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SKEW DERIVATIONS OF FINITE-DIMENSIONAL ALGEBRAS AND ACTIONS OF THE DOUBLE OF THE TAFT HOPF ALGEBRA

By

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1. Introduction

In this paper, k is a field containing a primitive n-th root of unity ω, where $n > 1$. We first study skew derivations of certain finite-dimensional k-algebras, in particular the algebra $A = k[u \mid u^n = \beta]$, for $\beta \in k$. Studying these skew derivations is equivalent to studying the actions of the n^2-dimensional Taft Hopf algebra $H = T_{n^2}(\omega)$ [T] on such an algebra A, where

$$H = k\langle g, x \mid g^n = 1, x^n = 0, xg = \omega g x\rangle, \tag{1.1}$$

$g \in G(H)$, and $\Delta(x) = x \otimes 1 + g \otimes x$, $\varepsilon(x) = 0$.

As an application, we show that actions of $u_q(sl_2)$ on A are determined by the action of a Borel subalgebra. This result gives a quantum analog of classical work of Jacobson on the Witt algebra in characteristic $p \neq 0$. It also gives an analog for q a root of 1 of [MSm] on actions of $U_q(sl_2)$ in the generic case.

We then determine the actions of the Drinfel'd double $D(H)$ on A, and as a consequence, those actions of H on A which give Yetter-Drinfel'd structures, answering a question from [CFM].

More specifically, in Section 2 we study the actions of H. First note that H is the (Hopf) homomorphic image of the infinite-dimensional Hopf algebra

$$\tilde{H} := k\langle g, \tilde{x} \mid g^n = 1 \text{ and } \tilde{x}g = \omega g \tilde{x}\rangle, \tag{1.2}$$

where $g \in G(\tilde{H})$ and $\Delta(\tilde{x}) = \tilde{x} \otimes 1 + g \otimes \tilde{x}$.

By definition, H-module algebras are the $\tilde{H}$-module algebras where $\tilde{x}^n$ acts trivially. If A is a left $\tilde{H}$-module algebra, the action of $\tilde{H}$ is called non-trivial if the skew-derivation $\tilde{x}\cdot$ of A is non-zero.

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Our main results on $\tilde{H}$-actions are:

(I) We consider non-trivial actions of $\tilde{H}$ on an n-dimensional algebra $A = k(u)$ such that u is an eigenvector under the action of g. All such actions are completely determined by the action of $\tilde{x}$ on u, and the possible values are exactly $\tilde{x} \cdot u = \delta u^s$, $0 \neq \delta \in k$, $0 \leq s \leq n-1$, $1-s$ invertible mod n.

In this case, $A = k[u \mid u^n = \beta]$ for some $\beta \in k$, and $\tilde{x}^n$ acts trivially, that is A is an H-module algebra (see Theorem 2.2).

(II) If A is an arbitrary algebra with no non-zero nilpotent elements, then for any non-trivial $\tilde{H}$-action on A, also $\tilde{x}^{n-1}$ acts non-trivially, and each power ω^i, $0 \leq i \leq n-1$, is an eigenvalue of the action of g. Moreover if A is n-dimensional, we are in the situation of (I) (see Theorem 2.3).

In Section 3 we extend (I) and (II) to describe the actions of $u_q(sl_2)$ on $A = k[u]$, for q a $2n^{th}$ root of 1; here $u_q(sl_2) = k[E, F, K]$ is the finite-dimensional quotient of $U_q(sl_2)$ defined by Lusztig. The actions are completely determined by the action of a Borel subalgebra, that is by the action of E (or F) on u (see Theorem 3.1). As noted above, this result can be viewed as a quantum analog of classical work of Jacobson [J] on restricted Lie algebras (see Remark 3.3); it is also the analog at a root of 1 of [MSm], which described actions of $U_q(sl_2)$ on the polynomial ring $A = \mathbb{C}[X]$, when q was not a root of 1 (see Remark 3.4).

As an application we construct smash products $A\sharp u_q(sl_2)$ which are not semisimple, and where A is a commutative field such that the action of $u_q(sl_2)$ is outer on A (that is, the action of any non-trivial Hopf subalgebra of $u_q(sl_2)$ acts non-trivially on A, hence is not inner). Thus the old result of Azumaya which says that if $H = kG$ is the group algebra of a finite group acting on an algebra A with outer G-action, then $A\sharp H$ is simple, does not generalize in a straightforward way to actions of pointed Hopf algebras. Our examples seem to be the first of this type where the smash product is not even semisimple.

In Section 4, we show that any action of H on $A = k[u \mid u^n = \beta]$ as in (I) extends uniquely to an action of the Drinfel'd double $D(H)$. As a consequence, in Section 5 we see that for each H-action there is a unique left H-comodule algebra structure on A, such that A is a Yetter-Drinfel'd algebra over H. In particular, if $g \cdot u = \omega u$ and $x \cdot u = \gamma 1$ for some $\gamma \in k$, then the unique left H-comodule algebra structure on A is given by the explicit formula

$$\rho(u) = \sum_{m=0}^{n-1} \gamma^{-m}(\omega - 1)^m \omega^{m(m+1)/2} x^m g^{-(m+1)} \otimes u^{m+1},$$

answering a question in [CFM, Remark 2.9].

For general references on Hopf algebras, see [M] or [K]. Throughout H denotes a Hopf algebra with comultiplication $\Delta : H \otimes H \to H$, counit $\varepsilon : H \to k$, and antipode S. We use the usual summation notation for Δ and denote the inverse of S (under compostion) by $\bar{S}$.

2. Skew Derivations and Actions of the Taft Algebra

In this section we determine the skew derivations associated to certain automorphisms of algebras $A = k(u)$, and then apply these results to the actions of the Taft algebras. Recall that for a given automorphism σ of an algebra A, D is called a *σ-skew derivation* of A if for all $a, b \in A$,

$$D(ab) = D(a)b + \sigma(a)D(b).$$

We usually write $\sigma(a) = \sigma \cdot a$ and $D(a) = D \cdot a$.

We first need a technical lemma.

LEMMA 2.1. *Let $A = k[u \mid u^n = \beta]$, for some $\beta \in k$, and let μ be a primitive n^{th} root of 1 in k. Choose $0 \neq \delta \in k$ and $0 \leq s, t \leq n-1$ such that $t(1-s) \equiv 1 \pmod{n}$. Let $\sigma \in Aut_k(A)$ be given by $\sigma \cdot u = \alpha u$, where $\alpha = \mu^t$. Define D on u by $D \cdot u = \delta u^s$. Then D extends to a σ-skew derivation of A satisfying $D\sigma = \mu\sigma D$ and $D^n = 0$. Moreover $D^{n-1} \neq 0 \Leftrightarrow \beta \neq 0$.*

In addition the following identities hold, for all p with $0 \leq p \leq n-1$:

(a) $D \cdot u^p = \delta(\sum_{i=0}^{p-1} \alpha^i)u^{p+s-1}$;

(b) $D^q \cdot u^p = \delta^q(\sum_{i_1=0}^{p-1} \alpha^{i_1})(\sum_{i_2=0}^{(p-1)+(s-1)} \alpha^{i_2}) \cdots (\sum_{i_q=0}^{(p-1)+(q-1)(s-1)} \alpha^{i_q})u^{p+q(s-1)}$

(c) *Let q_0 be the (unique) q such that $1 \leq q_0 < n$ and $q_0 \equiv pt \pmod{n}$. Then $D^q \cdot u^p = 0$ for $q_0 < q \leq n$. If $\beta \neq 0$, then $D^q \cdot u^p \neq 0$ for $1 \leq q \leq q_0 < n$; if $\beta = 0$, then $D^{q_0} \cdot u^p = 0$.*

PROOF. Clearly σ acts as an automorphism of A. To see that D acts as a σ-skew derivation, where in fact $D \cdot u^p$ is as in (a), it suffices to show that the map $\Phi : A \to M_2(A)$ given by $\Phi(a) = \begin{pmatrix} \sigma \cdot a & D \cdot a \\ 0 & a \end{pmatrix}$ is an algebra map. Since the only relation in A is $u^n = \beta$, it suffices to check that $\Phi(u)^n = \beta I$. We set $M := \Phi(u) = \begin{pmatrix} \alpha u & \delta u^s \\ 0 & u \end{pmatrix}$. It is easy to check by induction that

$$\Phi(u)^p = M^p = \begin{pmatrix} \alpha^p u^p & \delta(\sum_{i=0}^{p-1} \alpha^i)u^{p+s-1} \\ 0 & u^p \end{pmatrix}$$

Thus $\Phi(u)^n = \beta I$ since $\alpha^n = 1$ implies $\sum_{i=0}^{n-1} \alpha^i = 0$. Moreover formula (a) is also clear from this computation.

