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SOME DIFFERENTIAL-GEOMETRIC PROPERTIES OF R-SPACES

By

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§0. Introduction

Let G/K be an irreducible Riemannian symmetric space, where G is a connected compact semisimple Lie group and K its closed subgroup. The adjoint representation group $\mathrm{Ad}(K)$ acts on the tangent space $T_o(G/K)$ of G/K at the origin o as an isometry group. Let S denote a unit hypersphere in the $T_o(G/K)$ centered at the origin o. For each point a of S, the orbit $\mathrm{Ad}(K)a$ of a under $\mathrm{Ad}(K)$ is called an *R-space*. The R-spaces form an abundant class of homogeneous Riemannian manifolds and have several distinguished properties as submanifolds of S, and so they have been investigated by many authors from the point of view of differential geometry. (e.g., [5], [10], [12], [13], [16], [17], [21], [22], [24], [31], [32], [33])

In this paper, for these R-spaces we shall study the following:

(I) In the case where G/K is Hermitian, we investigate some relations between the complex structure and the restricted root system with respect to G/K.

(II) We express the covariant derivative of the second fundamental form of every R-space in S with respect to the Lie brackets in the Lie algebra of G.

As an application of (I), we obtain many new examples of homogeneous CR-submanifold in a complex projective space, which is stated as Theorem 3.2. As an application of (II), we can give a partial solution to the S. Maeda's Problem, which is stated as Corollary 4.5.

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§1. Preliminaries

In this paper, let G/K be an irreducible Riemannian symmetric space of compact type once and for all, where G is a connected compact semisimple Lie group

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and K its closed subgroup. Let $\mathfrak{g}$ and $\mathfrak{k}$ denote Lie algebras of G and K, respectively. Then G/K gives rise to an involutive automorphism θ of $\mathfrak{g}$ such that $\mathfrak{k} = \{X \in \mathfrak{g} \mid \theta(X) = X\}$. Put $\mathfrak{p} = \{X \in \mathfrak{g} \mid \theta(X) = -X\}$. Then we have

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p} \quad \text{(direct sum)}, \quad [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}.$$

We can identify $\mathfrak{p}$ with the tangent space $T_o(G/K)$ of G/K at the origin o. Let B denote the Killing form of $\mathfrak{g}$. We may assume that the metric g on G/K is given by $g_o = -B|_{\mathfrak{p} \times \mathfrak{p}}$.

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ and $\mathfrak{a}^*$ denote the dual space of $\mathfrak{a}$. For each $\lambda \in \mathfrak{a}^*$, we define subspaces $\mathfrak{k}_\lambda$ and $\mathfrak{p}_\lambda$ of $\mathfrak{g}$ as follows:

$$\mathfrak{p}_\lambda = \{X \in \mathfrak{p} \mid (\operatorname{ad} H)^2(X) = -\lambda(H)^2 X \text{ for all } H \in \mathfrak{a}\},$$

$$\mathfrak{k}_\lambda = \{X \in \mathfrak{k} \mid (\operatorname{ad} H)^2(X) = -\lambda(H)^2 X \text{ for all } H \in \mathfrak{a}\}.$$

Then $\mathfrak{p}_\lambda = \mathfrak{p}_{-\lambda}$, $\mathfrak{k}_\lambda = \mathfrak{k}_{-\lambda}$, $\mathfrak{p}_0 = \mathfrak{a}$ and $\mathfrak{k}_0$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. An element λ of $\mathfrak{a}^*$ is called a *restricted root* of $\mathfrak{g}$ with respect to $\mathfrak{a}$ if $\dim \mathfrak{p}_\lambda \neq 0$. We select a suitable ordering in $\mathfrak{a}^*$ and denote by Δ the set of all positive restricted roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$. Then we have

$$\mathfrak{p} = \mathfrak{a} + \sum_{\lambda \in \Delta} \mathfrak{p}_\lambda \quad \text{(orthogonal direct sum)}, \quad \mathfrak{k} = \mathfrak{k}_0 + \sum_{\lambda \in \Delta} \mathfrak{k}_\lambda, \tag{1.1}$$

$$[\mathfrak{a}, \mathfrak{k}_\lambda] = \mathfrak{p}_\lambda \quad \text{and} \quad [\mathfrak{a}, \mathfrak{p}_\lambda] = \mathfrak{k}_\lambda, \quad \lambda \in \Delta. \tag{1.2}$$

The following facts are fundamental (cf. [9]).
If $\lambda, \mu \in \Delta \cup \{0\}$, then

$$\begin{aligned} [\mathfrak{k}_\lambda, \mathfrak{k}_\mu] &\subset \mathfrak{k}_{\lambda+\mu} + \mathfrak{k}_{\lambda-\mu}, \\ [\mathfrak{k}_\lambda, \mathfrak{p}_\mu] &\subset \mathfrak{p}_{\lambda+\mu} + \mathfrak{p}_{\lambda-\mu}, \\ [\mathfrak{p}_\lambda, \mathfrak{p}_\mu] &\subset \mathfrak{k}_{\lambda+\mu} + \mathfrak{k}_{\lambda-\mu}. \end{aligned} \tag{1.3}$$

Moreover, if $\lambda + \mu \in \Delta \cup \{0\}$ or $\lambda - \mu \in \Delta \cup \{0\}$, then

$$[\mathfrak{k}_\lambda, \mathfrak{p}_\mu] \neq 0. \tag{1.4}$$

Let S denote a unit hypersphere in $\mathfrak{p}$ centered at the origin o. The adjoint representation group $\operatorname{Ad}(K)$ acts on $\mathfrak{p}$ as an isometry group. For any $a \in S$, the orbit $\operatorname{Ad}(K)a$ of a under $\operatorname{Ad}(K)$ is a submanifold in S, which is called an *R-space*. For any a $(\neq 0)$ in $\mathfrak{p}$, we put $M_a = \operatorname{Ad}(K)a$ for simplicity. For any real number $\xi \neq 0$, an R-space $M_{\xi a}$ is similar to an R-space M_a. On the other hand, every orbit in $\mathfrak{p}$ under $\mathrm{Ad}(K)$ meets $\mathfrak{a}$ ([32]). Therefore, we can say that all R-spaces M_a with $a \in S \cap \mathfrak{a}$ exhaust all R-spaces.

