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THE CHEN INVARIANTS OF WARPED PRODUCTS OF HYPERBOLIC PLANES AND THEIR APPLICATIONS TO IMMERSIBILITY PROBLEMS

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Abstract. The classical obstruction to minimal isometric immersions into Euclidean space is $Ric \geq 0$. In this article we construct examples of Riemannian manifolds with $Ric < 0$ which don't admit any minimal isometric immersion into Euclidean space for any codimension, by applying Chen invariants.

1. Introduction

Let h denote the second fundamental form of an isometric immersion of a Riemannian n-manifold M^n into an ambient Riemannian space $\overline{M}^{n+m}$. Then the mean curvature vector field is $H = (1/n)$ trace h. The immersion is called minimal if its mean curvature vector field H vanishes identically.

The following is a classical basic problem in Riemannian geometry:

PROBLEM. *When a given Riemannian manifold M admits (or does not admit) a minimal immersion into a Euclidean space of arbitrary dimension?*

For a minimal submanifold M in a Euclidean space the Gauss equation implies that the Ricci tensor of the minimal submanifold satisfies:

$$Ric(X, X) = -\sum_{i=1}^{n} |h(X, e_i)|^2 \leq 0, \tag{1.1}$$

where $e_1, \dots, e_n$ is an orthonormal local frame field on M. This gives rise to the first solution to the Problem above; namely, the Ricci tensor of a minimal submanifold M of a Euclidean space is negative semi-definite, and a Ricci-flat minimal submanifold of a Euclidean space is totally geodesic.

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The second solution to the Problem mentioned above was obtained by B. Y. Chen as an immediate application of his fundamental inequality and his invariants [3, 5]. Based on these facts, it is interesting to construct precise examples of Riemannian manifolds with $Ric < 0$, but which do not admit any minimal isometric immersion into a Euclidean space for any codimension.

Let M^n be a Riemannian n-manifold. For any orthonormal basis $e_1, \dots, e_n$ of the tangent space T_pM, the scalar curvature at p is defined to be $scal(p) = 2\sum_{i<j} sec(e_i \wedge e_j)$. Let us denote by $\tau(p) = (1/2)\, scal(p)$. For any r-dimensional subspace of T_pM denoted L with orthonormal basis $e_1, \dots, e_r$ one may define

$$\tau(L) = \sum_{1 \le i < j \le r} sec(e_i \wedge e_j). \tag{1.2}$$

In [5], Chen considered the finite set $S(n)$ of k-tuples $(n_1, \dots, n_k)$ with $k \ge 0$ which satisfy the conditions: $n_1 < n$, $n_i \ge 2$ and $n_1 + \cdots + n_k \le n$. For each $(n_1, \dots, n_k) \in S(n)$ he introduced the following Riemannian invariants:

$$\delta(n_1, \dots, n_k)(p) = \tau(p) - \inf\{\tau(L_1) + \cdots + \tau(L_k)\}(p), \tag{1.3}$$

where infimum is taken for all possible choices of orthogonal subspaces $L_1, \dots, L_k$, satisfying $n_j = \dim L_j$, $(j = 1, \dots, k)$. Recall that the Chen invariant with $k = 0$ is nothing but half the scalar curvature.

As in [5], we put

$$c(n_1, \dots, n_k) = \frac{n^2(n + k - 1 - \sum n_j)}{2(n + k - \sum n_j)},$$

$$b(n_1, \dots, n_k) = \frac{1}{2}\{(n(n-1) - \sum_{j=1}^{k} n_j(n_j - 1)\}.$$

Chen's fundamental inequalities obtained in [5] can be stated as follows:

THEOREM 1. *For any n-dimensional submanifold M of a Riemannian space form $R^{n+m}(\varepsilon)$ of constant sectional curvature ε and for any k-tuple $(n_1, \dots, n_k) \in S(n)$, we have*:

$$\delta(n_1, \dots, n_k) \le c(n_1, \dots, n_k)|H|^2 + b(n_1, \dots, n_k)\varepsilon. \tag{1.4}$$

The equality case of the inequality above holds at a point $p \in M$ if and only if there exists an orthonormal basis $e_1, \dots, e_{n+m}$ at p such that the shape operators of M in $R^{n+m}(\varepsilon)$ at p take the following forms: $S_r = \mathrm{diag}(A_1^r, \dots, A_k^r, \mu_r, \dots, \mu_r)$ *for* $r = n+1, \dots, m$, *where each* A_j^r *is a symmetric* $n_j \times n_j$ *submatrix such that* $\mathrm{trace}(A_1^r) = \cdots = \mathrm{trace}(A_k^r) = \mu_r$.

