes S(b^{(-1)})b^{(1)}, \quad B &\rightarrow B \otimes H, \end{aligned}$$

where $a, b \in B$, $h \in H$.

Similarly, Theorem 2.4 can be reformulated in a symmetric form, which will give a construction of bialgebras in ${}^H_H\mathcal{YD}$. These reformulated statements look simpler than the original, although here one has to assume that the antipode S of H is bijective.

§3. Braided Hopf Algebras

Let H be a Hopf algebra with a bijective antipode S , and B a Hopf algebra with an antipode S_B . In this section we give a sufficient condition for the braided bialgebras as defined in §2 to be a braided Hopf algebra. At first, we assume that the following condition (B) is satisfied:

CONDITIONS (B).

$$S_B(h \rightarrow b) = h \rightarrow S_B(b) \quad (3.1)$$

$$S_B(h \rightarrow b) = S^{-2}h \rightarrow S_B(b) \quad (3.2)$$

$$(S_B(b))^{(0)} \otimes (S_B(b))^{(1)} = S_B b_{(0)} \otimes S^{-2}(b_{(1)}) \quad (3.3)$$

$$(S_B(b))_{(0)} \otimes (S_B(b))_{(1)} = S_B b^{(0)} \otimes b^{(1)}, \quad (3.4)$$

where the S^{-2} denotes the $(S^{-1})^2$.

PROPOSITION 3.1. *In the situation of the Theorem 2.3. Assume that the condition (B) holds. If B has an antipode then \bar{B} has an antipode in the category ${}^H_H\mathcal{YD}_2^H$. It is defined by*

$$\bar{S}(b) = b_{(1)} \rightarrow S_B(b_{(0)}).$$

PROOF. We need to prove that \bar{S} is a morphism in ${}^H_H\mathcal{YD}_2^H$. For this, we have

$$\begin{aligned} \bar{S}(h \rightarrow b) &= (h \rightarrow b)_{(1)} \rightarrow S_B((h \rightarrow b)_{(0)}) \\ &\stackrel{(2.9)(1.8)}{=} (h_2 \rightarrow b)_{(1)} \rightarrow S_B(h_1 \rightarrow (h_2 \rightarrow b)_{(0)}) \\ &\stackrel{(1.1)}{=} h_4 b_{(1)} S^{-1}(h_2) \rightarrow S_B(h_1 \rightarrow h_3 \rightarrow b_{(0)}) \end{aligned}$$

$$\begin{aligned}
 &\stackrel{(2.9)}{=} h_5 b_{(1)1} S^{-1}(h_3) \rightarrow h_6 b_{(1)2} S^{-1}(h_2) \rightarrow S_B(h_1 \rightarrow h_4 \rightarrow b_{(0)}) \\
 &\stackrel{(3.1)-(3.4)}{=} h_5 b_{(1)1} S^{-1}(h_3) \rightarrow h_6 b_{(1)2} S^{-1}(h_2) h_1 \rightarrow S^{-2}(h_4) \rightarrow S_B b_{(0)} \\
 &\stackrel{(2.1)}{=} h_1 b_{(1)1} \rightarrow h_2 b_{(1)2} \rightarrow S_B b_{(0)} \\
 &\stackrel{(2.9)}{=} h \rightarrow \bar{S}(b),
 \end{aligned}$$

and this prove \bar{S} is H -module map.