For a manifold L and a point l of L we denote by $T_l(L)$ the tangent space of L at l. If Q is a submanifold in L and q is a point of Q, then we denote by $T_q^N(Q)$ the normal space of Q in L at q.

For a point b of M_a, let $T_b^N(M_a)$ denote the normal space of M_a in S at b. Any vector X in $\mathfrak{p}$ can be uniquely written as $X = A + B + C$, where $A \in \boldsymbol{R}b$, $B \in T_b(M_a)$, $C \in T_b^N(M_a)$. Then we put

$$X_{S_b} = B + C, \quad X_{M_b} = B \quad \text{and} \quad X^{N_b} = C.$$

In particular, we put

$$X_S = X_{S_a}, \quad X_M = X_{M_a} \quad \text{and} \quad X^N = X^{N_a}.$$

Each vector X in $\mathfrak{k}$ induces a vector field X^* on $\mathfrak{p}$ as follows:

$$X_Y^* = \frac{d}{dt}\bigg|_0 \mathrm{Ad}(\exp tX)Y = [X, Y], \quad Y \in \mathfrak{p}. \tag{1.5}$$

Let a symbol X^* stand for a vector field $X^*|_S$ on S or a vector field $X^*|_{M_a}$ on M_a for simplicity. We put

$$a^\perp = \{X \in \mathfrak{a} \mid g_o(X, a) = 0\} \quad \text{and} \quad \Delta_a = \{\lambda \in \Delta \mid \lambda(a) = 0\}.$$

Then from (1.5) we have

$$\begin{cases} T_a(M_a) = [\mathfrak{k}, a] = \sum_{\lambda \in \Delta - \Delta_a} \mathfrak{p}_\lambda, \\ T_a^N(M_a) = a^\perp + \sum_{\lambda \in \Delta_a} \mathfrak{p}_\lambda. \end{cases} \tag{1.6}$$

From the definition of Δ_a we know easily the following:

(i) If $\lambda, \mu \in \Delta_a$ and $\lambda + \mu \in \Delta$, then

$$\lambda + \mu \in \Delta_a. \tag{1.7}$$

(ii) If $\lambda \in \Delta_a$, $\mu \in \Delta - \Delta_a$ and $\lambda + \mu \in \Delta$, then

$$\lambda + \mu \in \Delta - \Delta_a. \tag{1.8}$$

Let ∇ and $\bar{\nabla}$ denote the Riemannian connections of M_a and S, respectively. Let h denote the second fundamental form of M_a in S. Then we have the following fundamental formulas:

$$\begin{aligned}\bar{\nabla}_{X_a^*} Y^* &= \left(\frac{d}{dt}\bigg|_0 Y^*_{\mathrm{Ad}(\exp tX)a}\right)_S \\ &= \left(\frac{d}{dt}\bigg|_0 [Y, \mathrm{Ad}(\exp tX)a]\right)_S \\ &= [Y, [X, a]]_S, \\ \nabla_{X_a^*} Y^* &= \left(\frac{d}{dt}\bigg|_0 Y^*_{\mathrm{Ad}(\exp tX)a}\right)_M \\ &= \left(\frac{d}{dt}\bigg|_0 [Y, \mathrm{Ad}(\exp tX)a]\right)_M \\ &= [Y, [X, a]]_M, \quad X, Y \in \mathfrak{k}.\end{aligned}$$

From these we have

$$\begin{aligned}h(X_a^*, Y_a^*) &= (\bar{\nabla}_{X_a^*} Y^*)^N \\ &= [Y, [X, a]]^N, \quad X, Y \in \mathfrak{k}.\end{aligned} \tag{1.9}$$

From now on we assume that a symmetric space G/K is Hermitian, unless otherwise stated. We put $p = \dim \mathfrak{a}$. In the case where $p = 1$, since any R-space M_a is very simple, we can easily compute various geometrical quantities on M_a which we want to know in this paper. So we assume that $p \geq 2$.

Now we note the following fact.

LEMMA 1.1 ([8, p. 528]). *There are two possibilities Δ_1 and Δ_2 for Δ as follows. There exists a base $\{\lambda_1, \ldots, \lambda_p\}$ of $\mathfrak{a}^*$ such that*

$$\Delta_1 = \{2\lambda_i, \lambda_i \pm \lambda_j \mid 1 \leq i < j \leq p\},$$

$$\Delta_2 = \{\lambda_i, 2\lambda_i, \lambda_i \pm \lambda_j \mid 1 \leq i < j \leq p\}.$$

If Δ can be expressed as Δ_1 (resp. Δ_2), then Δ is called of *type C* (resp. *type BC*). We put $I = \{1, \ldots, p\}$. Let I_p denote the set of all permutations of I. Put

$$\varepsilon_i = \pm 1, \quad 1 \leq i \leq p.$$

For any $\sigma \in I_p$, we put

$$\mu_i = \varepsilon_i \lambda_{\sigma(i)}, \quad 1 \leq i \leq p.$$

Introduce a new lexicographic ordering in $\mathfrak{a}^*$ with respect to the basis $\{\mu_1, \ldots, \mu_p\}$. Then the set Δ' of all new positive restricted roots coincides with the set obtained from Δ by exchanging every symbol λ in Δ by the symbol μ. In this case, we shall say that *we took a reorder in* $\mathfrak{a}^*$, or *reordered* $\mathfrak{a}^*$.

Let J be the complex structure on G/K at the origin o, and put $\dim \mathfrak{p} = 2n+2$. Then we can consider $\mathfrak{p}$ a complex vector space $\boldsymbol{C}^{n+1}$. We denote by $P_n(\boldsymbol{C})$ the complex projective space, and by π the natural projection of S onto $P_n(\boldsymbol{C})$. The complex structure and the Fubini-Study metric on $P_n(\boldsymbol{C})$ can be naturally induced from J and g_o through π. We denote them by $\tilde{J}$ and $\langle\,,\rangle$, respectively. We denote the image $\pi(M_a)$ of an R-space M_a under consideration by $\tilde{M}_a$, which we shall call an *$\tilde{R}$-space*. Obviously every $\tilde{R}$-space is a homogeneous submanifold in $P_n(\boldsymbol{C})$.