The invariants $\delta(n_1,\dots,n_k)$ became known as the Chen invariants and inequality (1.4) as Chen's fundamental inequality. Chen's fundamental inequality has many nice applications; for example, one has the following important result as an immediate consequence.

THEOREM 2. *Let M be a Riemannian n-manifold. If there exists a k-tuple $(n_1,\dots,n_k)\in S(n)$ and a point $p\in M$ such that*

$$\delta(n_1,\dots,n_k)(p) > \frac{1}{2}\{n(n-1) - \sum n_j(n_j-1)\}\varepsilon, \tag{1.5}$$

then M admits no minimal isometric immersion into any Riemannian space form $R^m(\varepsilon)$ with arbitrary codimension.

In particular, if $\delta(n_1,\dots,n_k)(p) > 0$ at a point for some k-tuple $(n_1,\dots,n_k)\in S(n)$, then M admits no minimal isometric immersion into any Euclidean space for any codimension.

We will use the second part of this theorem in our applications.

2. Warped Product Spaces

We use the warped product metrics introduced by Kručkovič in 1957 and by Bishop and O'Neill in [2] in our constructions in sections 2 and 3. (A reference on warped product metrics is in [1], which is in particular useful in the calculation on Ricci curvature of a warped product metric. Another reference is in [6]. A discussion in the context of manifolds with nonpositive curvature, based mainly on [2], can be found in [7].)

Let us consider two copies of the hyperbolic plane (H^2, g_0). The first has coordinates (x,y) with $y>0$ and has metric $g_0 = (1/y^2)(dx^2+dy^2)$. Let u and v denote the coordinates of the second copy of the hyperbolic plane with $v>0$. We consider the open subset $U = \{(x,y)\in H^2 \mid y>\varepsilon/2\}$, for sufficiently small $\varepsilon>0$. On the product manifold $(U\times_f H^2, g)$ we consider the warped product metric $g = g_0 + f^2 g_0$, i.e.,

$$g = \frac{1}{y^2}(dx^2+dy^2) + \frac{f^2(x,y)}{v^2}(du^2+dv^2), \tag{2.1}$$

where f is a positive differentiable function. We use the subscripts 1, 2, 3, 4 corresponding to the coordinates x,y,u,v, respectively. At every point $p\in M$, we denote the tangent vectors

$$\frac{\partial}{\partial x} = \partial_x, \quad \frac{\partial}{\partial y} = \partial_y, \quad \frac{\partial}{\partial u} = \partial_u, \quad \frac{\partial}{\partial v} = \partial_v.$$

We claim the following: *There exist differentiable functions f on $(U \times_f H^2, g)$ such that $Ric < 0$ and $\delta(2,2) > 0$ everywhere.*

A straightforward computation gives

$$sec(\partial_x \wedge \partial_y) = -1, \tag{2.2}$$

$$sec(\partial_x \wedge \partial_u) = sec(\partial_x \wedge \partial_v) = \frac{y}{f(x,y)}\left(\frac{\partial f}{\partial y} - y\frac{\partial^2 f}{\partial x^2}\right) \tag{2.3}$$

$$sec(\partial_y \wedge \partial_u) = sec(\partial_y \wedge \partial_v) = -\frac{y}{f(x,y)}\left(\frac{\partial f}{\partial y} + y\frac{\partial^2 f}{\partial y^2}\right) \tag{2.4}$$

$$sec(\partial_u \wedge \partial_v) = -\frac{1}{f^2(x,y)} - \frac{y^2}{f^2(x,y)}\left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right] \tag{2.5}$$