Also, one has

$$\begin{aligned}
 \xi_B \circ \bar{S}(b) &= (b_{(1)} \rightarrow S_B b_{(0)})_{(0)}^{(0)} \otimes (b_{(1)} \rightarrow S_B b_{(0)})_{(1)} (b_{(1)} \rightarrow S_B b_{(0)})_{(0)}^{(1)} \\
 &= (b_{(1)1} \rightarrow b_{(1)3} \rightarrow (S_B b_{(0)})_{(0)})^{(0)} \otimes b_{(1)4} (S_B b_{(0)})_{(1)} \\
 &\quad S^{-1} b_{(1)2} (b_{(1)1} \rightarrow b_{(1)3} \rightarrow (S_B b_{(0)})_{(0)})^{(1)} \\
 &= (b_{(1)5} \rightarrow b_{(1)2} \rightarrow S_B (b_{(0)})_{(0)}^{(0)}) \otimes b_{(1)4} b_{(0)}^{(1)} \\
 &\quad S^{-1} (b_{(1)4}) b_{(1)3} S^{-2} (b_{(0)})_{(1)}^{(0)} S^{-1} (b_{(1)1}) \\
 &= b_{(0)}^{(0)(1)} \rightarrow S_B (b_{(0)})_{(0)(0)} \otimes b_{(0)}^{(0)} b_{(1)}^{(1)} \\
 &= (\bar{S} \otimes id) (b_{(0)}^{(0)} \otimes b_{(0)}^{(0)} b_{(1)}^{(1)}) = (\bar{S} \otimes id) \xi_B(b),
 \end{aligned}$$

where the second equation follows (1.8) and (1.1); the third follows using the (2.1), (1.8) and (1.1); the forth follows equations (2.5) and (2.9), completing the \bar{S} is a morphism in ${}^H\mathcal{YD}_2^H$.

Using (2.6), one has that $\bar{S}b_1 \star b_2 = \varepsilon(b)$ holds. We also have

$$\begin{aligned}
 b_1 \star \bar{S}(b_2) &= [(b_{2(1)} \rightarrow S_B b_{2(0)})^{(1)} \rightarrow b_1] (b_{2(1)} \rightarrow S_B b_{2(0)})^{(0)} \\
 &= [(b_{2(1)3} (S_B b_{2(0)})^{(1)} S^{-1} b_{2(1)1} \rightarrow b_1] [b_{2(1)4} \rightarrow b_{2(1)2} \rightarrow (S_B b_{2(0)})^{(0)}] \\
 &= [(b_{2(1)2} \rightarrow b_{2(1)3} \rightarrow b_1] [b_{2(1)3} \rightarrow b_{2(1)1} \rightarrow S_B b_{2(0)}] \\
 &= b_{2(1)} \rightarrow b_1 S_B b_{2(0)} \stackrel{(2.8)}{=} b_{(1)} \rightarrow b_{(0)1} S_B b_{(0)2} = \varepsilon(b),
 \end{aligned}$$

here the second identity follows the formulae (2.9), (1.1), and (1.8); the third follows (3.3) and (2.9); the forth is given via the formulae (2.1), (1.2), (1.5) and (2.9). As required.

This completes the proof of the proposition 3.1.

Similarly, we postulate the following

CONDITIONS (C).

$$h \rightarrow S_B b = S_B(h \rightarrow b) \tag{3.5}$$

$$S^{-2}h \rightarrow S_B b = S_B(h \rightarrow b) \tag{3.6}$$

$$(S_B(b))^{(0)} \otimes (S_B(b))^{(1)} = S_B b_{(0)} \otimes b_{(1)} \tag{3.7}$$

$$(S_B(b))_{(0)} \otimes (S_B(b))_{(1)} = S_B b^{(0)} \otimes S^{-2}b^{(1)} \tag{3.8}$$

hold. Thus we have the following result similar to the proposition 3.1.

PROPOSITION 3.2. *In the situation of the theorem 2.4. Assume that the condition (C) hold. If B has an antipode then \underline{B} has an antipode in the category ${}^H\mathcal{YD}_1^H$. It is defined by*

$$\underline{S}(b) = b^{(1)} \succ S_B(b^{(0)}).$$

PROOF. The proof is similar to that of Proposition 3.1.

§4. Applications and Examples

In this section we give some of the braided Hopf algebras in the category ${}^H\mathcal{M}$ for a quasitriangular Hopf algebra H and the category \mathcal{M}^H for a coquasitriangular Hopf algebra H .

When (H, R) is quasitriangular, By [CFW], the category ${}^H\mathcal{M}$ of the left modules over the quasitriangular Hopf algebra (H, R) is endowed with the following structure:

$$\tau''(m \otimes n) = R^{(2)} \cdot n \otimes R^{(1)} \cdot m, \quad h \cdot (m \otimes n) = (h_1 \cdot m) \otimes (h_2 \cdot n),$$

for any $m \in M \in {}^H\mathcal{M}$, $n \in N \in {}^H\mathcal{M}$, and $h \in H$.