Generally, let $\tilde{L}$ be a submanifold of $P_n(\boldsymbol{C})$ and put $L = \pi^{-1}(\tilde{L})$. Then L is a submanifold in S. For $q \in L$ and $\tilde{X} \in T_{\pi(q)}(\tilde{L})$, there exists a unique $\tilde{X}' \in T_q(L)$ such that $\tilde{X}' \in V_q$ and $\pi_{*_q}\tilde{X}' = \tilde{X}$, where V_q denotes the orthogonal complement of $J(q)$ in $T_q(L)$ and π_{*_q} the differential map of π at q. This $\tilde{X}'$ is called the *horizontal lift of* $\tilde{X}$ at q. Then we have $(\tilde{J}\tilde{X})' = J\tilde{X}'$. We denote by $T^N_{\pi(q)}(\tilde{L})$ the normal space of $\tilde{L}$ in $P_n(\boldsymbol{C})$ at $\pi(q)$ and put

$$\tilde{J}\tilde{X} = (\tilde{J}\tilde{X})_{\tilde{L}} + (\tilde{J}\tilde{X})^N,$$

where $(\tilde{J}\tilde{X})_{\tilde{L}} \in T_{\pi(q)}(\tilde{L})$ and $(\tilde{J}\tilde{X})^N \in T^N_{\pi(q)}(\tilde{L})$.

Let ∇ and $\tilde{\nabla}$ denote the Riemannian connections of L and $\tilde{L}$, respectively. We denote by h and $\tilde{h}$ the second fundamental forms of L in S and $\tilde{L}$ in $P_n(\boldsymbol{C})$, respectively. Then there is a following relation between covariant derivatives of h and $\tilde{h}$ (e.g., cf. [1])

$$(1.10)\qquad (\nabla_{\tilde{X}'}h)(\tilde{Y}', \tilde{Z}') = ((\tilde{\nabla}_{\tilde{X}}\tilde{h})(\tilde{Y}, \tilde{Z}) + \langle(\tilde{J}\tilde{X})_{\tilde{L}}, \tilde{Y}\rangle(\tilde{J}\tilde{Z})^N + \langle(\tilde{J}\tilde{X})_{\tilde{L}}, \tilde{Z}\rangle(\tilde{J}\tilde{Y})^N)', \quad \tilde{X}, \tilde{Y}, \tilde{Z} \in T_{\pi(q)}(\tilde{L}).$$

From this we see

$$(1.11)\qquad \nabla h = 0 \quad \text{on } V_q \Leftrightarrow \mathfrak{S}\tilde{\nabla}\tilde{h} = 0 \quad \text{on } T_{\pi(q)}(\tilde{L})$$

where $\mathfrak{S}$ denotes the cyclic sum.

Now we recall the notation of CR-submanifolds owing to A. Bejancu ([1]).

DEFINITION. A submanifold $\tilde{L}$ in $P_n(\boldsymbol{C})$ is called a *CR-submanifold* if there are two subbundles $\mathfrak{D}$ and $\mathfrak{D}^\perp$ of $T(\tilde{L})$ such that

(i) $T_{\tilde{l}}(\tilde{L}) = \mathfrak{D}_{\tilde{l}} + \mathfrak{D}_{\tilde{l}}^{\perp}$ (orthogonal sum) for each $\tilde{l} \in \tilde{L}$,

(ii) $\tilde{J}\mathfrak{D} = \mathfrak{D}$, $\tilde{J}\mathfrak{D}^{\perp} \subset T^N(\tilde{L})$,

where $T^N(\tilde{L})$ denotes the normal bundle of $\tilde{L}$ in $P_n(C)$.

If a CR-submanifold $\tilde{L}$ satisfies $\mathfrak{D} = 0$ (resp. $\mathfrak{D}^{\perp} = 0$), then $\tilde{L}$ is called *totally real* (resp. *holomorphic*). If a CR-submanifold $\tilde{L}$ satisfies $\tilde{J}\mathfrak{D}^{\perp} = T^N(\tilde{L})$, then $\tilde{L}$ is called *anti-holomorphic*.

§2. Some Basic Lemmas

Through this paper we preserve notations in §1. First we give some basic Lemmas for later use.

LEMMA 2.1. *Let G/K be a symmetric space and a be any point in $\mathfrak{a} \cap S$. Then the following holds.*

(i) *If $\lambda \in \Delta - \Delta_a$ and $\mu \in \Delta$, then*

$$[\mathfrak{k}_\lambda, \mathfrak{p}_\mu^N]^N = 0. \tag{2.1}$$

(ii) *If $X, Y \in \sum_{\lambda \in \Delta} \mathfrak{k}_\lambda$ and $Z \in \sum_{\lambda \in \Delta - \Delta_a} \mathfrak{k}_\lambda$, then*

$$[Z, [Y, [X, a]]^N]^N = 0. \tag{2.2}$$

(iii) *If $\lambda + \mu \in \Delta_a$ or $\lambda - \mu \in \Delta_a$, then*

$$[\mathfrak{k}_\lambda, \mathfrak{p}_\mu]^N \neq 0. \tag{2.3}$$

PROOF. (i) In the case where $\mu \in \Delta_a$ (resp. $\mu \in \Delta - \Delta_a$), from (1.6) we have $\mathfrak{p}_\mu^N = \mathfrak{p}_\mu$ (resp. $\mathfrak{p}_\mu^N = 0$). Hence from (1.3) we have

$$[\mathfrak{k}_\lambda, \mathfrak{p}_\mu^N] = [\mathfrak{k}_\lambda, \mathfrak{p}_\mu] \subset \mathfrak{p}_{\lambda+\mu} + \mathfrak{p}_{\lambda-\mu}.$$

On the other hand, since $\lambda + \mu, \lambda - \mu \in \Delta - \Delta_a$, we have from (1.6)

$$\mathfrak{p}_{\lambda+\mu}^N = 0 = \mathfrak{p}_{\lambda-\mu}^N,$$

which completes the proof of (i).