Since D acts as a σ-skew derivation, also $\sigma^{-1}D\sigma$ is a σ-skew derivation. Now by (a), for all p,

$$(\sigma^{-1}D\sigma)\cdot u^p = \alpha^p\sigma^{-1}(D\cdot u^p) = \alpha^p\delta\left(\sum_{i=0}^{p-1}\alpha^i\right)\sigma^{-1}\cdot u^{p+s-1}$$
$$= \alpha^{p-(p+s-1)}\delta\left(\sum \alpha^i\right)u^{p+s-1} = \mu D\cdot u^p$$

since $\alpha^{1-s} = \mu$, using (a). Thus $\sigma^{-1}D\sigma = \mu D$ on all of A, which is the desired relation between σ and D.

To see (b) and (c), we fix p and proceed by induction on q; note that (a) is the case $q = 1$. Assume the result is true for $q - 1$. Then

$$D^q\cdot u^p = D^{q-1}\cdot(D\cdot u^p) = \delta\left(\sum_{i=0}^{p-1}\alpha^i\right)D^{q-1}\cdot u^{p+s-1}$$
$$= \delta\left(\sum_{i=0}^{p-1}\alpha^i\right)\delta^{q-1}\left(\sum_{j_1=0}^{(p+s-2)}\alpha^{j_1}\right)\cdots\left(\sum_{j_{q-1}=0}^{(p+s-2)+(q-2)(s-1)}\alpha^{j_{q-1}}\right)u^{(p+s-1)+(q-1)(s-1)}$$
$$= \delta^q\left(\sum_{i_1=0}^{p-1}\alpha^{i_1}\right)\left(\sum_{i_2=0}^{(p-1)+(s-1)}\alpha^{i_2}\right)\cdots\left(\sum_{i_q=0}^{p-1+(q-1)(s-1)}\alpha^{i_q}\right)u^{p+q(s-1)}$$

which is the desired form. However, the difficulty with this formal computation is that it may equal zero for some q's. To see when this happens, we first consider the set of possible exponents of u, namely $\{p + q(s-1) \mid 0 \le q \le n-1\}$.

Since $s - 1$ is invertible mod n and p is fixed, this set contains n distinct elements mod n. Thus for exactly one q, $p + q(s-1) \equiv 0 \pmod{n}$, namely for $q = q_0 \equiv pt \pmod{n}$. For this q_0, $u^{p+q_0(s-1)} = \beta^b$ for some b; thus $D^{q_0}\cdot u^p = \delta^{q_0}\lambda_{q_0,p}\beta^b$, where $\lambda_{q,p}$ is the product of the summation terms.

We must also consider the coefficients $\lambda_{q,p}$. A given summation term $\sum_{i_r=0}^{(p-1)+(r-1)(s-1)}\alpha^{i_r}$ equals 0 only if the number of summands is divisible by n; equivalently $p + (r-1)(s-1) \equiv 0 \pmod{n}$. By the previous computation, this means that $r - 1 = q_0$, or that $r = q_0 + 1$. Thus $\lambda_{q,p} \ne 0$ for $q \le q_0$, and $\lambda_{q_0+1,p} = 0$. Consequently $D^q\cdot u^p = 0$ for $q_0 < q \le n$, and so $D^n\cdot A = 0$.

Now if $\beta = 0$, then $D^{q_0}\cdot u^p = 0$; in particular $D^{n-1} = 0$. However if $\beta \ne 0$, then $D^q\cdot u^p \ne 0$ for $q \le q_0$. This finishes the proof. □

THEOREM 2.2. *Assume that $A = k(u)$ is an n-dimensional k-algebra. Let σ be an automorphism of A such that $\sigma^n = id$ and $\sigma \cdot u = \alpha u$, for some $\alpha \in k$, and let D be a non-trivial σ-skew derivation of A such that $D\sigma = \mu\sigma D$, for μ a primitive n^{th} root of 1. Then there exists $\beta, \delta \in k$ with $\delta \neq 0$ and $0 \leq s, t \leq n-1$ with $t(1-s) \equiv 1 \pmod{n}$ such that*

(a) $u^n = \beta$

(b) $\alpha = \mu^t$ *and σ has order n in* $\mathrm{Aut}_k(A)$;

(c) $D \cdot u = \delta u^s$.

(d) $D^n = 0$, *and* $D^{n-1} \neq 0 \Leftrightarrow \beta \neq 0$.

Conversely, if $u^n = \beta \in k$, then for any such choice of $0 \neq \delta \in k$ and s, t with $t(1-s) \equiv 1 \pmod{n}$, there is an automorphism σ of A given by $\sigma \cdot u = \mu^t u$ and a σ-skew derivation of A given by $D \cdot u = \delta u^s$ satisfying $D\sigma = \mu\sigma D$ and $D^n = 0$.

PROOF. First note that since $\dim(A) = n$, the elements $\{1, u, \ldots, u^{n-1}\}$ are a basis for A. Thus for some $\delta_i \in k$, $D \cdot u = \sum_{i=0}^{n-1} \delta_i u^i$.

Now $D\sigma = \mu\sigma D$ implies that $\alpha(\sum_i \delta_i u^i) = \mu(\sum_i \delta_i \alpha^i u^i)$, and so $\delta_i(\alpha^{1-i} - \mu) = 0$ for all i. Since $D \cdot u \neq 0$, we must have $\delta_s \neq 0$ for some s, and so $\mu = \alpha^{1-s}$. Since $\sigma^n = id$, $\alpha^n = 1$ and so $1 - s$ is relatively prime to n. Thus $o(\alpha) = n$ and σ has order n in $\mathrm{Aut}_k(A)$. Moreover if t is the inverse of $1 - s \bmod n$, then $\alpha = \mu^t$.

Also $\mu = \alpha^{1-s} = \alpha^{1-i}$ is impossible if $i \neq s$, and so $\delta_i = 0$ if $i \neq s$. Thus $D \cdot u = \delta u^s$, where $\delta = \delta_s$.

We claim that $u^n = \beta$ for some $\beta \in k$. To see this, let $f(x) = x^n + \alpha_{n-1}x^{n-1} + \cdots + \alpha_0$, where all $\alpha_i \in k$, be the minimum polynomial of u. Then $f(\sigma(u)) = f(u) = 0$. Since α is a primitive n^{th} root of 1, it follows that $\alpha_1, \ldots, \alpha_{n-1} = 0$. Thus $f(x) = x + \alpha_0$; now set $\beta = -\alpha_0$.

Part (d) follows from Lemma 2.1 (c), and the converse statement also follows from Lemma 2.1. □

We see next that when A has no nilpotent elements and is n-dimensional, we may assume that A is of the form $A = k(u)$ where $\sigma \cdot u = \alpha u$, and so Theorem 2.2 applies.

THEOREM 2.3. *Let A be an arbitrary k-algebra with no non-zero nilpotent elements and assume that $\sigma \in \mathrm{Aut}_k(A)$ such that $\sigma^n = id$. Let D be a non-trivial σ-skew derivation of A such that $D\sigma = \mu\sigma D$, where μ is a primitive n^{th} root of 1. Then $D^{n-1} \neq 0$, and all the powers μ^i, $0 \leq i \leq n-1$, are eigenvalues of σ.*

If moreover A is n-dimensional, then there exist $u \in A$ and $0 \neq \beta$, $\gamma \in k$ such that

(a) $A = k(u)$ *and* $u^n = \beta \in k$;

(b) $\sigma \cdot u = \mu u$, *and so* σ *has order* n *in* $\mathrm{Aut}_k(A)$;

(c) $D \cdot u = \gamma 1$, $D^{n-1} \neq 0$, *and* $D^n = 0$.

In particular A *is commutative.*

Moreover if $u' \in A$ *and* β', $\gamma' \in k$ *also satisfy* (a), (b), *and* (c), *then for some* $0 \neq \nu \in k$, *we have* $u' = \nu u$, $\beta' = \nu^n \beta$, *and* $\gamma' = \nu\gamma$.

Before proving the theorem, we first make some preliminary observations. Since $\sigma^n = id$ and k contains a primitive n^{th} root of 1, σ can be diagonalized in $\mathrm{End}_k(A)$. Thus

$$A = \bigoplus_{i=0}^{n-1} A_i,$$

where $A_i := \{a \in A \mid \sigma \cdot a = \mu^i a\}$. Note that $D \cdot A_i \subseteq A_{i-1}$, for all i. For, if $a \in A_i$, then $\sigma \cdot (D \cdot a) = \mu^{-1} D \cdot (\sigma \cdot a) = \mu^{i-1}(D \cdot a)$; thus $D \cdot a \in A_{i-1}$.