In this case, the category ${}^H\mathcal{YD}_2^H$ contains this braided monoidal category ${}^H\mathcal{M}$ as its subcategory. In our setting, ${}^H\mathcal{M}_2 = {}^H\mathcal{M}$ with the well known coaction: $\rho(m) = R^{(2)} \cdot m \otimes R^{(1)}$.

Similarly, we have a braided monoidal subcategory ${}^H\mathcal{M}_1$ with the well known coaction: $\rho(m) = R^{(2)} \cdot m \otimes R^{(1)}$, and the braiding $\tau' : m \otimes n \mapsto R^{(1)} \cdot n \otimes R^{(2)} \cdot m$, and the monoidal structure $h \cdot (m \otimes n) = h_2 \cdot m \otimes h_1 \cdot n$ for any $m \in M \in {}^H\mathcal{M}_1$, $n \in N \in {}^H\mathcal{M}_1$ and $h \in H$.

When $(H, \langle | \rangle)$ is coquasitriangular, the category ${}^H\mathcal{YD}_2^H$ contains a braided monoidal subcategory \mathcal{M}_2^H which is endowed with the following structure, $h \cdot m = \langle h | m_{(1)} \rangle m_{(0)}$, $\tau'' : m \otimes n \mapsto \langle n_{(1)} | m_{(1)} \rangle n_{(0)} \otimes m_{(0)}$, $\rho(m \otimes n) = m_{(0)} \otimes n_{(0)} \otimes n_{(1)} m_{(1)}$ for any $m \in M \in \mathcal{M}_2^H$ and $n \in N \in \mathcal{M}_2^H$.

Similarly, we have a braided monoidal subcategory \mathcal{M}_1^H armed with the following structure, $h \cdot m = \langle h|m_{(1)} \rangle m_{(0)}$, $\tau' : m \otimes n \mapsto \langle m^{(1)}|n^{(1)} \rangle n^{(0)} \otimes m^{(0)}$, $\delta(m \otimes n) = m^{(0)} \otimes n^{(0)} \otimes m^{(1)}n^{(1)}$ for any $m \in M \in \mathcal{M}_1^H$ and $n \in N \in \mathcal{M}_1^H$.

In what follows, we construct two class of braided Hopf algebra in the categories ${}_H\mathcal{M}_1$, ${}_H\mathcal{M}_2$, and \mathcal{M}_1^H , and \mathcal{M}_2^H .

Let (H, R) be a quasitriangular Hopf algebra. Let (B, H, τ) be a Hopf pairing. We define $h \rightarrow b = \tau(Sb_1, h)b_2$ and $h \dashv b = \tau(b_2, h)b_1$ for all $b \in B$, $h \in H$. In ${}_H\mathcal{M}_2$, it is natural that we have $\delta(b) = R^{(2)} \dashv b \otimes R^{(1)} \stackrel{def.}{=} b^{(0)} \otimes b^{(1)}$ and $\rho(b) = R^{(2)} \triangleright b \otimes R^{(1)} \stackrel{def.}{=} b_{(0)} \otimes b_{(1)}$.

It is easy to verify that (B, \dashv, δ) is an algebra in ${}_H\mathcal{YD}_1^H$, and that (B, \dashv, ρ) is an algebra in ${}_H\mathcal{YD}_2^H$. Obviously, (B, \dashv, ρ) and (B, \dashv, δ) are objects in Long dimodule category ${}_H\mathcal{L}^H$.

Thus, by (2.9)–(2.10) and the proposition 3.1, we obtain

$$h \rightarrow b = h_1 \dashv (h_2 \dashv b) = \tau(S(b_1)b_3, h)b_2 \tag{4.1}$$

$$a \star b = (b^{(1)} \dashv a)b^{(0)} = \tau(S(a_1)a_3, R^{(1)})a_2b_2\tau(S(b_1), R^{(2)}) \tag{4.2}$$

$$\bar{S}(b) = b_{(1)} \dashv S_B(b_{(0)}) = \tau(b_2, R^{(2)})(R^{(1)} \dashv S_B(b_1)), \tag{4.3}$$

for all $a, b \in B$ and $h \in H$.