(ii) It suffices to prove that

$$[Z, [Y, [X, a]]^N]^N = 0 \quad \text{for } X \in \mathfrak{k}_\lambda, \ Y \in \mathfrak{k}_\mu, \ Z \in \mathfrak{k}_\nu,$$

where $\lambda, \mu \in \Delta$ and $\nu \in \Delta - \Delta_a$. From (1.3) we have

$$[Y, [X, a]] \in \mathfrak{p}_{\lambda+\mu} + \mathfrak{p}_{\lambda-\mu}.$$

Moreover, since

$$\mathfrak{p}^N_{\lambda+\mu} = \begin{cases} \mathfrak{p}_{\lambda+\mu} & \text{if } \lambda+\mu \in \Delta_a \\ 0 & \text{if } \lambda+\mu \notin \Delta_a \end{cases}$$

and

$$\mathfrak{p}^N_{\lambda-\mu} = \begin{cases} \mathfrak{p}_{\lambda-\mu} & \text{if } \lambda-\mu \in \Delta_a \\ 0 & \text{if } \lambda-\mu \notin \Delta_a, \end{cases}$$

we have

$$[Y, [X, a]]^N \in \mathfrak{p}^N_{\lambda+\mu} + \mathfrak{p}^N_{\lambda-\mu}.$$

Now (2.2) follows from (i).

(iii) This follows from (1.3), (1.4), and (1.6). (Q.E.D.)

In the case where a symmetric space G/K is Hermitian, we denote by $\mathfrak{z}$ the center of $\mathfrak{k}$. Then, as for a complex structure J, the following fact is known ([8], p. 376).

LEMMA 2.2. (i) *There exists a unique* (*up to sign*) $\tilde{Z} \in \mathfrak{z}$ *such that*

$$J = \operatorname{ad} \tilde{Z}|_{\mathfrak{p}};$$

(ii) *The element* $\tilde{Z}$ *can be written as*

$$\tilde{Z} = Z_0 + \sum_{i=1}^{p} Z_{2\lambda_i},$$

where $Z_0 \in \mathfrak{k}_0$ *and* $0 \neq Z_{2\lambda_i} \in \mathfrak{k}_{2\lambda_i}$.

Using this, we shall prove a key lemma.

LEMMA 2.3. *We have the following equations*:

(2.4) $$J\mathfrak{p}_{\lambda_i \pm \lambda_j} = \mathfrak{p}_{\lambda_i \mp \lambda_j},$$

(2.5) $$J\mathfrak{p}_{\lambda_i} = \mathfrak{p}_{\lambda_i},$$

(2.6) $$J\mathfrak{a} = \sum_{i=1}^{p} \mathfrak{p}_{2\lambda_i},$$

(2.7) $$\sum_{i=1}^{p} J\mathfrak{p}_{2\lambda_i} = \mathfrak{a}.$$

PROOF. Since $\tilde{Z}$ satisfies $[\tilde{Z}, \mathfrak{k}] = 0$, we see from Lemma 2.2(ii) that

$$[Z_0, X_\lambda] + \sum_{i=1}^{p} [Z_{2\lambda_i}, X_\lambda] = 0, \quad X_\lambda \in \mathfrak{k}_\lambda.$$

Owing to (1.3), we have

$$[Z_0, X_\lambda] = 0 \quad \text{for } X_\lambda \in \mathfrak{k}_\lambda,$$

where $\lambda \in \Delta - \{\lambda_1, \dots \lambda_p\}$. From this equation and (1.2), we have

$$0 = [\mathfrak{a}, [Z_0, \mathfrak{k}_\lambda]] = [Z_0, [\mathfrak{a}, \mathfrak{k}_\lambda]] = [Z_0, \mathfrak{p}_\lambda], \quad \lambda \in \Delta - \{\lambda_1, \dots \lambda_p\}.$$

It follows from Lemma 1.1 and Lemma 2.2 that

$$J\mathfrak{p}_\lambda = \begin{cases} \mathfrak{p}_\lambda & \text{if } \lambda \in \{\lambda_1, \dots \lambda_p\} \\ \sum_{i=1}^{p} [Z_{2\lambda_i}, \mathfrak{p}_\lambda] & \text{if } \lambda \in \Delta - \{\lambda_1, \dots \lambda_p\}. \end{cases}$$

Now the Lemma follows from (1.3). (Q.E.D.)

§3. *CR*-Submanifolds in a Complex Projective Space $P_n(\boldsymbol{C})$

THEOREM 3.1. *Let G/K be a Hermitian symmetric space and a be any point in $\mathfrak{a} \cap S$. Then an $\tilde{R}$-space $\tilde{M}_a$ is a CR-submanifold in $P_n(\boldsymbol{C})$.*

REMARK. Y. Shimizu ([31]) showed that an $\tilde{R}$-space $\tilde{M}_a$ is a *CR*-submanifold in $P_n(\boldsymbol{C})$ if $\Delta_a = \varnothing$.

PROOF OF THEOREM 3.1. From (1.6) we have

$$T_a(M_a) = \sum_{\lambda \in \Delta - \Delta_a} \mathfrak{p}_\lambda.$$

Lemma 2.3 implies that there are elements λ and μ in $\Delta - \Delta_a$ such that

$$J\mathfrak{p}_\lambda = \mathfrak{p}_\lambda \quad \text{and} \quad J\mathfrak{p}_\mu \subset T^N(M_a).$$

Then we put

$$I_\pm = \{(i,j) \mid \lambda_i + \lambda_j, \lambda_i - \lambda_j \in \Delta - \Delta_a, 1 \leq i < j \leq p\},$$

$$\mathfrak{p}_{(i,j)} = \begin{cases} \mathfrak{p}_{\lambda_i + \lambda_j} + \mathfrak{p}_{\lambda_i - \lambda_j} & \text{if } (i,j) \in I_\pm \\ 0 & \text{if } (i,j) \notin I_\pm. \end{cases}$$

Moreover we put

$$\hat{\mathfrak{D}}_a = \begin{cases} \sum_{(i,j)\in I_\pm} \mathfrak{p}_{(i,j)} & \text{if } \Delta \text{ is of type } C \\ \sum_{(i,j)\in I_\pm} \mathfrak{p}_{(i,j)} + \sum_{\lambda_i\in\Delta-\Delta_a} \mathfrak{p}_{\lambda_i} & \text{if } \Delta \text{ is of type } BC. \end{cases} \tag{3.1}$$