LEMMA 2.4. *Let* $a \in A_i$ *such that* $D \cdot a \neq 0$ *but* $D^2 \cdot a = 0$. *Then for all* m *with* $1 \leq m < n$,

(i) $D^m \cdot a^m = \gamma_m (D \cdot a)^m$, *for some* $0 \neq \gamma_m \in k$;

(ii) $D^m \cdot a^{m-1} = 0$.

Consequently $D^{n-1} \cdot A \neq 0$ *and* $A_j \neq 0$ *for all* j.

PROOF. By induction on m. The case $m = 1$ is given, as is the case $m = 2$ for (ii). We first show that if (i) is true for $m - 1$ and m, and (ii) is true for m, then (ii) is true for $m + 1$. Now

$$\begin{aligned} D^{m+1} \cdot a^m &= D \cdot (D^m \cdot a^m) = \gamma_m D \cdot (D \cdot a)^m \\ &= \gamma_m[(D \cdot (D \cdot a))(D \cdot a)^{m-1} + (\sigma \cdot (D \cdot a))(D \cdot (D \cdot a)^{m-1})] \\ &= \gamma_m[(D^2 \cdot a)(D \cdot a)^{m-1}) + (\sigma \cdot (D \cdot a))\gamma_{m-1}^{-1}(D \cdot (D^{m-1} \cdot a^{m-1}))] \\ &= 0. \end{aligned}$$

Here we have used (i) for m in the first line, (i) for $m - 1$ in the next to last line, $D^2 \cdot a = 0$, and (ii) for m to see that $D \cdot (D^{m-1} \cdot a^{m-1}) = D^m \cdot a^{m-1} = 0$.

We now show that if (i) is true for m and (ii) is true for $m + 1$, then (i) is true for $m + 1$. Since D is a σ-derivation with $D\sigma = \mu\sigma D$, for $b, c \in A$ we have

$$D^p(bc) = \sum_{j=0}^{p} \binom{p}{j}_{\mu} \sigma^j D^{p-j}(b) D^j(c)$$

where $\binom{p}{j}_\mu$ is the q-binomial coefficient with $q = \mu$. Thus

$$\begin{aligned} D^{m+1} \cdot a^{m+1} &= \sum_{j=0}^{m+1} \binom{m+1}{j}_\mu \sigma^j (D^{m+1-j} \cdot a^m)(D^j \cdot a) \\ &= \binom{m+1}{0}_\mu (D^{m+1} \cdot a^m) a + \binom{m+1}{1}_\mu \sigma(D^m \cdot a^m)(D \cdot a) \text{ since } D^2 \cdot a = 0 \\ &= \binom{m+1}{1}_\mu \sigma(D^m \cdot a^m)(D \cdot a) \text{ since } D^{m+1} \cdot a^m = 0 \text{ by (ii) for } m+1 \\ &= \binom{m+1}{1}_\mu \gamma_m \sigma(D \cdot a)^m (D \cdot a) \text{ by (i) for } m \\ &= \gamma_{m+1} (D \cdot a)^{m+1}, \end{aligned}$$

where $\gamma_{m+1} = \binom{m+1}{1}_\mu \gamma_m \mu^{m(i-1)} \neq 0$, since $D \cdot a \in A_{i-1}$.

This proves parts (i) and (ii).

Now if $D^{n-1} \cdot A = 0$, we may choose a such that $D \cdot a \neq 0$ but $D^2 \cdot a = 0$, where $a \in A_i$ for some i. By (i) above, $D^m \cdot a^m = \gamma_m (D \cdot a)^m$, for all $m < n$. If $m = n - 1$, then $D^{n-1} \cdot a^{n-1} = 0 = \gamma_{n-1}(D \cdot a)^{n-1}$, with $\gamma_{n-1} \neq 0$. But then $(D \cdot a)^{n-1} = 0$. Since A has no non-zero nilpotent elements, $D \cdot a = 0$, a contradiction. Thus $D^{n-1} \cdot A \neq 0$.

We may thus choose $a \in A_i$, for some i, such that $D^{n-1} \cdot a \neq 0$. Now $0 \neq D^j \cdot a \in A_{i-j}$, for all $j = 0, \ldots, n-1$, and so $A_j \neq 0$ for all j. This proves the lemma. □

To finish the proof of Theorem 2.3 we can now assume that A is n-dimensional.

Proof. By Lemma 2.4, A_j is 1-dimensional for all j. Choose any $0 \neq u \in A_1$. Then $\sigma \cdot u = \mu u$, and $\sigma \cdot u^i = \mu^i u^i$. All $u^i \neq 0$ since A has no nilpotent elements, and thus the elements $\{1, u, \ldots, u^{n-1}\}$ are a k-basis for A. In particular $A = k(u)$ and $A_0 = k1$. Now $u^n \in A_n = A_0$ and so $u^n = \beta \in k$. Note here that any such u must be in A_1, as it is a μ-eigenvector for σ, and so is unique up to a scalar multiple.

It remains only to show part (c). Since $A_0 = k1$, $D \cdot u = \gamma 1$ for some $\gamma \in k$, and $D^n = 0$. By Lemma 2.4, $D^{n-1} \neq 0$. □

We now rephrase the above theorems is terms of actions of the Taft algebras on A. Recall the definitions of $H = T_{n^2}(\omega)$ and $\tilde{H}$ in (1.1), (1.2).

THEOREM 2.5. *Let A be an n-dimensional k-algebra with no non-zero nilpotent elements, and assume that A is an H-module algebra such that $\tilde{x} \cdot A \neq 0$. Then there exists $u \in A$ and $0 \neq \beta, \gamma \in k$ such that $A = k(u)$, $u^n = \beta$, $g \cdot u = \omega u$, and $\tilde{x} \cdot u = \gamma 1$. Moreover A is also an H-module algebra.*

If for some $v \in A$, also $A = k(v)$ and $g \cdot v = \alpha v$ for some $\alpha \in k$, then $\alpha = \omega^t$ for some t relatively prime to n and $\tilde{x} \cdot v = \delta v^s$, where $t(1-s) \equiv 1 \pmod{n}$, for some $0 \neq \delta \in k$.

The theorem follows directly from Theorems 2.2 and 2.3, noting that $\tilde{x}^n \cdot u = 0$ gives an induced action of H on A.

We close this section with an example to show that the assumption of no non-zero nilpotent elements is essential in Theorem 2.5 (and in Theorem 2.3) in order to show $A = k(u)$.

EXAMPLE 2.6. (1) Inner actions of the Taft Hopf algebra $T_{n^2}(\omega)$ on an arbitrary algebra A are given as follows. Let $g \in A$ be an invertible element and $\sigma : A \to A$, $\sigma(b) = gbg^{-1}$ for all $b \in A$, the inner automorphism defined by g. Then the order of σ is the smallest integer $m \geq 1$ such that g^m is central in A. For an element $a \in A$, let $D : A \to A$ be the inner skew-derivation defined by a and σ, that is $D(b) = ab - \sigma(b)a$ for all $b \in A$. D is non-zero if and only if $g^{-1}a$ is not central in A.

Assume that $\sigma(a) = \omega^{-1}a$. Then $D\sigma = \omega\sigma D$, and if n is odd, σ has order n and a^n is central, then $D^n = 0$.

PROOF. The equality $D\sigma = \omega\sigma D$ follows easily. Write $D = L_a - R_a\sigma \in \operatorname{End} A$, where L_a, R_a denote left and right multiplication by a. Then $\omega(R_a\sigma)L_a = L_a(R_a\sigma)$, and we see from the q-bimomial formula that for all $b \in A$,

$$D^n(b) = L_a^n(b) + (-R_a\sigma)^n(b) = a^n b + (-1)^n \omega^{-n(n-1)/2} \sigma^n(b) a^n = 0. \qquad \square$$

(2) All the assumptions in (1) can be realized for matrix rings $A = M_r(k)$. Assume $n > 2$ is odd. Let e_{ij}, $1 \leq i, j \leq r$, be the matrix units of A. Take $g = \omega e_{11} + \sum_{i=2}^r e_{ii}$, and $a = e_{1r}$. Then the inner automorphism σ defined by g has order n, $\sigma(a) = \omega a$, $a^n = 0$ is central, and $g^{-1}a = \omega^{-1}a$ is not central. Thus $T_{n^2}(\omega)$ acts on A with non-zero skew-derivation $x\cdot$. However $A \neq k(u)$ whenever $r > 1$. Hence we see (with $n = r^2$) that Theorem 2.5 is not true if A has non-zero nilpotent elements.