We now have the following:

THEOREM 4.1. *Let H be quasitriangular. With the notation above. Then there is a braided Hopf algebra \bar{B} in ${}_H\mathcal{M}_2$, $\bar{B} = B$ as a linear space and as an object in ${}_H\mathcal{M}_2$ with a H -module structure defined by (4.1). The coalgebra structure and unit in \bar{B} coincide with those of B . The multiplication is given by (4.2). Its antipode is given by (4.3).*

PROOF. Firstly, in order to apply the Theorem 2.3, we need to check the Condition (B) hold. A routine check can show that the conditions (2.1)–(2.4) are satisfied. Then, by definition, and using (2.1), we have

$$\begin{aligned} b^{(1)} \otimes b^{(0)}_{(1)} \otimes b^{(0)}_{(0)} &= R^{(1)} \otimes (R^{(2)} \dashv b)_{(1)} \otimes (R^{(2)} \dashv b)_{(0)} \\ &= R^{(1)} \otimes r^{(1)} \otimes r^{(2)} \dashv (R^{(2)} \dashv b), \\ &= R^{(1)} \otimes r^{(1)} \otimes R^{(2)} \dashv (r^{(2)} \dashv b) \\ &= (r^{(2)} \dashv b)^{(1)} \otimes r^{(1)} \otimes (r^{(2)} \dashv b)^{(0)} = b_{(0)}^{(1)} \otimes b_{(1)} \otimes b_{(0)}^{(0)}, \end{aligned}$$

and the formula (2.5) is proven.

The following computation:

$$\begin{aligned} b_{1(0)} \otimes b_2^{(0)} \otimes b_2^{(1)} b_{1(1)} &= R^{(2)} \rightarrow b_1 \otimes r^{(2)} \rightarrow b_2 \otimes r^{(1)} R^{(1)} \\ &\stackrel{(QT2)}{=} R^{(1)}_1 \rightarrow b_1 \otimes R^{(1)}_2 \rightarrow b_2 \otimes R^{(1)} \stackrel{(2.2)}{=} b_1 \otimes b_2 \otimes 1, \end{aligned}$$

shows the equation (2.6).

Then, using definitions, and (2.4), we can obtain:

$$\begin{aligned} b_1^{(0)} \otimes b_2 \otimes b_1^{(1)} &= R^{(2)} \rightarrow b_1 \otimes b_2 \otimes R^{(1)} \\ &= (R^{(2)} \rightarrow b)_1 \otimes (R^{(2)} \rightarrow b)_2 \otimes R^{(1)} = b^{(0)}_1 \otimes b^{(0)}_2 \otimes b^{(1)}, \end{aligned}$$

and this checks the (2.7), and similarly, one has the (2.8). It is easy to that $\bar{S}(b) = b^{(2)} \rightarrow S_B(b^{(1)}) = \tau(b_4, R^{(2)})\tau(S^2(b_3)S(b_1), R^{(1)})b_2$.

In final, it is not hard to check the condition (B) hold, concluding the proof.

For a quasitriangular Hopf algebra H . Let (B, H, τ) be a Hopf pairing. Similarly, we can define: $h \rightarrow b = \tau^{-1}(b_1, h)b_2$ and $h \triangleright b = \tau^{-1}(Sb_2, h)b_1$ for all $b \in B$, $h \in H$. In ${}_H\mathcal{M}_1$, it is natural that we have $\delta(b) = R^{(2)} \rightarrow b \otimes R^{(1)} \stackrel{def.}{=} b^{(0)} \otimes b^{(1)}$ and $\rho(b) = R^{(2)} \rightarrow b \otimes R^{(1)} \stackrel{def.}{=} b^{(0)} \otimes b^{(1)}$.