By (1.1) and Lemma 2.2, we see that $\hat{\mathfrak{D}}_a$ and $J(a)$ are mutually orthogonal. Let $\hat{\mathfrak{D}}_a^\perp$ denote the orthogonal complement of $\hat{\mathfrak{D}}_a + J(a)$ in $T_a(M_a)$. Then we have

$$T_a(M_a) = \hat{\mathfrak{D}}_a + \hat{\mathfrak{D}}_a^\perp + \boldsymbol{R}J(a) \quad \text{(direct sum)}. \tag{3.2}$$

Since π is a submersion, a space $\hat{\mathfrak{D}}_a + \hat{\mathfrak{D}}_a^\perp$ can be identified with $T_{\pi(a)}(\tilde{M}_a)$. By the action of $\mathrm{Ad}(K)$ we can construct two subbundles $\mathfrak{D}$ and $\mathfrak{D}^\perp$ on $\tilde{M}_a$ such that

$$\begin{gathered} \pi_*(\hat{\mathfrak{D}}_a) = \mathfrak{D}_{\pi(a)}, \quad \pi_*(\hat{\mathfrak{D}}_a^\perp) = \mathfrak{D}_{\pi(a)}^\perp, \\ \tilde{J}\mathfrak{D} = \mathfrak{D}, \quad \tilde{J}\mathfrak{D}^\perp \subset T^N(\tilde{M}_a). \end{gathered} \tag{3.3}$$

Since $J = \mathrm{ad}\,\tilde{Z}|_{\mathfrak{p}}$, the bundles $\mathfrak{D}$ and $\mathfrak{D}^\perp$ are well-defined. These $\mathfrak{D}$ and $\mathfrak{D}^\perp$ are the desired subbundles of $T(\tilde{M}_a)$. (Q.E.D.)

Now we can find a class of R-spaces with a distinguished property:

THEOREM 3.2. *Let G/K be a Hermitian symmetric space and a be any point in $\mathfrak{a}\cap S$. Then*

(i) *An $\tilde{R}$-space $\tilde{M}_a$ is anti-holomorphic if and only if for a suitable reordering in $\mathfrak{a}^*$ the set Δ_a is a subset of $\{\lambda_i - \lambda_j \mid 1 \le i < j \le p\}$.*

(ii) *An $\tilde{R}$-space $\tilde{M}_a$ is totally real if and only if Δ is of type C and for a suitable reordering in $\mathfrak{a}^*$ the set Δ_a can be expressed as $\{\lambda_i - \lambda_j \mid 1 \le i < j \le p\}$.*

(iii) *An $\tilde{R}$-space $\tilde{M}_a$ is holomorphic if and only if for a suitable reordering in $\mathfrak{a}^*$ the set Δ_a is given by*

$$\{2\lambda_i, \lambda_i \pm \lambda_j \mid 2 \le i < j \le p\} \qquad \text{if } \Delta \text{ is of type } C$$

$$\{\lambda_i, 2\lambda_i, \lambda_i \pm \lambda_j \mid 2 \le i < j \le p\} \quad \text{if } \Delta \text{ is of type } BC.$$

PROOF. (i) Let $\tilde{M}_a$ be anti-holomorphic. First we assert

$$\lambda_i(a) \neq 0, \quad i = 1, \ldots, p. \tag{3.4}$$

In fact, assume that $\lambda_i(a) = 0$ for some index i. Then from (1.6) and (3.3) we have

$$\mathfrak{p}_{2\lambda_i} \subset T_a^N(M_a) \quad \text{and} \quad J\mathfrak{p}_{2\lambda_i} \subset JT_a^N(M_a) = \hat{\mathfrak{D}}_a^\perp.$$

On the other hand, since $J\mathfrak{p}_{2\lambda_i} \subset \mathfrak{a}$, we have from (1.1) and (1.6)

$$J\mathfrak{p}_{2\lambda_i} \not\subset \hat{\mathfrak{D}}_a^{\perp},$$

which is a contradiction. Thus (3.4) was proved. Since the case where $\Delta_a = \emptyset$ is trivial, let $\Delta_a \neq \emptyset$. Then by (3.4) there are indices i and j such that

$$\lambda_i + \lambda_j \in \Delta_a \quad \text{or} \quad \lambda_i - \lambda_j \in \Delta_a.$$

For this i, we put

$$\Delta' = \{\lambda \in \Delta_a \,|\, \lambda = \lambda_i + \lambda_j \text{ or } \lambda = \lambda_i - \lambda_j \text{ for some } j\}$$

and denote by k the cardinal number of Δ'. Since for any i and j with $1 \leq i < j \leq p$ the case where both $\lambda_i + \lambda_j$ and $\lambda_i - \lambda_j$ belong to Δ_a can not occur by (3.4), we can reorder $\mathfrak{a}^*$ so that

$$\Delta' = \{\lambda_1 - \lambda_2, \ldots, \lambda_1 - \lambda_{k+1}\}.$$

Put

$$\Delta(1) = \{\lambda_i - \lambda_j \,|\, 1 \leq i < j \leq k+1\}.$$

Then $\Delta(1) \subset \Delta_a$. If $\Delta_a - \Delta(1) \neq \emptyset$, then we can continue this procedure for the set $\Delta_a - \Delta(1)$ and obtain a subset $\Delta(2)$ of $\Delta_a - \Delta(1)$ such that $\Delta(2)$ is given by the form $\{\lambda_i - \lambda_j \,|\, k+2 \leq i < j \leq l+1\}$, where $l-k-1$ is the cardinal number of $\Delta(2)$. By the induction, Δ_a is given by the subset of $\{\lambda_i - \lambda_j \,|\, 1 \leq i < j \leq p\}$. The converse is obvious from Lemma 2.3, (1.6) and (3.3).