3. An Application to Frobenius-Lusztig Kernels

We consider the special case of the Frobenius-Lusztig kernel associated to $U_q(sl_2)$, when q is a primitive $2n^{th}$ root of 1, for $n > 2$; it is a finite-dimensional quotient of $U_q(sl_2)$. See [K, IV.5.6]. This Hopf algebra is also called the *restricted* quantum enveloping algebra of sl_2. Specifically,

$$u_q = u_q(sl_2) := k\langle E, F, K\rangle$$

where E, F, K satisfy the relations $E^n = 0$, $F^n = 0$, $K^n = 1$, and

$$KE = q^2 EK, \quad KF = q^{-2} FK, \quad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$$

with coaction $\Delta E = E \otimes K + 1 \otimes E$, $\Delta F = F \otimes 1 + K^{-1} \otimes F$, and $\Delta K = K \otimes K$.

Note that u_q is generated by two different copies of the Taft algebra as in (1.1), although with two different choices for ω. Namely, $H_1 := k\langle K^{-1}, F\rangle \cong T_{n^2}(q^{-2})$ and $H_2 := k\langle K^{-1}, EK^{-1}\rangle \cong T_{n^2}(q^2)$. One can think of H_1 as b^-, the "Borel" subalgebra of u_q, and similarly of H_2 as b^+. We can now apply our results from Section 2 to u_q.

In the spirit of Theorem 2.2, we give the form of all possible actions of $u_q(sl_2)$ on $A = k(u)$ so that the action of K stabilizes the space ku.

THEOREM 3.1. *Let $A = k(u)$ be an n-dimensional k-algebra, for $n > 2$. Assume that A is a $u_q(sl_2)$-module algebra such that $K \cdot u = \alpha^{-1} u$, for some $\alpha \in k$, and that $F \cdot u \neq 0$ or $E \cdot u \neq 0$. Then there exist $\beta, \gamma, \delta \in k$ with $\gamma, \delta \neq 0$ and $0 \leq s, t, l \leq n-1$ with $t(1-s) \equiv 1 \pmod n$ and $s + l \equiv 2 \pmod n$ such that*

(a) *$u^n = \beta \in k$, $\alpha = q^{-2t}$ and K has order n in* Aut(A);

(b) *$F \cdot u = \gamma u^s$ and $E \cdot u = \delta u^l$;*

(c) *If $s = 0$ and $l = 2$, or $s = 2$ and $l = 0$, then $\delta\gamma = -q$. If both $s, l \geq 2$, then $s + l = n + 2$, $\beta \neq 0$, and*

$$\delta\gamma = \frac{\alpha^{s-2}(1-\alpha)}{\beta(q - q^{-1})(\sum_{j=0}^{s-2} \alpha^j)} = \frac{1 - \alpha^{-1}}{\beta(q - q^{-1})(\sum_{i=0}^{l-2} \alpha^i)}.$$

Conversely, any choice of β, γ, δ and s, t, l satisfying the above relations defines an $u_q(sl_2)$-module algebra structure on A.

PROOF. We assume that $F \cdot A \neq 0$; the case $E \cdot A \neq 0$ is similar. Apply Theorem 2.2 to H_1 with $\sigma = K^{-1}$, $D = F$, and $\mu = q^{-2}$ to find $\beta, \gamma \in k$ with $\gamma \neq 0$ and t, s with $t(1-s) \equiv 1 \pmod n$ to see that $u^n = \beta$, $\alpha = q^{-2t}$ and $F \cdot u = \gamma u^s$. This proves (a).

Now since $EF - FE = (K - K^{-1})/(q - q^{-1})$ and α is a primitive n^{th} root of 1, it follows that $(EF - FE) \cdot u \neq 0$, and so also $E \cdot A \neq 0$.

Next, we apply Theorem 2.2 to H_2, with $\sigma = K^{-1}$ but now with $D' = EK^{-1}$ and $\mu = q^2$; this gives us $0 \neq \delta' \in k$ and $0 \leq r, l \leq n-1$ with $\alpha = q^{2r}$ and $D' \cdot u = \delta' u^l$. Since $E = D'K$, it follows that $E \cdot u = \delta u^l$, where $\delta := \alpha^{-1}\delta'$. Moreover $q^{-2t} = q^{2r}$ implies $r \equiv -t \pmod n$ and thus that $s - 1 \equiv 1 - l \pmod n$; equivalently, $s + l \equiv 2 \pmod n$. Thus (b) is proved.

It remains only to check the relation between γ and δ. To see this we use the fact that the actions of $EF - FE$ and $(K - K^{-1})/(q - q^{-1})$ must agree on u. Trivially $(K - K^{-1})/(q - q^{-1}) \cdot u = (\alpha^{-1} - \alpha)/(q - q^{-1})u$, so we consider the action of $EF - FE$. Applying Lemma 2.1 with $\sigma = K^{-1}$ and $D = F$, we see that

$$F \cdot u^l = \gamma\left(\sum_{i=0}^{l-1} \alpha^i\right) u^{l+s-1}.$$

Similarly, use Lemma 2.1 with $\sigma = K^{-1}$ and $D' = EK^{-1}$ to see that

$$E \cdot u^s = \alpha^{1-s}\delta\left(\sum_{j=0}^{s-1} \alpha^j\right) u^{s+l-1}.$$

Combining these, we have

$$\begin{aligned}(EF - FE) \cdot y &= \gamma\delta\left[\alpha^{1-s}\sum_{j=0}^{s-1} \alpha^j - \sum_{i=0}^{l-1} \alpha^i\right] u^{s+l-1} \\ &= \gamma\delta\alpha^{1-s}\left[\sum_{j=0}^{s-1} \alpha^j - \alpha^{s-1}\sum_{i=0}^{l-1} \alpha^i\right] u^{s+l-1} \\ &= \gamma\delta\alpha^{1-s}\left[\sum_{j=0}^{s-2} \alpha^j - \alpha\left(\sum_{i=s-1}^{n-1} \alpha^i\right)\right] u^{s+l-1} \\ &= \gamma\delta\alpha^{1-s}(1+\alpha)\left(\sum_{j=0}^{s-2} \alpha^j\right) u^{s+l-1},\end{aligned}$$

assuming for the moment that $s, l \geq 2$, hence $s + l = n + 2$ and so $u^{s+l-1} = \beta u$. Thus we have

$$\frac{\alpha^{-1} - \alpha}{q - q^{-1}} = \gamma\delta\beta\alpha^{1-s}(1+\alpha)\left(\sum_{j=0}^{s-2} \alpha^j\right).$$

Since $\alpha^{-1} - \alpha = \alpha^{-1}(1 - \alpha^2)$, the first form of $\gamma\delta$ in (c) now follows by cancelling $1 + \alpha$ from both sides (note that $\alpha^2 \neq 1$ implies that $\beta \neq 0$). The second form then follows by noting that $\alpha^{s-1} \sum_{i=0}^{l-2} \alpha^i = -\sum_{j=0}^{s-2} \alpha^j$.

Now if $s = 0$, and so $t = 1$ and $l = 2$, $u^{s+l-1} = u$ and so β does not appear. In this case the second form of the formula for $\gamma\delta$, without the β, makes sense and so, using that now $\alpha = q^{-2t} = q^{-2}$, we see that $\gamma\delta = -q$. The case $s = 2$, $l = 0$ is similar, using $\alpha = q^2$ in the first form of the formula for $\gamma\delta$.

Conversely, suppose β, γ, δ and s, t, l are given as in the theorem. They define actions of H_1 and H_2 on $A = k[u \mid u^n = \beta]$ by Theorem 2.2. It remains to show that A is in fact an $u_q(sl_2)$-module, that is $(EF - FE) \cdot a = ((K - K^{-1})/(q - q^{-1})) \cdot a$ for all $a \in A$. This is true for $a = u$ by the previous argument. Hence it is true for all $a \in A$, since $EF - FE$ and $(K - K^{-1})/(q - q^{-1})$ both act as (σ, σ^{-1})-skew derivations. □

Corollary 3.2. *Let A be an n-dimensional k-algebra with no non-zero nilpotent elements, and assume that A is a $u_q(sl_2)$-module algebra such that $F \cdot A \neq 0$ (or that $E \cdot A \neq 0$). Then there exists $u \in A$ and $0 \neq \beta, \gamma, \delta \in k$ such that*

(a) $A = k(u)$, $u^n = \beta$, *and* $K \cdot u = q^2 u$;

(b) $F \cdot u = \gamma 1$ *and* $E \cdot u = \delta u^2$;

(c) $\gamma\delta = -q$.