Thus, by (2.12)–(2.13) and the proposition 3.2 we have:

$$h \triangleright b = h_1 \rightarrow (h_2 \rightarrow b) = \tau^{-1}(b_1 S b_3, h) b_2 \quad (4.4)$$

$$a \bar{\triangleright} b = a_{(0)}(a_{(1)} \triangleright b) = \tau^{-1}(b_1 S b_3, R^{(1)}) a_1 b_2 \tau^{-1}(S(a_2), R^{(2)}) \quad (4.5)$$

$$\underline{S}(b) = b^{(1)} \triangleright S_B(b^{(0)}) = \tau^{-1}(b_1, R^{(2)})(R^{(1)} \triangleright S_B(b_2)). \quad (4.6)$$

We now have the following

THEOREM 4.2. *Let H be quasitriangular. With the notation above. Then there is a braided Hopf algebra \underline{B} in ${}_H\mathcal{M}_1$, $\underline{B} = B$ as a linear space and as an object in ${}_H\mathcal{M}_1$ with a H -module structure defined by (4.4). The coalgebra structure and unit in \underline{B} coincide with those of B . The multiplication is given by (4.5). Its antipode is defined by (4.6)*

PROOF. Similar to Theorem 4.1.

Let $(H, \langle | \rangle)$ be coquasitriangular, and let B be a Hopf algebra. Assume that $f : B \rightarrow H$ is a Hopf algebra map. Define $\delta(b) = b_2 \otimes S^{-1}f(b_1) \stackrel{def.}{=}$

$b^{(0)} \otimes b^{(1)}$, and $\rho(b) = b_1 \otimes f(b_2) \stackrel{\text{def.}}{=} b_{(0)} \otimes b_{(1)}$ for $b \in B$. Then we have $h \rightharpoonup b = \langle h | S^{-1}f(b_1) \rangle b_2$ and $h \rightharpoonup b = \langle h | f(b_2) \rangle b_1$ for $h \in H, b \in B$. It is easy to check that $(B^{op}, \rightharpoonup, \delta)$ is an algebra in ${}^H\mathcal{YD}_1^H$, and $(B^{op}, \rightharpoonup, \rho)$ an algebra in ${}^H\mathcal{YD}_2^H$ such that $(B^{op}, \rightharpoonup, \rho)$ is in ${}^H\mathcal{L}^H$, and $(B^{op}, \rightharpoonup, \delta)$ is in ${}^H\mathcal{L}^H$.

Then, by (2.10)–(2.11) and the proposition 3.1, one has

$$\chi_B(b) = b_{(0)}^{(0)} \otimes b_{(1)}b_{(0)}^{(1)} = b_2 \otimes S^{-1}(f(b_1))f(b_3) \tag{4.7}$$

$$a \star b = (b^{(1)} \rightharpoonup a)b^{(0)} = \langle S^{-1}(f(b_1)) | f(a_3)S^{-1}(f(a_1)) \rangle b_2a_2 \tag{4.8}$$

$$\bar{S}(b) = b_{(1)} \rightharpoonup S_B(b_{(0)}) = \langle f(b_4) | f(b_3)f(Sb_1) \rangle S_B(b_2). \tag{4.9}$$

It is easy to show the condition (A) and the condition (B) are satisfied, and so by the theorem 2.3 and the proposition 3.1, we have:

THEOREM 4.3. *Let $(H, \langle | \rangle)$ be coquasitriangular, and B a Hopf algebra. Let $f : B \rightarrow H$ be a Hopf algebra map. Then there is a braided Hopf algebra \bar{B} in \mathcal{M}_2^H , $\bar{B} = B$ as a linear space and as an object in \mathcal{M}_2^H with a H -comodule structure defined by (4.7). The coalgebra structure and counit in \bar{B} coincide with those of B . The multiplication is given by (4.8). Its antipode is defined by (4.9).*

Let B be any bialgebra, and $f : H \rightarrow B$ a bialgebra map. If f is a convolution invertible map with an inverse f^{-1} , then $f^{-1} : H \rightarrow B$ is an anti-bialgebra map, i.e., $f^{-1}(hl) = f^{-1}(l)f^{-1}(h)$ and $\Delta_B f^{-1}(h) = f^{-1}(h_2) \otimes f^{-1}(h_1)$.

EXAMPLE. If H is a Hopf algebra, then $f^{-1} = fS_H$ is a convolution invertible map.