Therefore, the half of scalar curvature at $p = (x,y,u,v)$ is given by

$$\tau(p) = -1 - \frac{1}{f^2(x,y)} - \frac{2y^2}{f(x,y)}\left[\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right]$$

$$- \frac{y^2}{f^2(x,y)}\left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right] \tag{2.6}$$

Using eventually Proposition 9.106 from [1] and the fact that the components of the Hessian of a function ϕ are given in general by:

$$(h_\phi)_{jk} = \frac{\partial^2 \phi}{\partial x^j \partial x^k} - \frac{\partial \phi}{\partial x^r}\Gamma^r_{jk}$$

the values of the Ricci tensor are:

$$Ric(\partial_x, \partial_x) = -\frac{1}{y^2} + \frac{2}{yf(x,y)}\frac{\partial f}{\partial y} - \frac{2}{f(x,y)}\frac{\partial^2 f}{\partial x^2}, \tag{2.7}$$

$$Ric(\partial_y, \partial_y) = -\frac{1}{y^2} - \frac{2}{yf(x,y)}\left(\frac{\partial f}{\partial y} + y\frac{\partial^2 f}{\partial y^2}\right), \tag{2.8}$$

$$Ric(\partial_u, \partial_u) = Ric(\partial_v, \partial_v) = -\frac{1}{v^2} - \frac{y^2 f(x,y)}{v^2}\left[\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right]$$

$$-\frac{y^2}{v^2}\left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right] \tag{2.9}$$

$$Ric(\partial_x, \partial_y) = -\frac{2}{f(x,y)}\frac{\partial^2 f}{\partial y \partial x} - \frac{2}{yf(x,y)}\frac{\partial f}{\partial x} \tag{2.10}$$

$$Ric(\partial_x, \partial_u) = Ric(\partial_x, \partial_v) = Ric(\partial_y, \partial_u) = Ric(\partial_y, \partial_v) = Ric(\partial_u, \partial_v) = 0 \tag{2.11}$$

To complete our example, let us choose a function "close" to 1 which has the desired properties: $Ric < 0$ at every point $p = (x, y, u, v)$, but at least one of Chen invariants is strictly positive.

Let us consider $f(x, y) = e^{\varepsilon \arctan y}$. For this specific function one gets by direct computation that

$$Ric(\partial_u, \partial_u) = Ric(\partial_v, \partial_v) = \frac{-1}{v^2(1+y^2)^2}[(1+y^2)^2 + 2\varepsilon y^2(\varepsilon - y)e^{2\varepsilon \arctan y}] < 0. \tag{2.12}$$

This last conclusion shows us that the only minor we need to study is the one corresponding to subscripts 1 and 2.

The canonical base we've considered is not an orthonormal one. To complete the computation on an orthonormal basis let us take $e_1 = y\partial_x$, $e_2 = y\partial_y$, $e_3 = (v/f(x,y))\partial_u$, $e_4 = (v/f(x,y))\partial_v$. Then

$$Ric(e_1, e_1) = y^2\ Ric(\partial_x, \partial_x),$$

$$Ric(e_1, e_2) = y^2\ Ric(\partial x, \partial y),$$

$$Ric(e_2, e_2) = y^2\ Ric(\partial_y, \partial_y),$$

$$Ric(e_3, e_3) = (v^2/f^2)\ Ric(\partial_u, \partial_u) < 0,$$

$$Ric(e_4, e_4) = (v^2/f^2)\ Ric(\partial_v, \partial_v) < 0.$$

To see that $Ric < 0$, we have to study the 2×2 minor:

$$Ric(e_1, e_1) = -1 + \frac{2y}{f}\frac{\partial f}{\partial y} - \frac{2y^2}{f}\frac{\partial^2 f}{\partial x^2},$$

$$Ric(e_1, e_2) = Ric(e_2, e_1) = -\frac{2y^2}{f}\frac{\partial^2 f}{\partial y \partial x} - \frac{2y}{f}\frac{\partial f}{\partial x},$$

$$Ric(e_2, e_2) = -1 - \frac{2y}{f}\frac{\partial f}{\partial y} - \frac{2y^2}{f}\frac{\partial^2 f}{\partial y^2},$$

or, for the considered function:

$$Ric(e_1, e_1) = -1 + \frac{2\varepsilon y}{1+y^2},$$

$$Ric(e_1, e_2) = Ric(e_2, e_1) = 0,$$

$$Ric(e_2, e_2) = -1 - \frac{2\varepsilon y}{1+y^2} - \frac{2\varepsilon y^2(\varepsilon - 2y)}{(1+y^2)^2}.$$