Moreover u is unique up to a scalar multiple.

Proof. As before we may assume that $F \cdot A \neq 0$. We apply Theorem 2.5 to H_1 with $\omega = q^{-2}$, $g = K^{-1}$, and $x = F$ to find $u \in A$ and $\beta, \gamma \in k$ such that $A = k(u)$, $K^{-1} \cdot u = q^{-2} u$, and $F \cdot u = \gamma 1$. Theorem 3.1 now applies, as we are in the case $s = 0$, $l = 2$. □

Remark 3.3. Theorem 3.1 gives a quantum analog of classical work of Jacobson [J] on the Witt algebra. See also [Z]. That is, assume that k has characteristic $p \neq 0, 2$, and let $A = k[v \mid v^p = 0]$. We may write $A = k[u \mid u^p = 1]$ by setting $u = v - 1$. Then $\mathrm{Der}_k(a)$ is spanned by all e_i, $i = 0, \ldots, p - 1$, where e_i is determined by setting $e_i \cdot u = u^{i+1}$. One may check that $[e_i, e_j] = (i - j)e_{i+j}$. Thus, setting $h := e_0$, it follows that $[e_i, h] = ie_i$ and that $[e_i, e_{-i}] = 2ih$. Thus $\mathrm{Der}_k(A)$ contains a copy of sl_2 for each $i \neq 0 \bmod p$.

Remark 3.4. Corollary 3.2 also gives an analog of [MSm, 3.8], in which it was shown that when q is not a root of 1, there are essentially two module-algebra actions of $U_q(sl_2)$ on the polynomial ring $\mathbb{C}[X]$. To state this result precisely, we recall the Drinfel'd-Jimbo formulation of $\tilde{U} = U_q(sl_2)$ [Ji], [D]. Assume $k = \boldsymbol{C}$ and that $0 \neq \tilde{q}$ is not a root of 1. Then

$$\tilde{U} := \boldsymbol{C}\langle \tilde{E}, \tilde{F}, \tilde{K}, \tilde{K}^{-1} \rangle$$

where $\tilde{E}, \tilde{F}, \tilde{K}$ satisfy the relations

$$\tilde{K}\tilde{E} = \tilde{q}^2 \tilde{E}\tilde{K}, \quad \tilde{K}\tilde{F} = \tilde{q}^{-2}\tilde{F}\tilde{K}, \quad \tilde{E}\tilde{F} - \tilde{F}\tilde{E} = \frac{\tilde{K}^2 - \tilde{K}^{-2}}{\tilde{q}^2 - \tilde{q}^{-2}}$$

with coaction $\Delta(\tilde{E}) = \tilde{E} \otimes \tilde{K}^{-1} + \tilde{K} \otimes \tilde{E}$, $\Delta(\tilde{F}) = \tilde{F} \otimes \tilde{K}^{-1} + \tilde{K} \otimes \tilde{F}$, and $\Delta\tilde{K} = \tilde{K} \otimes \tilde{K}$.

THEOREM 3.5. [MSm, 3.8] *Assume that the polynomial ring $\boldsymbol{C}[X]$ is a $\tilde{U}$-module algebra. Then there exists $Y \in \boldsymbol{C}[X]$ such that $\boldsymbol{C}[X] = \boldsymbol{C}[Y]$, and one of the following two possibilities holds*:

(a) $\tilde{K} \cdot Y = \tilde{q}^{-2} Y$, $\tilde{E} \cdot Y = \tilde{q}^2 1$, *and* $\tilde{F} \cdot Y = -\tilde{q}^{-2} Y^2$;

(b) $\tilde{K} \cdot Y = \tilde{q}^2 Y$, $\tilde{E} \cdot Y = -\tilde{q}^2 Y^2$, *and* $\tilde{F} \cdot Y = \tilde{q}^{-2} 1$.

Here both $\tilde{E}$ and $\tilde{F}$ act as $(\tilde{K}^{-1}, \tilde{K})$-derivations of $\boldsymbol{C}[Y]$.

We can translate $\tilde{U}$ into the currently more standard definition (as in [K]) of $U_q(sl_2)$ as follows: set $F := \tilde{F}\tilde{K}$, $E := \tilde{K}^{-1}\tilde{E}$, $K := \tilde{K}^{-2}$, and $q := \tilde{q}^{-2}$. It is then straightforward to check that

$$KE = q^2 EK, \quad KF = q^{-2} FK, \quad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$$

and that $\Delta(E) = E \otimes K + 1 \otimes E$ and $\Delta(F) = F \otimes 1 + K^{-1} \otimes F$, as in u_q. After doing this, we see that part (b) of Theorem 3.5 says that $K \cdot Y = q^2 Y$, $E \cdot Y = -qY^2 = \delta Y^2$, and $F \cdot Y = 1 = \gamma 1$. As in Corollary 3.2, $\delta\gamma = -q$.

It is interesting to note that in the generic case, there are two essentially distinct actions, whereas in the root of unity case, there is only one. The case corresponding to part (a) of Theorem 3.5 in our Corollary 3.2 is obtained from (b) by setting $u' := u^{n-1}$. Of course in the generic case, one cannot replace Y by Y^{n-1}.

We close this section with an application to ideals in smash products. When $H = T_{n^2}(\omega)$ acts on A as in Theorem 2.5 and A is a field, it was shown in [CFM] that $A \sharp H$ is semisimple (in fact, the proof there assumed that $\beta = \omega$ and that $\gamma = 1$, but the arguments work similarly for the more general constants). Here the action is outer in the sense that the action of no non-trivial Hopf subalgebra is inner. Hence this result may be viewed as a weak form of classical results on outer actions, as for example Azumaya's theorem, which says that $A\sharp H$ is simple if a group algebra H acts on a simple algebra A and the group action is outer.

One might hope that the same is true for the action of $u_q(sl_2)$ on a field A as in Corollary 3.2. The action is again outer but the smash product $A\sharp u_q(sl_2)$ is not even semisimple, as we will see.

First, u_q is unimodular, with non-zero integral $\Lambda = E^{n-1}F^{n-1}(\sum_{i=0}^{n-1} K^i)$. This can be verified directly; alternatively, it can be seen as a consequence of the fact that u_q is a Hopf image of the Drinfeld double of T_{n^2}, as in 4.7.

Corollary 3.6. *Let A be a $u_q(sl_2)$-module algebra as in Corollary* 3.2. *Then $A\sharp u_q(sl_2)$ has a non-zero nilpotent ideal $I := A\Lambda A$.*

Proof. First, for any $a \in A$, $h \in H$, it is easy to see that $ha\Lambda = (h \cdot a)\Lambda$ and $\Lambda ah = \Lambda(\bar{S}(h) \cdot a)$. It follows that I is an ideal of $A\sharp u_q$. We claim that $\Lambda \cdot A = 0$.

To see this, note that Lemma 2.1(b) implies that $F^{n-1} \cdot A \subseteq k$, since $s = 0$. But then $EF^{n-1} \cdot A = 0$ and so certainly $\Lambda \cdot A = 0$. It now follows that $I^2 = A\Lambda A\Lambda A = A(\Lambda \cdot A)\Lambda A = 0$. □

4. Actions of the Drinfel'd Double $D(H)$

For any finite-dimensional Hopf algebra H, we recall that the Drinfel'd double $D(H) = (H^*)^{cop} \bowtie H$ is given as follows: as a coalgebra, it is simply $H^{*cop} \otimes H$. The algebra structure is more complicated, and for our purposes we use a formula from [R]; see also [M. 10.3.11]. That is, for $f, f' \in H^*$ and $h, h' \in H$,

$$(f \bowtie h)(f' \bowtie h') = \sum f(h_1 \rightharpoonup f' \leftharpoonup \bar{S}h_3) \bowtie h_2 h'.$$

In particular we have

$$(4.1) \qquad (\varepsilon \bowtie h)(f \bowtie 1) = \sum \langle f_3, h_1\rangle\langle f_1, \bar{S}h_3\rangle f_2 \bowtie h_2.$$

For simplicity we write $h = (\varepsilon \bowtie h)$ and $f = (f \bowtie 1)$.

We now specialize to the n^2-dimensional Taft algebra $H = T_{n^2}(\omega)$ as in 1.1. In this case it is known that $H^* \cong H$; thus we may write

$$(4.2) \qquad H^* = k\langle G, X \mid G^n = \varepsilon, X^n = 0, XG = \omega GX\rangle$$

where $\Delta(G) = G \otimes G$, $\Delta(X) = X \otimes \varepsilon + G \otimes X$, $\langle G, 1\rangle = 1$, and $\langle X, 1\rangle = \varepsilon_{H^*}(X) = 0$.