Similarly, let $(H, \langle | \rangle)$ be coquasitriangular, and B a Hopf algebra. Let $B \rightarrow H$ be a Hopf algebra map. Define $\delta(b) = b_2 \otimes Sf(b_1) \stackrel{\text{def.}}{=} b^{(0)} \otimes b^{(1)}$, and $\rho(b) = b_1 \otimes f(b_2) \stackrel{\text{def.}}{=} b_{(0)} \otimes b_{(1)}$ for $b \in B$. Naturally, we get: $h \rightharpoonup b = \langle h | Sf(b_1) \rangle b_2$ and $h \rightharpoonup b = \langle h | f(b_2) \rangle b_1$ for $h \in H, b \in B$. It is easy to check that $(B^{op}, \rightharpoonup, \delta) \in {}^H\mathcal{YD}_1^H$ is an algebra, and $(B^{op}, \rightharpoonup, \rho) \in {}^H\mathcal{YD}_2^H$ an algebra such that $(B^{op}, \rightharpoonup, \rho)$ is in ${}^H\mathcal{L}^H$, and $(B^{op}, \rightharpoonup, \delta)$ is in ${}^H\mathcal{L}^H$.

Thus, by (2.13)–(2.14) and the proposition 3.2, one has

$$\xi_B(b) = b_{(0)}^{(0)} \otimes b_{(0)}^{(1)}b_{(1)} = b_2 \otimes S(f(b_1))f(b_3) \tag{4.10}$$

$$a \bar{\star} b = a_{(0)}(a_{(1)} \succ b) = \langle S(f(b_1)) | f(a_3)S(f(a_1)) \rangle b_2a_2 \tag{4.11}$$

$$\underline{S}(b) = b^{(1)} \succ S_B(b^{(0)}) = \langle f(b_1) | f(b_2)f(Sb_4) \rangle S_Bb_3. \tag{4.12}$$

In final, it is not hard to the conditions (2.1)–(2.2), (2.4)–(2.5), (2.7)–(2.8),

(2.15), (2.16) and the condition (C) are satisfied. By the theorem 2.4 and the proposition 3.2, we have:

THEOREM 4.4. *Let $(H, \langle | \rangle)$ be coquasitriangular and B a Hopf algebra. Let $f : B \rightarrow H$ be a Hopf algebra map. Then there is a braided Hopf algebra \underline{B} in \mathcal{M}_1^H , $\underline{B} = B$ as a linear space and as an object in \mathcal{M}_1^H with a H -comodule structure defined by (4.10). The coalgebra structure and counit in \underline{B} coincide with those of B . The multiplication is given by (4.11). Its antipode is defined by (4.12).*

By the theorem 4.1, we give an example explicitly as follows.

REMARK 4.5. If (H, R) is a quasitriangular Hopf algebra, then H^{cop} is a Hopf algebra, which has the quasitriangular structure $R^t = R^{(2)} \otimes R^{(1)}$. The braided category ${}_H\mathcal{M}_1$ is identified with ${}_{H^{cop}}\mathcal{M}_2$, and hence with $\mathcal{M}_{H^{bop2}}$, the 2nd kind (or the natural) braided category of right modules over $H^{bop} := (H^{op})^{cop}$. In addition, if $\tau : B \otimes H \rightarrow k$ is a Hopf pairing, then $\tau^{-1} : B \otimes H^{bop} \rightarrow k$ is a Hopf pairing as shown by $(DP1)'$ and $(DP2)'$. Therefore, it is not hard to check that Theorem 4.2 follows from a variation of Theorem 4.1 which gives a construction of Hopf algebras in \mathcal{M}_{H2} .

A similar investigation and verification would be given for Theorem 4.3 and 4.4.

