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SOME CONDITIONS ON THE WEINGARTEN ENDOMORPHISM OF REAL HYPERSURFACES IN QUATERNIONIC SPACE FORMS

By

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Abstract. We classify the real hypersurfaces in a non-flat quaternionic space form satisfying several conditions on their Weingarten endomorphism. Firstly, we study on the maximal quaternionic distribution of the real hypersurface a relationship between a certain metric tensor and the restriction of the metric of the ambient manifold. Secondly, we consider some formulae relating the Weingarten endomorphism to the curvature operator.

Introduction

Let $QM^m(c)$, $m \geq 2$, $c \neq 0$, be a non-flat quaternionic space form endowed with the metric g of constant quaternionic sectional curvature $c \neq 0$. For sake of simplicity, we will use $c = \pm 4$. When $c = 4$, we will call it the quaternionic projective space, QP^m, and when $c = -4$, the quaternionic hyperbolic space, QH^m. Let M be a connected real hypersurface in $QM^m(c)$ without boundary, and we choose a locally defined unit normal vector field N on M. We denote by A and $\boldsymbol{D}$ the Weingarten endomorphism associated with N and the maximal quaternionic distribution on M respectively. The restriction of the metric g to TM will be also denoted by g.

A totally umbilical real hypersurface should satisfy that there exists a smooth function λ on M such that $g(AX, Y) = \lambda g(X, Y)$, for any $X, Y \in TM$. But a classical result of Tashiro and Tachibana (see [4]) states that there do not exist totally umbilical real hypersurfaces in not-flat quaternionic space forms. Thus, an interesting problem is to classify those real hypersurfaces in $QM^m(c)$ satisfying a weaker condition. We copy the above formula, but we do not choose all tangent

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vectors in the tangent bundle TM. Instead, we consider the quaternionic distribution $\boldsymbol{D}$ of M, which is also a vector bundle on M. We continue denoting by g the restriction of the metric to $\boldsymbol{D}$. Besides, we define the symmetric tensor h on the vector bundle $\boldsymbol{D}$ by $h(X, Y) = g(AX, Y)$ for any $X, Y \in \boldsymbol{D}$. Then, we classify in Theorem 1 those real hypersurfaces in $\boldsymbol{QM}^m(c)$, $m \geq 3$, $c \neq 0$, satisfying that there exists a smooth function λ on M such that $h = \lambda g$.

On the other hand, the Codazzi equation implies the non-existence of real hypersurfaces in $\boldsymbol{QM}^m(c)$ with parallel second fundamental form. It is therefore natural to consider the action of the curvature operator R of M acting as a derivation on A, that is to say $R \cdot A = 0$, that is weaker than $\nabla A = 0$, where ∇ is the Levi-Civita connection of M. From this, we are interested in studying several conditions that enable us to relate the Weingarten endomorphism to the curvature operator. In this way, we have obtained several characterizations of some of the examples of real hypersurfaces in $\boldsymbol{QM}^m(c)$ known at present, making a clear distinction among all real hypersurfaces in $\boldsymbol{QM}^m(c)$. More precisely, we consider the following two conditions

$$g(R(AX, Y)Z - AR(X, Y)Z, W) = 0, \quad \text{for any } X, Y, Z, W \in \boldsymbol{D} \tag{1}$$

$$g((R(X, Y)A)Z + (R(Z, X)A)Y + (R(Y, Z)A)X, W) = 0, \tag{2}$$

$$\text{for any } X, Y, Z, W \in \boldsymbol{D}$$

Obviously, the second one is weaker than $R \cdot A = 0$. We classify the real hypersurfaces in $\boldsymbol{QM}^m(c)$, $m \geq 3$, $c \neq 0$, satisfying these two conditions in Theorems 2 and 3 respectively. As a consequence, we can show that do not exist real hypersurfaces in $\boldsymbol{QM}^m(c)$, $m \geq 3$, $c \neq 0$, such that $R \cdot A = 0$.