The dual pairing between H and H^* is determined by

$$(4.3) \qquad \langle G, g\rangle = \omega^{-1}, \quad \langle G, x\rangle = 0, \quad \langle X, g\rangle = 0, \quad \text{and} \quad \langle X, x\rangle = 1.$$

LEMMA 4.4. *$D(H)$ is generated as an algebra by $\{x, g, X, G\}$. The relations among these generators, in addition to the relations in H and H^*, are as follows*:

$$gG = Gg, \quad xG = \omega^{-1}Gx, \quad Xg = \omega^{-1}gX, \quad \text{and} \quad xX - Xx = G - g.$$

PROOF. It is clear that the given elements generate $D(H)$. To check the relations, we use (4.1). First, $gG = \langle G, g\rangle\langle G, g^{-1}\rangle Gg = \langle G, 1\rangle Gg = Gg$.

For the next relation, we use $\Delta_2(x) = x \otimes 1 \otimes 1 + g \otimes x \otimes 1 + g \otimes g \otimes x$ and the fact that $\langle G, \bar{S}x\rangle = 0$. Then

$$xG = \langle G, x\rangle\langle G, 1\rangle G1 + \langle G, g\rangle\langle G, 1\rangle Gx + \langle G, g\rangle\langle G, \bar{S}x\rangle Gg = \omega^{-1}Gx.$$

The third relation is similar.

For the fourth relation, we again use $\Delta_2(x)$ as well as $\Delta_2(X)$. Then with $h = x$, $f = X$ in (4.1), we have

$$\begin{aligned} xX &= \langle \varepsilon, x\rangle\langle X, 1\rangle \varepsilon 1 + \langle \varepsilon, g\rangle\langle X, 1\rangle \varepsilon x + \langle \varepsilon, g\rangle\langle X, \bar{S}x\rangle \varepsilon g \\ &\quad + \langle \varepsilon, x\rangle\langle G, 1\rangle X1 + \langle \varepsilon, g\rangle\langle G, 1\rangle Xx + \langle \varepsilon, g\rangle\langle G, \bar{S}x\rangle Xg \\ &\quad + \langle X, x\rangle\langle G, 1\rangle G1 + \langle X, g\rangle\langle G, 1\rangle Gx + \langle X, g\rangle\langle G, \bar{S}x\rangle Gg \\ &= 0 + 0 - g + 0 + Xx + 0 + G + 0 + 0 \\ &= Xx + G - g. \end{aligned}$$

□

We now apply the results of Section 2 to $D(H)$-actions on the algebra A. As in the last section, we apply Theorem 2.2 and Theorem 2.5 to two different copies of the Taft algebra in $D(H)$.

THEOREM 4.5. *Let A be an n-dimensional k-algebra with no non-zero nilpotent elements, and let H be the Taft algebra $T_{n^2}(\omega)$. Assume that A is a $D(H)$-module algebra such that $x \cdot A \neq 0$ or that $X \cdot A \neq 0$. Then there exists $u \in A$ and $0 \neq \beta$, $\gamma \in k$ such that*

(a) *$A = k(u)$ and $u^n = \beta \in k$;*

(b) *$g \cdot u = \omega u$ and $G \cdot u = \omega^{-1}u$, and so both g and G have order n in* $\mathrm{Aut}_k(A)$;

(c) *$x \cdot u = \gamma 1$ and $X \cdot u = \lambda u^2$, where $\gamma\lambda = \omega^{-1} - 1$.*

In particular, any $D(H)$-module algebra action on A is determined by the action of $H \subset D(H)$ on A.

Conversely given A as in (a) *and $0 \neq \beta, \gamma, \delta \in k$ such that* (b) *and* (c) *are satisfied, then A becomes a left $D(H)$-module algebra with the given actions.*

PROOF. First assume that $x \cdot A \neq 0$. We apply Theorem 2.5 to $H \subset D(H)$ to find $u \in A$ and $\beta, \gamma \in k$ such that $A = k(u)$, $u^n = \beta$, $x \cdot u = \omega u$, and $x \cdot u = \gamma 1$. This proves (a) and the first parts of (b) and (c). Now by Lemma 4.4, $g(G \cdot u) = G(g \cdot u) = G(\omega u) = \omega(G \cdot u)$. Thus $G \cdot u \in A_1$, which is one-dimensional by Lemma 2.4; it follows that $G \cdot u = \alpha u$, for some $\alpha \in k$. But we also have $x(G \cdot u) = \omega^{-1} G(x \cdot u)$, or $\alpha x \cdot u = \alpha\gamma 1 = \omega^{-1}\gamma 1$. Thus $\alpha = \omega^{-1}$, finishing (b).

Now consider the action of $H^{*cop} \subset D(H)$. Since $\Delta^{cop}(X) = \varepsilon \otimes X + X \otimes G$, the element $D := XG^{-1}$ is an (ε, G^{-1})-primitive element and satisfies $DG^{-1} = \omega^{-1} G^{-1} D$. Thus $H^{*cop} = k\langle G^{-1}, D\rangle \cong T_{n^2}(\omega^{-1})$. Apply Theorem 2.2 to H^{*cop}, replacing σ with G^{-1}, μ with ω^{-1}, and using $G^{-1} \cdot u = \omega u$. Then $t = n - 1$ and so $s = 2$; thus there exists $\delta' \in k$ such that $XG^{-1} \cdot u = \delta' u^2$. Setting $\lambda := \omega^{-1}\delta'$, it follows that $X \cdot u = \lambda u^2$.

It remains only to check the relation between γ and λ. To do this, we apply the fourth relation in Lemma 4.4 to u. Thus

$$x(X \cdot u) - X(x \cdot u) = G \cdot u - g \cdot u,$$

or $x \cdot (\lambda u^2) - \gamma X \cdot 1 = (\omega^{-1} - \omega)u$. Now $x \cdot u^2 = \gamma(1 + \omega)u$ by Lemma 2.1, and so $\lambda\gamma(1 + \omega)u = (\omega^{-1} - \omega)u$. Thus $\lambda\gamma = \omega^{-1} - 1$.

This finishes the case $x \cdot A \neq 0$; note we have shown that $X \cdot A \neq 0$. Assuming $X \cdot A \neq 0$, we would obtain similarly that $x \cdot A \neq 0$, so the previous arguments would apply.

It is clear from the proof that the $D(H)$-action is determined by the H-action. The converse follows from Lemma 2.1, as A will be a module algebra for both H and H^{*cop}; by construction the relations in Lemma 4.4 are satisfied, and so A will be a $D(H)$-module. □

Now, analogously to Corollary 3.6, we show that $A \sharp D(H)$ is also not semisimple.

First, by a result of Radford [R] for any finite-dimensional $H, D(H)$ is unimodular with integral $0 \neq \Lambda = \lambda \bowtie t$, where λ is a left integral of H^* and t is a right integral of H. Thus when $H = T_{n^2}$ as above,

$$\Lambda = \left(\sum_{i=0}^{n-1} G^i\right) X^{n-1} \bowtie x^{n-1} \left(\sum_{j=0}^{n-1} g^j\right).$$

COROLLARY 4.6. *Let A be a $D(H)$-module algebra as in Theorem 4.5. Then $A \sharp D(H)$ has a non-zero nilpotent ideal $I := A\Lambda A$.*

PROOF. First, for any $a \in A$, $w \in D(H)$, it is easy to see that $wa\Lambda = (w \cdot a)\Lambda \in A\Lambda$; similarly, $\Lambda aw \in \Lambda A$. It follows that I is an ideal of $A\sharp D(H)$. We claim that $\Lambda \cdot A = 0$.

Now since $x \cdot u = \gamma 1$, it follows from Lemma 2.1(b) that $x^{n-1} \cdot A \subseteq k$. But then $Xx^{n-1} \cdot A = 0$ and so certainly $\Lambda \cdot A = 0$. It now follows that $I^2 = A\Lambda A\Lambda A = 0$. □

REMARK 4.7. An alternate approach to the results in Section 3 can be given using the results of this section.

First, it is well-known that $u_q(sl_2)$ is a Hopf homomorphic image of $D(H)$, for $H = T_{n^2}(\omega)$, by setting $\omega = q^{-2}$ and defining $\Phi : D(H) \to u_q$ as follows:

$$G \bowtie 1 \mapsto K, \quad \varepsilon \bowtie g \mapsto K^{-1}, \quad \varepsilon \bowtie x \mapsto F, \quad \text{and} \quad X \bowtie 1 \mapsto \bar{E} := -(q - q^{-1})E$$

Moreover $\mathrm{Ker}(\Phi) = D(H)k\mathcal{G}^+$, where $\mathcal{G} = \{G^i \otimes g^i \mid 0 \le i < n\}$. See [K, IX.6].