On the other hand, since on U we get $sec(\partial_x \wedge \partial_y) < sec(\partial_x \wedge \partial_u)$, $sec(\partial_x \wedge \partial_y) < sec(\partial_y \wedge \partial_u)$, $sec(\partial_u \wedge \partial_v) < sec(\partial_x \wedge \partial_u)$, $sec(\partial_u \wedge \partial_v) < sec(\partial_y \wedge \partial_v)$, the smallest values of $sec(e_i \wedge e_j)$ on the considered basis are $sec(\partial_x \wedge \partial_y)$ and $sec(\partial_u \wedge \partial_v)$, we have on U:

$$\begin{aligned}\delta(2,2) &\geq 2\,sec(\partial_x \wedge \partial_u) + 2\,sec(\partial_y \wedge \partial_u) \\ &= -\frac{y^2}{f(x,y)}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) = -\frac{2\varepsilon y^2(\varepsilon - 2y)}{(1+y^2)^2} > 0. \end{aligned} \tag{2.13}$$

The last inequality allows us to apply theorem 2 to obtain the following:

PROPOSITION 1. *For sufficiently small $\varepsilon > 0$, the Riemannian manifold*

$$M = (U \times H^2, g_0 + (e^{2\varepsilon \arctan y})g_0)$$

cannot be isometrically immersed in any Euclidean ambient space E^m as a minimal submanifold for any codimension, even though $Ric < 0$.

One may obtain similar result by applying the same construction with some other warping functions on an appropriate open set $U \subset H^2$.

Let us notice that one doesn't need a specific computation for $\delta(2,2)$ to apply theorem 2. An estimate as in the relation (2.13) is sufficient to obtain the obstruction to minimal immersions into a Euclidean space of any codimension.

3. Multiwarped Product Spaces

Let us now consider a *multiwarped product* of hyperbolic spaces defined as follows. Let us use a similar notation $U = \{(x,y) \in H^2 \mid y > n/2\varepsilon\}$ to the previous section. Consider the product manifold of U with n warped copies of the hyperbolic plane H^2, endowed with coordinates $(x, y, u_1, v_1, \dots, u_n, v_n)$ with $y, v_1, \dots, v_n > 0$. At an arbitrary point of the product manifold let $\eta_1, \eta_2, \dots, \eta_{2n+2}$ denote respectively the tangent vectors:

$$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial u_1}, \frac{\partial}{\partial v_1}, \ldots, \frac{\partial}{\partial u_n}, \frac{\partial}{\partial v_n}.$$

The multiwarped product metric on $(U \times_{f_1} H^2 \times_{f_2} \cdots \times_{f_n} H^2, g)$ is defined by

$$g = \frac{1}{y^2}(dx^2 + dy^2) + \sum_{i=1}^{n} \frac{f_i^2(x, y)}{v_i^2}(du_i^2 + dv_i^2) \tag{3.1}$$

where $f_1(x, y), \ldots, f_n(x, y)$ are positive differentiable functions.

We claim the following: *There are some choices of $f_1, \ldots, f_n$ such that $Ric < 0$ everywhere, but some Chen invariant is positive.*