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1. Preliminaries

As $\boldsymbol{QM}^m(c)$ is a quaternionic manifold, there is a three dimensional fiber bundle $\mathscr{V} \subset EndTM$, called the quaternionic structure, such that given $\{J_1, J_2, J_3\}$ a basis of $\mathscr{V}$ defined on a suitable open subset of M, the following properties are satisfied

$$\begin{gathered} J_1^2 = J_2^2 = J_3^2 = -Id, \quad J_1J_2 = -J_2J_1 = J_3 \\ g(J_kX, Y) + g(X, J_kY) = 0, \quad k = 1,2,3 \\ \bar{\nabla}_X J_i = -q_j(X)J_k + q_k(X)J_j, \quad (dq_i + q_j \wedge q_k)(X, Y) = 4g(X, J_iY) \end{gathered} \tag{3}$$

for any X, Y tangent to $QM^m(c)$, where $\bar{\nabla}$ is the Levi-Civita connection of $QM^m(c)$, (i,j,k) is a cyclic permutation of $(1,2,3)$ and q_1, q_2, q_3 are local 1-forms. In [2], we see that another basis $\{J_1', J_2', J_3'\}$ of the fiber bundle $\mathscr{V}$, which is defined on the same open subset, can be obtained by considering $P \in SO(3)$, and then computing $PJ_k = J_k'$, for $k = 1,2,3$.

Let M be a connected real hypersurface of $QM^m(c)$ without boundary. Let X be a tangent vector field to M. We write $J_kX = \phi_kX + f_k(X)N$, $k = 1,2,3$, where ϕ_kX is the tangential component of J_kX and $f_k(X) = g(X, U_k)$, $k = 1,2,3$. Then we have

$$\phi_k^2X = -X + f_k(X)U_k, \quad f_k(\phi_kX) = 0, \quad \phi_kU_k = 0, \quad k = 1,2,3 \tag{4}$$

for any X tangent to M, and

$$\begin{gathered} \phi_iX = \phi_j\phi_kX - f_k(X)U_j = -\phi_k\phi_jX + f_j(X)U_k, \quad i = 1,2,3 \\ f_i(X) = f_j(\phi_kX) = -f_k(\phi_jX) \end{gathered} \tag{5}$$

for any X tangent to M, where (i,j,k) is a cyclic permutation of (1,2,3). It is also easy to check

$$\begin{gathered} \phi_iU_j = -\phi_jU_i = U_k \\ g(\phi_iX, Y) + g(X, \phi_iY) = 0, \quad g(\phi_iX, \phi_iY) = g(X, Y) - f_i(X)f_i(Y) \end{gathered} \tag{6}$$

for any X, Y tangent to M, $i = 1,2,3$, (i,j,k) being a cyclic permutation of $(1,2,3)$. From the expression of the curvature tensor of $QM^m(c)$, [2], we have the Codazzi equation

$$(\nabla_XA)Y - (\nabla_YA)X = \frac{c}{4}\sum_{k=1}^{3}\{f_k(X)\phi_kY - f_k(Y)\phi_kX - 2g(\phi_kX, Y)U_k\} \tag{7}$$

for any X, Y tangent to M, where ∇ is the Levi-Civita connection of M and A is the Weingarten endomorphism. The eigenfunctions and eigenvectors of A are called principal curvatures and principal vectors respectively. If $p \in M$, we will write $W_\lambda(p) = \{X \in T_pM : A_pX = \lambda(p)X\}$, which is called the principal curvature distribution associated to the principal curvature λ at p. In general, dim W_λ is not constant, although it is possible to find a dense open subset of M where dim W_λ is constant. We also define $B \in \hom(\boldsymbol{D}, \boldsymbol{D})$ by the following. Given a point $p \in M$, and a vector $X \in \boldsymbol{D}_p$, define B_pX as the orthogonal projection of A_pX on $\boldsymbol{D}_p$. Clearly, $g(BX, Y) = g(X, BY)$ for any $X, Y \in \boldsymbol{D}$.