COROLLARY 4.8. *Each action of $D(H)$ on $A = k(u)$ as in Theorem 4.5 induces a corresponding action of $u_q(sl_2)$ on A via*

$$\bar{E} \cdot u := X \cdot u, \quad F \cdot u := x \cdot u, \quad \text{and} \quad K \cdot u := g^{-1} \cdot u.$$

Conversely any action of u_q on A lifts to an action of $D(H)$ on A.

PROOF. We must only check that for an action as in Theorem 4.5, $\mathrm{Ker}(\Phi) \cdot A = 0$. However for any such action, $(G \bowtie g) \cdot u = G \cdot (\omega u) = u$. Consequently $(\varepsilon \otimes 1 - G^i \otimes g^i) \cdot u = 0$, for all i, and so $k\mathcal{G}^+ \cdot A = 0$. Thus $\mathrm{Ker}(\Phi) \cdot A = 0$ and the u_q-action is well-defined. □

It follows that in some sense, Corollary 3.2 and Theorem 4.5 are equivalent. We use this method to describe the other possible actions of $D(H)$ on $A = k(u)$, analogously to Theorem 3.1.

COROLLARY 4.9. *Let $A = k(u)$ be an n-dimensional k-algebra. Assume that A is a $D(H)$-module algebra such that $g \cdot u = \alpha u$, for some $\alpha \in k$, and such that $x \cdot A \neq 0$. Then there exist $0 \neq \beta, \gamma, \delta \in k$ and $0 \le s, t, l \le n-1$ with $t(1-s) \equiv 1 \pmod n$ and $s + l \equiv 2 \pmod n$ such that*

(a) $u^n = \beta$, $\alpha = \omega^t$, $G \cdot u = \alpha^{-1}u$, *and g and G have order n in* $\mathrm{Aut}(A)$;

(b) $x \cdot u = \gamma u^s$ *and* $X \cdot u = \lambda u^l$;

(c) $\lambda = -(q - q^{-1})\delta$, *where $\omega = q^{-2}$ and $\gamma\delta$ is given as in Theorem* 3.1.

Conversely, any choice of β, γ, δ and s, t, l satisfying the above relations defines a $D(H)$-module algebra structure on A.

PROOF. We use the previous corollary to get the corresponding induced action of $u_q(sl_2)$ on A. Now apply Theorem 3.1. $\square$

We can also give an alternate proof of Corollary 3.6, as follows. Given an action of $D(H)$ on A, and the corresponding action of $u_q(sl_2)$, the homomorphism in Remark 4.7 induces a corresponding homomorphism of the smash product $A\sharp D(H)$ onto $A\sharp u_q(sl_2)$. In particular, if Λ is the integral in $D(H)$, then $\Phi(\Lambda)$ is the integral in $u_q(sl_2)$. It follows that the nilpotent ideal of $A\sharp D(H)$ constructed in Corollary 4.6 has as a non-zero image the ideal of $A\sharp u_q(sl_2)$ considered in Corollary 3.6, which must therefore be nilpotent.

5. Yetter-Drinfel'd Module Algebras

In this section we give some general results about Yetter-Drinfeld module algebras for any Hopf algebra H, and then specialize to the Taft algebras.

Let M be a left H-module and a left H-comodule via $\rho : M \to H \otimes M$. The usual Yetter-Drinfel'd condition is

$$\rho(h \cdot m) = \sum (h \cdot m)_{-1} \otimes (h \cdot m)_0 = \sum h_1 m_{-1} S h_3 \otimes h2 \cdot m_0 \tag{5.1}$$

for all $m \in M$, $h \in H$. We will just refer to this as the *YD-condition*, and the category of (left, left) H-Yetter-Drinfel'd modules as ${}^H_H\mathcal{YD}$.

The left H-comodule structure dualizes as usual to a right H^*-module structure on M, via $m \circ f = \sum \langle f, m_{-1} \rangle m_0$, all $f \in H^*$, $m \in M$. This in turn induces a left H^*-module structure on M using $\bar{S}$; that is,

$$f \cdot m = \sum \langle \bar{S} f, m_{-1} \rangle m_0 \quad \forall m \in M,\ f \in H^*. \tag{5.2}$$

Conversely, if M is a finite-dimensional left H^*-module, then M becomes a left H-comodule using $\bar{S}$. Note, however, that $\bar{S}$ induces the coopposite module algebra structure; that is, if $M = A$, a finite-dimensional k-algebra, then using (5.2), A is a left H-comodule algebra $\Leftrightarrow$ A is a left H^{*cop}-module algebra.

The next lemma is due to S. Majid, see [M, 10.6.16]. We give a short proof for completeness, and to fix our notation.

LEMMA 5.3. *Let M be a left H-module and a left H-comodule. Then $M \in {}^H_H\mathcal{YD} \Leftrightarrow M$ is a left $D(H)$-module, via $(f \bowtie h) \cdot m := f \cdot (h \cdot m)$, using $f \cdot m$ as in* (5.2), *for all $f \in H^*$, $h \in H$, $m \in M$.*

PROOF. As in Section 3, we write $hf = (\varepsilon \bowtie h)(f \bowtie 1)$. Using (4.1), we see that

$$\begin{aligned}(hf) \cdot m &= \sum \langle f_3, h_1 \rangle \langle f_1, \bar{S}h_3 \rangle f_2 \cdot (h_2 \cdot m) \\ &= \sum \langle \bar{S}f_3, Sh_1 \rangle \langle \bar{S}f_1, h_3 \rangle \langle \bar{S}f_2, (h_2 \cdot m)_{-1} \rangle (h_2 \cdot m)_0 \\ &= \sum \langle \bar{S}f, (Sh_1)(h_2 \cdot m)_{-1} h_3 \rangle (h_2 \cdot m)_0\end{aligned}$$

Also $h \cdot (f \cdot m) = \sum h \cdot \langle \bar{S}f, m_{-1} \rangle m_0 = \sum \langle \bar{S}f, m_{-1} \rangle h \cdot m_0$.

It follows that M is a left $D(H)$-module $\Leftrightarrow$ $(hf) \cdot m = h \cdot (f \cdot m)$, for all $h \in H$, $f \in H^*$, $m \in M$ $\Leftrightarrow$ $\sum m_{-1} \otimes h \cdot m_0 = \sum (Sh_1)(h_2 \cdot m)_{-1} h_3 \otimes (h_2 \cdot m)_0$, for all $h \in H$, $m \in M$.

This last condition is easily seen to be equivalent to the YD-condition (5.1). This proves the Lemma. □

We now specialize to $H = T_{n^2}(\omega) = k\langle g, x \rangle$ and its dual $H^* = k\langle G, X \rangle$. We first note that (4.3) extends to all of H and H^*; the proof is a straightforward induction, and is presumably well-known.

LEMMA 5.4. *$\langle X^q G^r, x^i y^j \rangle = \delta_{i,q} \prod_{z=1}^{q} \binom{z}{1}_{\omega^{-1}} \omega^{-jr}$ where $\binom{z}{1}_{\omega^{-1}} = 1 + \omega^{-1} + \cdots + \omega^{-(z-1)}$.*

Next we consider our algebra A as in Sections 2–4 and determine the left H-coactions corresponding to the possible left H^*-actions, as in (5.2). Recall from the proof of Theorem 4.5 that $H^{*cop} = k\langle \sigma, D \rangle \cong T_{n^2}(\omega^{-1})$, by setting $\sigma = G^{-1}$ and $D = XG^{-1}$.

PROPOSITION 5.5. *Let $A = k[u \mid u^n = \beta]$, $\beta \in k$. Then A is a left H^{*cop}-module algebra via $G \cdot u = \omega^t u$ and $X \cdot u = \delta u^s$, where $0 \neq \delta \in k$, $t(1 - s) \equiv 1 \pmod{n}$, $0 \leq s, t \leq n - 1$. This action determines a unique left H-comodule algebra structure on A satisfying (5.2), given by*

$$\rho(u) = \sum_{m=0}^{t} a_m x^m g^{-m+t} \otimes u^{m(s-1)+1}.$$

For all s, $a_0 = 1$. When $s \geq 2$ and $m \geq 1$,

$$a_m = (-\delta)^m \omega^{m(1-t)} \frac{\left(\sum_{j_1=0}^{s-1} \omega^{-j_1 t}\right) \cdots \left(\sum_{j_{m-1}=0}^{(m-1)(s-1)} \omega^{-j_{m-1} t}\right)}{\prod_{z=1}^{m-1} (1 + \omega^{-1} + \cdots + \omega^{-z})}$$

When $s = 0$, then $\rho(u) = g \otimes u - \delta x \otimes 1$.