(ii) Let $\tilde{M}_a$ be totally real. By (2.5), Δ is of type C. First we assert that $2\lambda_i \in \Delta - \Delta_a$ for any index i. In fact, assume that there exists an index j such that $2\lambda_j \in \Delta_a$. Since a is nonzero, there exists an index k such that $2\lambda_k \in \Delta - \Delta_a$. Then for these indices j and k, we have from (1.8)

$$\lambda_j + \lambda_k, \;\; \lambda_j - \lambda_k \in \Delta - \Delta_a,$$

which contradicts (2.5). Thus the assertion was proved. Since for any indices i and j

$$\lambda_i + \lambda_j \in \Delta_a \Leftrightarrow \lambda_i - \lambda_j \in \Delta - \Delta_a,$$

we can reorder $\mathfrak{a}^*$ so that Δ_a is given by

$$\{\lambda_i - \lambda_j \,|\, 1 \leq i < j \leq p\}.$$

The converse follows from Lemma 2.3, (1.6) and (3.3).

(iii) Let $\tilde{M}_a$ be holomorphic. First we assert that there exists an only index i such that $\lambda_i(a) \neq 0$. In fact, if there exist two indices i and j such that $\lambda_i(a) \neq 0$ and $\lambda_j(a) \neq 0$, a 2-dimensional subspace $J(\mathfrak{p}_{2\lambda_i} + \mathfrak{p}_{2\lambda_j})$ of $T_a(M_a)$ must contain a nonzero element of $\mathfrak{a}$, which contradict (1.6). Hence the assertion was proved. Then we have only to reorder $\mathfrak{a}^*$ so that $\lambda_1(a) \neq 0$. The converse follows from Lemma 2.3, (1.6) and (3.3). (Q.E.D.)

REMARK. Recently Choe, Ki and Takagi ([4]) and Ki, Song and Takagi ([15]) gave some examples of CR-submanifolds in $P_n(\boldsymbol{C})$. These examples form a class of $\tilde{R}$-spaces constructed from Theorem 3.2.

REMARK. For every totally real $\tilde{R}$-space $\tilde{M}_a$, we have

$$\dim T(\tilde{M}_a) = \dim T^N(\tilde{M}_a).$$

This is already pointed out by S. Kobayashi ([17]).

§4. Second Fundamental Forms of R-Spaces and Its Covariant Derivatives

For a while, we do not assume that a symmetric space G/K is Hermitian. We define the covariant derivative ∇h of h on $T_a(S)$ as follows:

$$(\nabla_{X_a^*} h)(Y_a^*, Z_a^*) := (\bar{\nabla}_{X_a^*} h(Y_a^*, Z_a^*))^N - h(\nabla_{X_a^*} Y^*, Z_a^*) - h(Y_a^*, \nabla_{X_a^*} Z^*).$$

THEOREM 4.1. *Let G/K be a symmetric space and a be any point in $\mathfrak{a} \cap S$. Let ∇ and h denote the Riemannian connection of an R-space M_a and the second fundamental form of M_a in S, respectively. Then we have*

$$(4.1) \qquad (\nabla_{X_a^*} h)(Y_a^*, Z_a^*) = -[X, [Z, [Y, a]]_M]^N - [Y, [Z, [X, a]]_M]^N,$$

where $X, Y, Z \in \mathfrak{k}$.

PROOF. First we calculate $h(\nabla_{X_a^*} Y^*, Z_a^*)$. From (1.9), we have

$$h(\nabla_{X_a^*} Y^*, Z_a^*) = (\bar{\nabla}_L Z^*)^N,$$

where $L = \nabla_{X_a^*} Y^*$. This L can be written as

$$L = \sum_{\lambda \in \Delta - \Delta_a} L_\lambda,$$

where $L_\lambda \in \mathfrak{p}_\lambda$. By Takagi and Takahashi ([32]), we see that

$$L = [Q, a] = [Y, [X, a]]_M,$$

where $Q = \sum_{\lambda \in \Delta - \Delta_a} (1/\lambda(a)^2)[a, L_\lambda]$. From the equation above we have

$$\begin{aligned}
\bar{\nabla}_L Z^* &= \left(\frac{d}{dt}\bigg|_0 Z^*_{\mathrm{Ad}(\exp tQ)a} \right)_S \\
&= \left(\frac{d}{dt}\bigg|_0 [Z, \mathrm{Ad}(\exp tQ)a] \right)_S \\
&= [Z, [Q, a]]_S \\
&= [Z, [Y, [X, a]]_M]_S, \quad X, Y, Z \in \mathfrak{k}.
\end{aligned}$$

Hence we obtain

$$h(\nabla_{X_a^*} Y^*, Z_a^*) = [Z, [Y, [X, a]]_M]^N.$$

Next, we have

$$\begin{aligned}
(\bar{\nabla}_{X_a^*} h(Y_a^*, Z_a^*))^N &= \left(\frac{d}{dt}\bigg|_0 (\bar{\nabla}_{Y^*} Z^* - \nabla_{Y^*} Z^*)_{a(t)} \right)^N \\
&= \left(\frac{d}{dt}\bigg|_0 \bar{\nabla}_{Y^*_{a(t)}} Z^* \right)^N - \left(\frac{d}{dt}\bigg|_0 (\bar{\nabla}_{Y^*_{a(t)}} Z^*)_{M_{a(t)}} \right)^N,
\end{aligned}$$

where $a(t) = \mathrm{Ad}(\exp tX)a$. As for the first term, we have

$$\begin{aligned}
\left(\frac{d}{dt}\bigg|_0 \bar{\nabla}_{Y^*_{a(t)}} Z^* \right) &= \frac{d}{dt}\bigg|_0 [Z, Y^*_{a(t)}]_{S_{a(t)}} \\
&= \frac{d}{dt}\bigg|_0 [Z, [Y, a(t)]]_{S_{a(t)}} \\
&= [Z, [Y, [X, a]]_S]_S.
\end{aligned}$$

As for the second term, we have

$$\begin{aligned}
\frac{d}{dt}\bigg|_0 (\bar{\nabla}_{Y^*_{a(t)}} Z^*)_{M_{a(t)}} &= \frac{d}{dt}\bigg|_0 [Z, Y^*_{a(t)}]_{M_{a(t)}} \\
&= \frac{d}{dt}\bigg|_0 [Z, [Y, a(t)]]_{M_{a(t)}} \\
&= \frac{d}{dt}\bigg|_0 \mathrm{Ad}(\exp tX)[\mathrm{Ad}(\exp -tX)Z, [\mathrm{Ad}(\exp -tX)Y, a]]_M \\
&= [X, [Z, [Y, a]]_M]_S - [[X, Z], [Y, a]]_M - [Z, [[X, Y], a]]_M.
\end{aligned}$$