If R denotes the curvature tensor of M, the Gauss equation takes the form

$$R(X,Y)Z = \frac{c}{4}\left\{g(Y,Z)X - g(X,Z)Y + \sum_{k=1}^{3}\{g(\phi_k Y, Z)\phi_k X - g(\phi_k X, Z)\phi_k Y - 2g(\phi_k X, Y)\phi_k Z\}\right\} + g(AY,Z)AX - g(AX,Z)AY \tag{8}$$

for any X, Y, Z tangent to M. If $\bar{R}$ denotes the curvature tensor of $QM^m(c)$, M is called curvature-adapted if its normal Jacobi operator $K_N = \bar{R}(-, N)N$ commutes with the Weingarten endomorphism A of M. J. Berndt obtained the following three theorems in [1].

THEOREM A. *Let M be a curvature-adapted real hypersurface in $\boldsymbol{QP}^m$, $m \geq 2$. Then M is orientable and an open subset of a tube of radius $0 < r < \pi/2$ over one of the following*:

1. *a totally geodesic $\boldsymbol{QP}^k$, $k \in \{1, \ldots, m-1\}$,*
2. *a totally geodesic $\boldsymbol{CP}^m$.*

REMARK 1.1. The tube of radius $0 < r < \pi/2$ over a totally geodesic $\boldsymbol{QP}^{m-1}$ is also a tube of radius $\pi/2 - r$ over a point. These model spaces are known as geodesic hyperspheres.

THEOREM B. *Let M be a curvature-adapted real hypersurface in $\boldsymbol{QH}^m$, $m \geq 2$, with constant principal curvatures. Then M is an open subset of one of the following*:

1. *a tube of radius $r > 0$ over a totally geodesic $\boldsymbol{QH}^k$, $k \in \{0, \ldots, m-1\}$,*
2. *a tube of radius $r > 0$ over a totally geodesic $\boldsymbol{CH}^m$,*
3. *a horosphere.*

THEOREM C. *Let M be a curvature adapted real hypersurface in $\boldsymbol{QH}^m$. Suppose that M has non-constant principal curvatures. Then*

1. *$AZ = 2Z$ for any $Z \in \boldsymbol{D}^{\perp}$,*
2. *the constant 1 is a principal curvature,*
3. *if ρ is another principal curvature different from 2 and 1, then $\phi_k W_\rho \subset W_1$ for $k = 1, 2, 3$.*

A real hypersurface is also curvature adapted if $A\boldsymbol{D} \subseteq \boldsymbol{D}$ or equivalently, $A\boldsymbol{D}^{\perp} \subseteq \boldsymbol{D}^{\perp}$ (see [1]). Sometimes, it is useful to consider a pointwise version of this definition when we are making a proof. Indeed, given P any non-empty subset of M, we will say M is curvature adapted on P if for each point $p \in P$, $A_p\boldsymbol{D}_p \subseteq \boldsymbol{D}_p$, equivalently, $A_p\boldsymbol{D}_p^{\perp} \subseteq \boldsymbol{D}_p^{\perp}$. Clearly, we are considering $P = M$ if we simply say M is curvature adapted.

We recall that a real hypersurface in $\boldsymbol{Q}M^m(c)$ is ruled if $\boldsymbol{D}$ is integrable. We can call the horosphere a real hypersurface of type A_0, the tubes over a point or over a totally geodesic $\boldsymbol{Q}M^{m-1}(c)$, real hypersurfaces of type A_1, and the tubes over a totally geodesic $\boldsymbol{Q}M^k(c)$, $1 \le k \le m-2$, real hypersurfaces of type A_2.

2. Main Results

The reader should refer to the introduction to recall the definition of g and h.