EXAMPLE 4.6. Let H_4 be the Sweedler's 4-dimensional Hopf algebra, i.e., H_4 is a free k -module with basis $1, x, y, z$ and its Hopf algebra structure is defined by

$$\begin{aligned} x^2 &= 1, & y^2 &= 0, & xy &= z, & xz &= -zx = y \\ \Delta(x) &= x \otimes x, & \Delta(y) &= y \otimes x + 1 \otimes y, & \Delta(z) &= z \otimes 1 + x \otimes z \\ \varepsilon(x) &= 1, & \varepsilon(y) &= \varepsilon(z) = 0, & S(x) &= x, & S(y) &= z, & S(z) &= -y \end{aligned}$$

For any $\alpha \in k$. Then (H_4, σ_α) has uniquely coquasitriangular Hopf algebra, where σ_α is given via:

σ_α	1	x	y	z
1	1	1	0	0
x	1	-1	0	0
y	0	0	α	$-\alpha$
z	0	0	α	α

Note that H_4 is also a quasitriangular Hopf algebra with

$$R_\alpha = 1/2(1 \otimes 1 + 1 \otimes x + x \otimes 1 - x \otimes x) + \alpha/2(y \otimes y + y \otimes z + z \otimes z - z \otimes y).$$

Then two actions H_4 on H_4^{cop} is respectively defined by:

\rightarrow	1	x	y	z		\rightarrow	1	x	y	z
1	1	x	y	z	and	1	1	x	y	z
x	1	$-x$	$-y$	z		x	1	$-x$	y	$-z$
y	0	0	$\alpha 1$	$-\alpha x$		y	0	0	αx	$\alpha 1$
z	0	0	$\alpha 1$	αx		z	0	0	$-\alpha x$	$\alpha 1$

Thus, by the formula (4.1), the H_4^{cop} is a left H_4 -module where the H_4 -module structure is given by:

\rightarrow	1	x	y	z
1	1	x	y	z
x	1	x	$-y$	$-z$
y	0	0	$\alpha(x+1)$	$\alpha(x+1)$
z	0	0	$\alpha(x+1)$	$\alpha(x+1)$

By the equation (4.2), the multiplication on H_4^{cop} is obtained by the following table:

\star	1	x	y	z
1	1	x	y	z
x	x	1	z	y
y	y	z	$\alpha^3(x+1)$	$\alpha^3(x+1)$
z	z	y	$\alpha^3(x+1)$	$\alpha^3(x+1)$

Therefore, by the theorem 4.1, the $(H_4, \Delta_{H_4}^{cop}, \star)$ is a braided Hopf algebra in $H_4\mathcal{M}_2$. Its antipode is defined by

$$\bar{S}(1) = 1, \quad \bar{S}(x) = x, \quad \bar{S}(y) = z, \quad \bar{S}(z) = y.$$

Similarly, applying the theorem 4.2, we have

EXAMPLE 4.6. Let H_4 be the Sweedler's 4-dimensional Hopf algebra. Then the $(H_4, \Delta_{H_4}^{cop}, \bar{\star})$ is a braided Hopf algebra in $H_4\mathcal{M}_2$. Its antipode is defined by

$$\underline{S}(1) = 1, \quad \underline{S}(x) = x, \quad \underline{S}(y) = -z, \quad \underline{S}(z) = -y.$$

The H_4 -module structure and the multiplication on H_4^{cop} is given respectively by the following tables:

γ	1	x	y	z		$\bar{\kappa}$	1	x	y	z
1	1	x	y	z	and	1	1	x	y	z
x	1	x	$-y$	$-z$		x	x	1	$-z$	$-y$
y	0	0	$\alpha(x+1)$	$-\alpha(x+1)$		y	y	$-z$	$\alpha^3(x+1)$	$-\alpha^3(x+1)$
z	0	0	$\alpha(x+1)$	$-\alpha(x+1)$		z	z	$-y$	$-\alpha^3(x+1)$	$\alpha^3(x+1)$

REMARK. In the example 4.6, our two H_4 -module structures associated to the H_4 -module structure \succ is given by respectively:

\rightarrow	1	x	y	z		\rightarrow	1	x	y	z
1	1	x	y	z	and	1	1	x	y	z
x	1	$-x$	$-y$	z		x	1	$-x$	y	$-z$
y	0	0	$\alpha 1$	$-\alpha x$		y	0	0	$-\alpha x$	$-\alpha 1$
z	0	0	$\alpha 1$	αx		z	0	0	αx	$-\alpha 1$

Acknowledgment

The author would like to thank Professor M. Takeuchi for the valuable suggestions. I am indebted to the referees for valuable comments, in particular Remark 2.5 and 4.5.

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