By direct computation we have, for $i = 1, \ldots, n$:

$$sec(\eta_1 \wedge \eta_2) = -1, \tag{3.2}$$

$$sec(\eta_1 \wedge \eta_{2i+1}) = sec(\eta_1 \wedge \eta_{2i+2}) = \frac{y}{f_i(x, y)}\left(\frac{\partial f_i}{\partial y} - y\frac{\partial^2 f_i}{\partial x^2}\right), \tag{3.3}$$

$$sec(\eta_2 \wedge \eta_{2i+1}) = sec(\eta_2 \wedge \eta_{2i+2}) = -\frac{y}{f_i(x, y)}\left(\frac{\partial f_i}{\partial y} + y\frac{\partial^2 f_i}{\partial y^2}\right), \tag{3.4}$$

$$sec(\eta_{2i+1} \wedge \eta_{2i+2}) = -\frac{1}{f_i^2(x, y)} - \frac{y^2}{f_i^2(x, y)}\left[\left(\frac{\partial f_i}{\partial x}\right)^2 + \left(\frac{\partial f_i}{\partial y}\right)^2\right], \tag{3.5}$$

$$\begin{aligned} sec(\eta_{2i+2} \wedge \eta_{2j+2}) &= sec(\eta_{2i+2} \wedge \eta_{2j+1}) = sec(\eta_{2i+1} \wedge \eta_{2j+1}) \\ &= -\frac{y^2}{f_i(x, y) f_j(x, y)}\left[\frac{\partial f_i}{\partial x}\frac{\partial f_j}{\partial x} + \frac{\partial f_i}{\partial y}\frac{\partial f_j}{\partial y}\right], \end{aligned} \tag{3.6}$$

$$\begin{aligned} \tau(p) = &-1 - \sum_{i=1}^{n}\left\{\frac{1}{f_i^2} + \frac{y^2}{f_i^2}\left[\left(\frac{\partial f_i}{\partial x}\right)^2 + \left(\frac{\partial f_i}{\partial y}\right)^2\right]\right\} \\ &- 2y^2 \sum_{i=1}^{n}\left(\frac{\partial^2 f_i}{\partial x^2} + \frac{\partial^2 f_i}{\partial^2 y}\right) - 4y^2 \sum_{i,j=1; i \neq j}^{n} \frac{1}{f_i f_j}\left[\frac{\partial f_i}{\partial x}\frac{\partial f_j}{\partial x} + \frac{\partial f_i}{\partial y}\frac{\partial f_j}{\partial j}\right], \end{aligned} \tag{3.7}$$

$$Ric(\eta_1, \eta_1) = -\frac{1}{y^2} + \frac{2}{y}\sum_{i=1}^{n}\frac{1}{f_i}\left(\frac{\partial f_i}{\partial y} - y\frac{\partial^2 f_i}{\partial x^2}\right) \tag{3.8}$$

$$Ric(\eta_2, \eta_2) = -\frac{1}{y^2} - \frac{2}{y}\sum_{i=1}^{n}\frac{1}{f_i}\left(\frac{\partial f_i}{\partial y} + y\frac{\partial^2 f_i}{\partial y^2}\right) \tag{3.9}$$

$$Ric(\eta_{2i+2},\eta_{2i+2}) = Ric(\eta_{2i+1},\eta_{2i+1})$$
$$= -\frac{1}{v_i^2} - \frac{y^2}{v_i^2}\left[\left(\frac{\partial f_i}{\partial x}\right)^2 + \left(\frac{\partial f_i}{\partial y}\right)^2\right] - \frac{y^2 f_i}{v_i^2}\left(\frac{\partial^2 f_i}{\partial x^2} + \frac{\partial^2 f_i}{\partial y^2}\right)$$
$$- \frac{2y^2 f_i^2}{v_i^2}\sum_{i=1;i\neq j}^{n}\frac{1}{f_i f_j}\left[\frac{\partial f_i}{\partial x}\frac{\partial f_j}{\partial x} + \frac{\partial f_i}{\partial y}\frac{\partial f_j}{\partial y}\right]. \tag{3.10}$$

A long computation yields the other terms of *Ric* matrix. Let us explain how to compute $Ric(\eta_1,\eta_2)$. We need to compute terms of the type R^k_{1k2}. We distinguish three cases: $k=1$, $k=2$ and $k\neq 1,2$. Then

$$R^1_{112} = 0, \quad R^2_{122} = 0, \quad R^k_{1k2} = -\frac{1}{f_k}\frac{\partial^2 f_k}{\partial y \partial x} - \frac{1}{y f_k}\frac{\partial f_k}{\partial x}.$$