PROOF. We first check that the given action of G and X determines an H^{*cop}-module algebra action. First, $G \cdot u = \omega^t u$ gives that $\sigma \cdot u = \omega^{-t}u = \mu^t u$. Thus by Theorem 2.2, for any $0 \neq \delta' \in k$, there is an H^{*cop}-module algebra action on A given by $D \cdot u = \delta' u^s$, where $t(1-s) \equiv 1 \bmod n$. Since $D = XG^{-1}$, it follows that $X \cdot u = \delta u^s$ does give an H^{*cop}-module algebra on A by setting $\delta' := \omega^{-t}\delta$.

From (5.2), the given $\rho(u) = \sum u_{-1} \otimes u_0$ is the correct coaction if and only if, for all $f \in H^*$, $f \cdot u = \sum \langle \bar{S}f, u_{-1} \rangle u_0$. Since S is bijective, we may check instead that

$$(*) \quad Sf \cdot u = \sum \langle f, u_{-1} \rangle u_0.$$

It suffices to check this for all $f \in \{X^q G^r\}$, a basis of H^*.

First assume $q = 0$. Then, using Lemma 5.4, and $f = G^r$,

$$G^{-r} \cdot u = \omega^{-tr}u = a_0 \langle G^r, g^t \rangle u + \sum_{m>0} a_m \langle G^r, x^m g^{-m+t} \rangle u^{m(s-1)+1} = a_0 \omega^{-tr} u.$$

Thus $a_0 = 1$. We may therefore assume that $q \geq 1$.

Now assume $s \geq 2$. For $f = X^q G^r$, note that $Sf = G^{-r}(-G^{-1}X)^q = G^{-r}(-\omega D)^q = (-\omega)^q G^{-r} D^q$. Using $D \cdot u = \delta' u^s = \omega^{-t}\delta u^s$, apply Lemma 2.1(b) with $p = 1$ and $\alpha = \omega^{-t}$ to see that the left-hand side of (*) is

$$\begin{aligned} Sf \cdot u &= (-\omega)^q G^{-r} D^q \cdot u \\ &= (-\omega)^q (\omega^{-tr})^{q(s-1)+1} (\omega^{-t}\delta)^q \left(\sum_{i_2=0}^{s-1} \omega^{-ti_2} \right) \cdots \left(\sum_{i_q=0}^{(q-1)(s-1)} \omega^{-ti_q} \right) u^{1+q(s-1)} \\ &= (-)^q \delta^q \omega^{q(1-t)} \omega^{r(q-t)} \left(\sum_{j_1=0}^{s-1} \omega^{-j_1 t} \right) \cdots \left(\sum_{j_{q-1}=0}^{(q-1)(s-1)} \omega^{-j_{m-1} t} \right) u^{q(s-1)+1} \end{aligned}$$

where we have changed the index of summation. Moreover $G^{-r}D^q \cdot u \neq 0$ for all q such that $1 \leq q \leq t$, and $= 0$ for all $q > t$. For the right-hand side of (*), using Lemma 5.4, it follows that

$$\begin{aligned} \sum \langle X^q G^r, u_{-1} \rangle u_0 &= \sum_{m=0}^{n-1} a_m \langle X^q G^r, x^m g^{-m+t} \rangle u^{m(s-1)+1} \\ &= a_q \prod_{z=1}^{q} \binom{z}{1}_{\omega^{-1}} \omega^{-(-q+t)r} u^{q(s-1)+1} = a_q \omega^{r(q-t)} \prod_{z=1}^{q} \binom{z}{1}_{\omega^{-1}} u^{q(s-1)+1} \end{aligned}$$

We see that the power of u is the same on both sides of (*). We may also cancel $\omega^{r(q-t)}$ from both sides. Writing out $\binom{z}{1}_{\omega^{-1}}$ and changing the index of summation gives precisely the desired formula for a_q.

When $s = 0$ (and so $t = 1$), then $D \cdot u \in k$ implies $D^q \cdot u = 0$ for all $q \geq 2$, and so, as in the first part of the previous argument, $a_q = 0$ for all $q \geq 2$. Thus $\rho(u) = a_0 g \otimes u + a_1 x \otimes 1$. We have already seen that $a_0 = 1$, using $q = 0$. When $q = 1$,

$$SX \cdot u = -\omega D \cdot u = -\omega(\omega^{-1}\delta)1 = a_0\langle X, g\rangle u + a_1\langle X, x\rangle 1 = 0 + a_1 = a_1.$$

Thus $a_1 = -\delta$, finishing the proof. □

EXAMPLE 5.6. Consider the special ("generic") case arising in Theorem 4.5, that is, $G \cdot u = \omega^{-1}u$ and $X \cdot u = \delta u^2$. In that case, $s = 2$ and $t = n - 1$. Thus the left H-coaction corresponding to this action is given by

$$\rho(u) = \sum_{m=0}^{n-1} a_m x^m g^{-(m+1)} \otimes u^{m+1}$$

where $a_m = (-\delta)^m \omega^m \omega^{m(m+1)/2}$.

For, in this case, each factor in the numerator of a_m of the form $\sum_{j_i=0}^{i} \omega^{j_i}$ is paired with the term $1 + \omega^{-1} + \cdots + \omega^{-i}$ in the denominator, for $i = 1, \ldots, m - 1$, to give a quotient of ω^i. Substituting into the formula for a_m in Proposition 5.5, we see that $a_m = (-\delta)^m \omega^{2m} \omega^{m(m-1)/2} = (-\delta)^m \omega^m \omega^{m(m+1)/2}$.

We can now determine the left YD-structures on our ring A.

THEOREM 5.7. *Let A be an n-dimensional k-algebra with no non-zero nilpotent elements, and let $H = T_{n^2}(\omega)$. Assume that A is a left H-module algebra such that $x \cdot A \neq 0$. Then for each such action, there exists a unique left H-comodule algebra structure on A such that $A \in {}^H_H\mathcal{YD}$.*

In particular, assume the H-action is as described as in Theorem 2.5; that is, $A = k(u)$ for $u^n = \beta \in k$, and $g \cdot u = \omega u$, $x \cdot u = \gamma 1$. Then the unique left H-comodule algebra structure on A, such that $A \in {}^H_H\mathcal{YD}$, is given by

$$\rho(u) = \sum_{m=0}^{n-1} \gamma^{-m}(\omega - 1)^m \omega^{m(m+1)/2} x^m g^{-(m+1)} \otimes u^{m+1}.$$

PROOF. Apply Theorem 4.5 to get a unique left $D(H)$-module algebra structure on A; by Proposition 5.5 the left H^{*cop}-module algebra structure gives a unique left H-coaction on A. That this implies $A \in {}_H^H\mathcal{YD}$ follows from Lemma 5.3.

Now consider the particular "generic" case in the theorem. By Theorem 4.5, we must have $G \cdot u = \omega^{-1}u$ and $X \cdot u = \delta u^2$ such that $\delta\gamma = \omega^{-1} - 1$. Thus $(-\delta)^m = \gamma^{-m}(1 - \omega^{-1})^m$. Since we are in the situation of Example 5.6, we use the simplified form of a_m found there to see that

$$a_m = \gamma^{-m}((1 - \omega^{-1})\omega)^m \omega^{m(m+1)/2},$$

which is the desired coefficient. The form of $\rho(u)$ also follows from Example 5.6. □

The theorem gives an answer to the question raised in [CFM, 2.2]; the coaction proposed there is in fact a coaction, as it is the special case of Theorem 5.7 with $\gamma = 1$. We note that this fact seems quite difficult to prove directly. For, one has to verify the commutativity of the comodule diagram; that is, that $(id \otimes \rho) \circ \rho = (\Delta \otimes id) \circ \rho$. To check this on $u \in A$, first note that ρ becomes an algebra map by defining $\rho(u^s) := \rho(u)^s$ for all s; this is shown in [CFM, 2.2]. But then one must still check that

$$\sum_{s=0}^{n-1} a_s x^s g^{-(s+1)} \otimes \rho(u)^{s+1} = \sum_{m=0}^{n-1} a_m \Delta(x^m g^{-(m+1)}) \otimes u^{m+1}.$$

This is the problem. The indirect argument via passing to H^*-modules as in Theorem 5.7 above avoids having to check this identity. The identity may be of independent interest.

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