THEOREM 1. *Let M be a connected real hypersurface in $\boldsymbol{Q}M^m(c)$, $m \ge 3$, $c \neq 0$, without boundary. Let us suppose there exists a smooth function $\lambda : M \to \boldsymbol{R}$ such that $h = \lambda g$. Then the function λ is constant, and M is one of the following*:

1. *ruled*, $\lambda = 0$,
2. *in $\boldsymbol{Q}P^m$, an open subset of a tube of radius r, $0 < r < \pi/2$, over a totally geodesic $\boldsymbol{Q}P^{m-1}$*, $\lambda = \cot(r)$,
3. *in $\boldsymbol{Q}H^m$*,
   (a) *an open subset of a tube of radius $r > 0$ over a totally geodesic $\boldsymbol{Q}H^{m-1}$*, $0 < \lambda = \tanh(r) < 1$,
   (b) *an open subset of a horosphere*, $\lambda = 1$,
   (c) *an open subset of a tube of radius $r > 0$ over a point*, $1 < \lambda = \coth(r)$.

PROOF. The hypothesis $h = \lambda g$ is equivalent to

$$AX = \lambda X + \sum_{l=1}^{3} f_l(AX)U_l \quad \text{for any } X \in \boldsymbol{D} \tag{9}$$

Let Γ be the set of points $p \in M$ such that M is not curvature-adapted at p and $\lambda(p) \neq 0$. Given a point $p \in \Gamma$, let Ω be a connected coordinate open neighbourhood of p. We consider the vector bundle $\mathrm{Hom}(\boldsymbol{D}^{\perp}, \boldsymbol{D}^{\perp})$. We define $A^0 \in \mathrm{Hom}(\boldsymbol{D}^{\perp}, \boldsymbol{D}^{\perp})$ by the following. Given any $q \in \Omega$ and any $X \in \boldsymbol{D}_q^{\perp}$, $A_q^0 X$ is the normal projection of $A_q X$ on $\boldsymbol{D}_q^{\perp}$. Clearly, $g(A^0X, Y) = g(X, A^0Y)$ for any $X, Y \in \boldsymbol{D}^{\perp}$. By restricting Ω if necessary, there exists an orthonormal basis of $\boldsymbol{D}^{\perp}$,

$\{U_1', U_2', U_3'\}$, defined on Ω such that $A^0 U_k' = \mu_k U_k'$ for certain smooth functions μ_k, $k = 1,2,3$, defined on Ω. As it is shown in [2], there exists $P \in SO(3)$ such that $PJ_k = J_k'$ for any $k = 1,2,3$, maybe after reordering the indices k. It is easy to check $U_k' = -J_k'N$. Moreover, $AU_k' = A^0 U_k' + E_k$, for certain tangent vector fields $\{E_1, E_2, E_3\}$ on Ω. As $E_k(q) \neq 0$ for some $k \in \{1,2,3\}$, we can restrict Ω to obtain that M is not curvature adapted on Ω. Therefore, Γ is open, so we can choose a connected open neighbourhood Ω included in Γ, and we retain the notations to compute the vector fields E_k, although for the sake of simplicity we will denote by U_k the new orthonormal basis of $\boldsymbol{D}^\perp$ defined on Ω. Also, for each $k, l \in \{1,2,3\}$, define the smooth function $a_{kl} = g(E_k, E_l)$ on Ω. In the sequel, all the computations will be made on Ω unless otherwise stated. Let us define $V = Span\{E_1, E_2, E_3\}$ and W the orthogonal complement of V in $\boldsymbol{D}$. By (9),

$$AX = \lambda X \quad \text{for any } X \in W \tag{10}$$

Given $X, Y \in W$ and $k \in \{1,2,3\}$, we develop $g((\nabla_X A)Y - (\nabla_Y A)X, E_k)$ bearing in mind (7), (9) and (10),

$$0 = \sum_{l=1}^{3} a_{kl}\, g(Y, \phi_l X) \tag{11}$$

for any X, Y in $W, k = 1,2,3$ on Ω. We can regard (11) as a homogeneous linear system whose coefficients are a_{kl}, so that we have to distinguish three cases. We define the matrix $G = (a_{kl})_{k,l=1,2,3}$.