Consequently using the equations above, (1.3), (1.6) and (2.2), we have

$$\begin{aligned}(\nabla_{X_a^*}h)(Y_a^*, Z_a^*) &= [Z,[Y,[X,a]]_S]^N - [X,[Z,[Y,a]]_M]^N \\ &\quad - [Z,[Y,[X,a]]_M]^N - [Y,[Z,[X,a]]_M]^N \\ &= -[X,[Z,[Y,a]]_M]^N - [Y,[Z,[X,a]]_M]^N. \qquad \text{(Q.E.D.)}\end{aligned}$$

D. Ferus ([5], [6]) proved the following facts.

(1) Let a be a point on S such that the endomorphism $(\operatorname{ad} a)^2$ of $\mathfrak{p}$ has eigenvalues $0, 1$. Then an R-space M_a is a parallel submanifold in S.

(2) All R-spaces M_a obtained in (1) exhaust all parallel submanifolds in S.

Kobayashi and Nagano ([18]) and T. Nagano ([21]) classified completely R-spaces satisfying (1). After some time, S. Kobayashi ([17]) realized a various class of symmetric R-spaces.

In the remainder of this paper, we assume that symmetric space G/K is Hermitian.

From (1), (2) and Theorem 3.2(ii) we have:

LEMMA 4.2. *An R-space M_a is parallel in S if and only if the corresponding $\tilde{R}$-space $\tilde{M}_a$ is totally real.*

Here we recall the natural projection $\pi : S \to P_n(\boldsymbol{C})$. For each $a \in \mathfrak{a} \cap S$, we have from π

$$T_a(M_a) = \boldsymbol{R}J(a) + V_a \quad \text{(orthogonal direct sum)}.$$

From (3.2) we have

$$V = \hat{\mathfrak{D}} + \hat{\mathfrak{D}}^{\perp}. \tag{4.2}$$

If an R-space M_a satisfies

$$\nabla h = 0 \quad \text{on } V,$$

then we shall call M_a *almost parallel.*

First we prepare the following Lemma:

LEMMA 4.3. *Let $a \in \mathfrak{a} \cap S$ satisfy*

$$2\lambda_k,\ 2\lambda_l,\ \lambda_k + \lambda_l,\ \lambda_k - \lambda_l \in \Delta - \Delta_a \quad \textit{for some } k, l\ (k \neq l).$$

Then an R-space M_a is not almost parallel.

PROOF. By (4.2), it suffices to show that there exist elements $X, Y, Z \in \sum_{\lambda \in \Delta - \Delta_a} \mathfrak{k}_\lambda$ such that

$$X_a^*, Y_a^*, Z_a^* \in \hat{\mathfrak{D}}_a + \hat{\mathfrak{D}}_a^\perp \quad \text{and} \quad (\nabla_{X_a^*} h)(Y_a^*, Z_a^*) \neq 0. \tag{4.3}$$

The author could not show the existence of elements X, Y and Z of $\sum_{\lambda \in \Delta - \Delta_a} \mathfrak{k}_\lambda$ satisfying (4.3) by a general method. But, according as every Hermitian symmetric space G/K we can find elements X, Y and Z of $\sum_{\lambda \in \Delta - \Delta_a} \mathfrak{k}_\lambda$ satisfying (4.3). In the following we show this for a typical Hermitian symmetric space G/K and abbreviate the proofs for every other Hermitian symmetric space since we have only to apply the same method.

Let $0 < p \leq q$ be integers and $M = SU(p+q)/S(U_p \times U_q)$ be a Hermitian symmetric space. Let E_{ij} denote $(p+q) \times (p+q)$ matrix with entry 1 where the ith row and jth column meet, all other entries being 0. Let I_p denote the unit matrix of order p. We put

$$I_{p,q} = \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix}.$$

Let $\mathfrak{g} = \mathfrak{su}(p+q)$ denote the Lie algebra of $SU(p+q)$ and θ the involutive automorphism of $\mathfrak{g}$ defined by $\theta(X) = I_{p,q} X I_{p,q}$ ([8, p. 454 and p. 347–p. 349]). Let $\mathfrak{k}$ (resp. $\mathfrak{p}$) be the eigenspace of θ for the eigenvalue $+1$ (resp. -1). Then

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \middle| \begin{matrix} A \in \mathfrak{u}(p), & B \in \mathfrak{u}(q) \\ \mathrm{Tr}(A+B) = 0 & \end{matrix} \right\} \quad \text{and}$$

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & Z \\ -{}^t\bar{Z} & 0 \end{pmatrix} \middle| Z : p \times q \text{ complex matrix} \right\}.$$

A maximal abelian subspace $\mathfrak{a}$ and the complex structure J on $\mathfrak{p}$ are given by

$$\mathfrak{a} = \sum_{i=1}^{p} \sqrt{-1}\boldsymbol{R}(E_{i,p+i} + E_{p+i,i}) \quad \text{and} \quad J = \mathrm{ad}\left(\sqrt{-1} \begin{pmatrix} \frac{q}{p+q} I_p & 0 \\ 0 & -\frac{p}{p+q} I_q \end{pmatrix} \right).$$

The positive restricted root system Δ is given by:

$$\{2\lambda_i, \lambda_i \pm \lambda_j \mid 1 \leq i < j \leq p\} \qquad \text{if } p = q$$

$$\{\lambda_i, 2\lambda_i, \lambda_i \pm \lambda_j \mid 1 \leq i < j \leq p\} \quad \text{if } p < q.$$

Here

$$\lambda_j(\sqrt{-1}(E_{i,p+i} + E_{p+i,i})) = \sqrt{-1}\delta_{ji}, \quad 1 \leq j \leq p.$$