A similar discussion is taking place for every element of the *Ric* matrix, to yield that all non-diagonal terms vanish everywhere, except

$$Ric(\eta_1,\eta_2) = -2\sum_{k=1}^{n}\left[\frac{1}{f_k}\frac{\partial^2 f_k}{\partial y\partial x} + \frac{1}{y f_k}\frac{\partial f_k}{\partial x}\right]. \tag{3.11}$$

For a specific example let us consider $f_i(x,y) = f(x,y) = e^{\varepsilon \arctan y}$ for $i=1,\dots,n$. To simplify the computations one may choose $0<\varepsilon\leq 1/n$. For the orthonormal basis we work with, let us denote as above $e_1 = y\eta_1$, $e_2 = y\eta_2$, and $e_{2k+1} = (v_k/f_k)\eta_{2k+1}$, $e_{2k+2} = (v_k/f_k)\eta_{2k+2}$, for $k=1,\dots,n$, respectively.

For the subscript 3 to $2n$, the *Ric* matrix is in diagonal form at every point. Through a direct computation, we obtain, for $i=1,\dots,n$, that

$$Ric(e_{2i+1},e_{2i+1}) = Ric(e_{2i+2},e_{2i+2})$$
$$= -\frac{1}{f^2} - \frac{(2n+1)y^2}{f^2}\left(\frac{\partial f}{\partial y}\right)^2 - \frac{y^2}{f}\frac{\partial^2 f}{\partial y^2} < 0. \tag{3.12}$$

In order to estimate Chen invariant, we compute the sectional curvatures as follows, for $i,j=1,\dots,n$, $i\neq j$:

$$sec(\eta_1\wedge\eta_2) = -1, \tag{3.13}$$

$$sec(\eta_1\wedge\eta_{2i+1}) = sec(\eta_1\wedge\eta_{2i+2}) = \frac{\varepsilon y}{1+y^2} > 0, \tag{3.14}$$

$$sec(\eta_2\wedge\eta_{2i+1}) = sec(\eta_2\wedge\eta_{2i+2}) = \frac{\varepsilon y(y^2-\varepsilon y-1)}{(1+y^2)^2} > 0, \tag{3.15}$$

$$sec(\eta_{2i+1} \wedge \eta_{2i+2}) = -\frac{1}{f^2} - \frac{\varepsilon^2 y^2}{(1+y^2)^2} < 0, \tag{3.16}$$

$$sec(\eta_{2i+1} \wedge \eta_{2j+1}) = sec(\eta_{2i+2} \wedge \eta_{2j+2}) = sec(\eta_{2i+1} \wedge \eta_{2j+2}) = -\frac{\varepsilon^2 y^2}{(1+y^2)^2} < 0. \tag{3.17}$$

In fact, one can easily obtain that

$$sec(\eta_{2i+1} \wedge \eta_{2i+2}) < sec(\eta_{2i+2} \wedge \eta_{2j+2}). \tag{3.18}$$

This allows us to obtain the estimate of the $(2, 2, \dots, 2)$-order Chen invariant (2 repeats $n+1$ times) such that

$$\begin{aligned} \delta(2, \dots, 2) &\geq \tau(p) - \left[sec(\eta_1 \wedge \eta_2) + \sum_{i=1}^{n} sec(\eta_{2i+1} \wedge \eta_{2i+2}) \right] \\ &= \frac{2\varepsilon n y^2}{(1+y^2)^2}(2y - \varepsilon n) > 0 \end{aligned} \tag{3.19}$$

Thus, by applying the theorem 2, we have proved the following:

PROPOSITION 2. *The Riemannian manifold* $(U \times H^2 \times \cdots \times H^2, g)$, *endowed with the metric given by* (3.1) *with* $f_i(x, y) = e^{\varepsilon \arctan y}$, $i = 1, \dots, n$, *cannot be isometrically immersed as a minimal submanifold into a Euclidean space for arbitrary dimension, even though* $Ric < 0$.

The same procedure with some other functions f_i may also give rise to other specific examples of Riemannian manifolds whose Chen's invariants obstruct to minimal immersions via theorem 2, although the classical invariants does not provide obstruction to minimal immersion.

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