CASE 1. Define $\Omega_1 = \{q \in \Omega : \dim V(q) = 3\} = \{q \in \Omega : \det G(q) \neq 0\}$, which is open. The matrix G has rank 3, so that the linear system (11) has the unique solution $0 = g(Y, \phi_l X)$, for any $l = 1,2,3$ and any $X, Y \in W$. Therefore, $\phi_1 W \subseteq V$, so that $3 = \dim V \geq \dim W = 4m - 7$, that is to say, $4m \leq 10$, which contradicts $m \geq 3$. Therefore Ω_1 is empty.

CASE 2. Define $\Omega_2 = \{q \in \Omega : \dim V(q) = 2\}$. We can suppose without losing any generality $V = \mathrm{Span}\{E_1, E_2\}$ on an open subset Ω_2^0 of Ω_2. We can omit the third equation of (11) and rewrite the others as

$$\begin{aligned} a_{11} g(Y, \phi_1 X) + a_{12} g(Y, \phi_2 X) &= -a_{13} g(Y, \phi_3 X) \\ a_{21} g(Y, \phi_1 X) + a_{22} g(Y, \phi_2 X) &= -a_{23} g(Y, \phi_3 X) \end{aligned} \tag{12}$$

for any $X, Y \in W$ on Ω_2^0. There are two subcases.

CASE 2.a. There exist a point $q \in \Omega_2^0$ and a unit vector $Z \in (W \cap \phi_3 W)(q)$. By computing at q, we put $X = Z$, $Y = \phi_3 Z$ in (12), obtaining $0 = a_{13} = a_{23}$. We introduce this information in (12), and we finish this case as in Case 1.

CASE 2.b. $W \cap \phi_3 W = \{0\}$ at some point $q \in \Omega_2^0$. As $(\phi_3 W \oplus W)(q) \subset \boldsymbol{D}(q) = (V \oplus W)(q)$, then $4m - 6 = \dim W = \dim \phi_3 W \leq \dim V = 2$, and therefore $m \leq 2$. This is a contradiction. We conclude the set of interior points of Ω_2 is empty.

CASE 3. Define $\Omega_3 = \{q \in \Omega : \dim V(q) = 1\}$. We can suppose without losing any generality

$$a_{11} g(Y, \phi_1 X) + a_{12} g(Y, \phi_2 X) + a_{13} g(Y, \phi_3 X) = 0$$

for any $X, Y \in W$ on Ω_3. Choose $k \in \{1, 2, 3\}$. Let us define $\tilde{W} = W \cap (V \oplus \phi_1 V \oplus \phi_2 V \oplus \phi_3 V)^{\perp}$. As $m \geq 3$, $\dim \tilde{W} \geq 4$ on Ω_3, so that we can choose a nonzero vector X lying in $\tilde{W}$. Now, take $Y = \phi_k X \in W$, in the above equation, and we obtain $0 = a_{1k}$ on Ω_3, $k = 1, 2, 3$, that is to say, M is curvature-adapted on $\Omega_3 \subset \Gamma$. This is a contradiction, and therefore Ω_3 is empty.

Summing up these three cases, Ω is an open subset, $\Omega = \Omega_2$, and Ω_2 has no interior points. Therefore, Ω is empty, which yields that Γ is empty. Next, let us define $\Delta = \{p \in M : \lambda(p) \neq 0\}$, which is open. As Γ is empty, M is curvature adapted on Δ. Therefore, Δ is a curvature adapted real hypersurface in $\boldsymbol{Q}M^m(c)$. Moreover, equation (9) becomes $AX = \lambda X$ for any $X \in \boldsymbol{D}$ tangent to Δ. If the ambient manifold is $\boldsymbol{Q}H^m$, let us suppose Δ has a non-constant principal curvature ρ on an open subset G contained in Δ. By Theorem C, the constant 1 is a principal curvature on G, and $W_1 \subset \boldsymbol{D}$. This contradicts $AX = \lambda X$ for any $X \in \boldsymbol{D}$. Therefore, we can resort to Theorems A and B to classify the real hypersurface Δ, so that λ is locally constant on Δ. Paper [1] contains a table with the principal curvatures of each model space, so we only have to refer to this. As M is connected, and λ is a continous function, λ must be constant on M and M is one of the real hypersurfaces of Theorems A and B. This concludes the proof. □

Next, we turn our attention to equation (1).

THEOREM 2. *Let M be a connected real hypersurface in $\boldsymbol{Q}M^m(c)$, $m \geq 3$, $c \neq 0$, without boundary. Suppose that M satisfies* (1). *Then M is one of the following*:

1. *ruled,*
2. *an open subset of a real hypersurface of type A_0 or A_1.*

PROOF. By (8), (1) is equivalent to

$$\begin{aligned}0 = (c/4)\{g(X,Z)g(AY,W) - g(AX,Z)g(Y,W) + \sum_{i=1}^{3}\{g(\phi_i X,Z)g(A\phi_i Y,W) \\ + g(\phi_i Y,Z)g((\phi_i A - A\phi_i)X,W) - g(\phi_i AX,Z)g(\phi_i Y,W) \\ - 2g(\phi_i AX,Y)g(\phi_i Z,W) + 2g(\phi_i X,Y)g(\phi_i Z,AW)\}\} \\ + g(AX,Z)g(A^2Y,W) - g(A^2X,Z)g(AY,W)\end{aligned} \tag{13}$$

for any $X, Y, Z, W \in \boldsymbol{D}$. Let p be a point of M, and choose G a connected open neighbourhood of p on which the basis $\{U_1, U_2, U_3\}$ is globally defined. Define $\{E_1, \dots, E_{4m-4}\}$ an orthonormal basis of $\boldsymbol{D}$ on G. Fix $i \in \{1,2,3\}$. Define the smooth function $a: G \to \boldsymbol{R}$ by $a = \sum_{k=1}^{4m-4} g(AE_k, E_k)$. In the sequel, all the computations will be made on G unless otherwise stated. We introduce $X = E_l$ and $Z = \phi_i E_l$ in (13), by (4), (5) and (6), and adding in $l \in \{1, \dots, 4m-4\}$ we obtain

$$\begin{aligned}0 = (4m-7)g(A\phi_i Y,W) + g(\phi_i AY,W) - ag(\phi_i Y,W) \\ - g(A\phi_k Y,\phi_j W) + g(A\phi_j Y,A\phi_k W)\end{aligned} \tag{14}$$

for any $Y, W \in \boldsymbol{D}$ tangent to G, being (i,j,k) a cyclic permutation of $\{1,2,3\}$. We exchange Y and W in (14), obtaining

$$\begin{aligned}0 = (4m-7)g(A\phi_i W,Y) + g(\phi_i AW,Y) - ag(\phi_i W,Y) \\ - g(A\phi_k W,\phi_j Y) + g(A\phi_j W,A\phi_k Y)\end{aligned}$$

and adding this equation to (14), we obtain $0 = (4m-8)g(A\phi_i Y,W) + (8-4m) \cdot g(\phi_i AY,W)$. As $m \geq 3$, $g(A\phi_i Y,W) = g(\phi_i AY,W)$. We introduce this information in (14), and the resulting equation is $0 = (4m-4)g(\phi_i AY,W) - ag(\phi_i Y,W)$, so that we obtain $g(AY,W) = \lambda g(Y,W)$ for any $Y, W \in \boldsymbol{D}$ tangent to G, where $\lambda : G \to \boldsymbol{R}$ is a smooth function. Conversely, any real hypersurface in $\boldsymbol{QM}^m(c)$ that has a smooth function λ such that $g(AX,Y) = \lambda g(X,Y)$ for any $X, Y \in \boldsymbol{D}$ satisfies (13). Finally, these real hypersurfaces are classified in Theorem 1. □

THEOREM 3. *Let M be a connected real hypersurface in $\boldsymbol{Q}M^m(c)$, $m \geq 3$, $c \neq 0$, satisfying* (2). *Then, M is one of the following,*

1. *ruled,*
2. *an open subset of a real hypersurface of type A_0 or A_1.*

PROOF. By (8) and the first Bianchi equation, equation (2) is equivalent to

$$\begin{aligned}0 = \sum_{k=1}^{3}\{&g((\phi_k A + A\phi_k)X, Y)g(\phi_k Z, W) + g((\phi_k A + A\phi_k)Y, Z)g(\phi_k X, W)\\ &+ g((\phi_k A + A\phi_k)Z, X)g(\phi_k Y, W) - 2g(\phi_k Z, X)g(\phi_k A Y, W)\\ &- 2g(\phi_k X, Y)g(\phi_k AZ, W) - 2g(\phi_k Y, Z)g(\phi_k AX, W)\}\end{aligned} \tag{15}$$

for any $X, Y, Z, W \in \boldsymbol{D}$. Choose a point $p \in \boldsymbol{D}$, and an orthonormal basis $\{E_1, \dots, E_{4m-4}\}$ of $\boldsymbol{D}$ defined on an open neighbourhood G of p. Given $i, k \in \{1, 2, 3\}$, we define the smooth function $\lambda_{ik} : G \to \boldsymbol{R}$ by $\lambda_{ik} = (1/2) \cdot \sum_{l=1}^{4m-4} g((A\phi_k + \phi_k A)E_l, \phi_i E_l)$. We should point out $\lambda_{ik} = -\mathrm{trace}(\phi_i(A\phi_k + \phi_k A))$, for any $i, k \in \{1, 2, 3\}$, so that if $i = k$, $\lambda_{ii} = \mathrm{trace}(B)$ and if $i \neq k$, $\lambda_{ik} = 0$. Given $i \in \{1, 2, 3\}$, $l \in \{1, \dots, 4m-4\}$, we introduce $Y = E_l$ and $Z = \phi_i E_l$ in (15), and adding in $l = 1, \dots, 4m-4$ we obtain

$$\begin{aligned}0 = \sum_{k=1}^{3}\{&g((\phi_k A + A\phi_k)X, \phi_i\phi_k W) - g(\phi_i A\phi_k X, \phi_k W) - g(\phi_i\phi_k AX, \phi_k W)\\ &- 2g(\phi_k A\phi_i\phi_k X, W) + 2g(\phi_i\phi_k X, A\phi_k W)\} + 2\lambda_{ii} g(\phi_i X, W)\\ &- 2(4m-4)g(\phi_i AX, W)\end{aligned}$$

for any $X, W \in \boldsymbol{D}$ tangent to G. By using (5), we develop the above equation, and we get

$$\begin{aligned}0 = (7-4m)&g(\phi_i AX, W) - g(A\phi_i X, W) + \lambda_{ii} g(\phi_i X, W)\\ &- g(A\phi_j X, \phi_k W) + g(A\phi_k X, \phi_j W)\end{aligned}$$

for any $X, W \in \boldsymbol{D}$ tangent to G, being (i, j, k) a cyclic permutation of (1, 2, 3). This equation is equal to (14), so we can repeat the proof of Theorem 2 to obtain that G is either a ruled real hypersurface or an open subset of a real hypersurface of type A_0 or A_1. Usual connection reasonings show that M is either a ruled real hypersurface or an open subset of a real hypersurface of type A_0 or A_1. Conversely, if M is either a ruled real hypersurface or a real hypersurface of type

A_0 or A_1, using Theorem 1, it is easy to see that M satisfies equation (15). This concludes the proof. □

COROLLARY 2.1. *Let M be a connected real hypersurface in $QM^m(c)$, $m \geq 3$, $c \neq 0$. The following statements are equivalent,*

1. *M satisfies $(R(X,Y)A)Z = 0$ for any $X, Y, Z \in \boldsymbol{D}$,*
2. *M satisfies $(R(X,Y)A)Z + (R(Y,Z)A)X + (R(Z,X)A)Y = 0$ for any $X, Y, Z \in \boldsymbol{D}$,*
3. *M is an open subset of a real hypersurface of type A_0 or A_1.*

PROOF. Clearly, statement 1) implies statement 2). Let M be a connected real hypersurface in $QM^m(c)$, $m \geq 3$, $c \neq 0$, satisfying statement 2). By Theorem 3, M is either a ruled real hypersurface or an open subset of a real hypersurface of type A_0 or A_1. Let M be a ruled real hypersurface. We develop $(R(X,Y)A)Z + (R(Y,Z)A)X + (R(Z,X)A)Y = 0$, bearing in mind (8), the First Bianchi equation and Theorem 1, obtaining

$$0 = \sum_{k=1}^{3}\{g(\phi_k Z, X)\phi_k AY + g(\phi_k X, Y)\phi_k AZ + g(\phi_k Y, Z)\phi_k AX\}$$

for any $X, Y, Z \in \boldsymbol{D}$. Given $\{E_1, \dots, E_{4m-4}\}$ a local basis of $\boldsymbol{D}$, choose $i \in \{1,2,3\}$. We substitute $Y = E_l$, $Z = \phi_i E_l$ in the above equation, and adding in $l = 1, \dots, 4m-4$, we obtain $0 = \phi_i AX$ for any $X \in \boldsymbol{D}$. Thus, $AX = f_i(AX)U_i$, for any $X \in \boldsymbol{D}$ and any $i = 1,2,3$. This yields $AX = 0$ for any $X \in \boldsymbol{D}$. Therefore, given $X, Y \in \boldsymbol{D}$, as M is ruled, $\boldsymbol{D}$ is integrable, so $(\nabla_X A)Y - (\nabla_Y A)X = \nabla_X AY - \nabla_Y AX - A[X,Y] = 0$. Introducing this in the Codazzi equation (7), we have $0 = \sum_{k=1}^{3} g(\phi_k X, Y)U_k$ for any $X, Y \in \boldsymbol{D}$. If we introduce $X \in \boldsymbol{D}$ unitary and $Y = \phi_1 X$ in this last equation a contradiction arises. Therefore, no ruled real hypersurface satisfies statement 2). Finally, if M is a real hypersurface of type A_0 or A_1, there exists a real constant λ such that $AX = \lambda X$ for any $X \in \boldsymbol{D}$. Thus, given $X, Y, Z \in \boldsymbol{D}$, by (8), $(R(X,Y)A)Z = R(X,Y)AZ - AR(X,Y)Z = R(X,Y)(\lambda Z) - \lambda R(X,Y)Z = 0$. This concludes the proof. □

COROLLARY 2.2. *There do not exist real hypersurfaces in $QM^m(c)$, $m \geq 3$, $c \neq 0$, satisfying*

$$(R(X,Y)A)Z + (R(Y,Z)A)X + (R(Z,X)A)Y = 0, \quad \textit{for any } X, Y, Z \in TM. \tag{16}$$

PROOF. Suppose a real hypersurface M in $QM^m(c)$, $m \geq 3$, $c \neq 0$, satisfies (16). We then resort to Corollary 2.1 to know that M is an open subset of a real hypersurface of type A_0 or A_1. In either case, there exist two real constants λ, μ such that $AX = \lambda X + \mu \sum_{k=1}^{3} f_k(X) U_k$ for any $X \in TM$. Introducing this information in (16), by the first Bianchi equation, we obtain $0 = \sum_{k=1}^{3} \{f_k(Z) \cdot R(X, Y) U_k + f_k(Y) R(Z, X) U_k + f_k(X) R(Y, Z) U_k\}$ for any $X, Y, Z \in TM$. Now we choose a unit vector $Y \in \boldsymbol{D}$, $Z = \phi_2 Y$ and $X = U_1$, and introduce them in the above equation, by (8), obtaining $0 = R(Y, \phi_2 Y) U_1 = (c/2) U_3$. This is a contradiction. $\square$

COROLLARY 2.3. *There do not exist real hypersurfaces in $QM^m(c)$, $m \geq 3$, $c \neq 0$, satisfying $R \cdot A = 0$.*

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