By a direct calculation, we have

$$\mathfrak{k}_{\lambda_i-\lambda_j} = \{x(E_{ij} + E_{p+i,p+j}) - \bar{x}(E_{ji} + E_{p+j,p+i}) \mid x \in \boldsymbol{C}\} \quad (1 \leq i < j \leq p)$$

$$\mathfrak{k}_{\lambda_i+\lambda_j} = \{x(E_{ij} - E_{p+i,p+j}) - \bar{x}(E_{ji} - E_{p+j,p+i}) \mid x \in \boldsymbol{C}\} \quad (1 \leq i < j \leq p)$$

$$\mathfrak{k}_{2\lambda_i} = \sqrt{-1}\boldsymbol{R}(E_{ii} - E_{p+i,p+i}), \quad (1 \leq i \leq p)$$

$$\mathfrak{k}_{\lambda_i} = \sum_{\alpha=1}^{q-p} \boldsymbol{R}(E_{p+i,2p+\alpha} - E_{2p+\alpha,p+i}) + \sum_{\alpha=1}^{q-p} \boldsymbol{R}\sqrt{-1}(E_{p+i,2p+\alpha} + E_{2p+\alpha,p+i}) \quad (1 \leq i \leq p)$$

$$\mathfrak{p}_{\lambda_i-\lambda_j} = \{x(E_{i,p+j} + E_{p+i,j}) - \bar{x}(E_{j,p+i} + E_{p+j,i}) \mid x \in \boldsymbol{C}\} \quad (1 \leq i < j \leq p)$$

$$\mathfrak{p}_{\lambda_i+\lambda_j} = \{x(E_{i,p+j} - E_{p+i,j}) + \bar{x}(E_{j,p+i} - E_{p+j,i}) \mid x \in \boldsymbol{C}\} \quad (1 \leq i < j \leq p)$$

$$\mathfrak{p}_{2\lambda_i} = \boldsymbol{R}(E_{i,p+i} - E_{p+i,i}) \quad (1 \leq i \leq p)$$

$$\mathfrak{p}_{\lambda_i} = \sum_{\alpha=1}^{q-p} \boldsymbol{R}(E_{i,2p+\alpha} - E_{2p+\alpha,i}) + \sum_{\alpha=1}^{q-p} \boldsymbol{R}\sqrt{-1}(E_{i,2p+\alpha} + E_{2p+\alpha,i}) \quad (1 \leq i \leq p).$$

Here we may put $k = 1$ and $l = 2$, that is,

$$a = \sum_{i=1}^{2} \sqrt{-1}a_i(E_{i,p+i} + E_{p+i,i}),$$

where $a_1 \neq 0$, $a_2 \neq 0$, $a_1^2 \neq a_2^2$, $a_i \in \boldsymbol{R}$. Then we see that

$$a^{\perp} = \boldsymbol{R}\sqrt{-1}(a_2(E_{1,p+1} + E_{p+1,1}) - a_1(E_{2,p+2} + E_{p+2,2})) + \sum_{i=3}^{p} \sqrt{-1}\boldsymbol{R}(E_{i,p+i} + E_{p+i,i}),$$

$$Ja^{\perp} = \boldsymbol{R}(a_2(E_{1,p+1} - E_{p+1,1}) - a_1(E_{2,p+2} - E_{p+2,2})) + \sum_{i=3}^{p} \boldsymbol{R}(E_{i,p+i} - E_{p+i,i}),$$

$$\hat{\mathfrak{D}}_a = \sum_{i=2}^{p} \mathfrak{p}_{\lambda_1 \pm \lambda_i} + \sum_{j=3}^{p} \mathfrak{p}_{\lambda_2 \pm \lambda_j}, \quad \hat{\mathfrak{D}}_a^{\perp} = Ja^{\perp}.$$

We put

$$X = a_2^2\sqrt{-1}(E_{11} - E_{p+1,p+1}) - a_1^2\sqrt{-1}(E_{22} - E_{p+2,p+2}) \in \mathfrak{k}_{2\lambda_1} + \mathfrak{k}_{2\lambda_2},$$

$$Y = y(E_{12} + E_{p+1,p+2}) - \bar{y}(E_{21} + E_{p+2,p+1}) \in \mathfrak{k}_{\lambda_1-\lambda_2},$$

$$Z = z(E_{12} - E_{p+1,p+2}) - \bar{z}(E_{21} - E_{p+2,p+1}) \in \mathfrak{k}_{\lambda_1+\lambda_2},$$

where $y\bar{z} \neq \bar{y}z$ and $y, z \in \boldsymbol{C}$. Then we have

$$[Z,[Y,a]]_M = -\sqrt{-1}(a_1 - a_2)(y\bar{z} - \bar{y}z)(E_{1,p+1} - E_{p+1,1} + E_{2,p+2} - E_{p+2,2}),$$

$$\begin{aligned}[X,[Z,[Y,a]]_M]_S &= -2a_2^2(a_1 - a_2)(y\bar{z} - \bar{y}z)(E_{1,p+1} + E_{p+1,1}) \\ &\quad + 2a_1^2(a_1 - a_2)(y\bar{z} - \bar{y}z)(E_{2,p+2} + E_{p+2,2}),\end{aligned}$$

$$\begin{aligned}[Z,[X,a]]_M &= 2a_1a_2z(a_1 - a_2)(E_{1,p+2} + E_{p+1,2}) \\ &\quad - 2a_1a_2\bar{z}(a_1 - a_2)(E_{2,p+1} + E_{p+2,1}),\end{aligned}$$

$$[Y,[Z,[X,a]]_M]_S = -2a_1a_2(a_1 - a_2)(y\bar{z} - \bar{y}z)(E_{1,p+1} + E_{p+1,1} - E_{2,p+2} - E_{p+2,2}).$$

Thus we have

$$\begin{aligned}&[X,[Z,[Y,a]]_M]_S + [Y,[Z,[X,a]]_M]_S \\ &\quad = -2a_2(a_1 - a_2)(a_1 + a_2)(y\bar{z} - \bar{y}z)(E_{1,p+1} + E_{p+1,1}) \\ &\qquad + 2a_1(a_1 - a_2)(a_1 + a_2)(y\bar{z} - \bar{y}z)(E_{2,p+2} + E_{p+2,2}) \\ &\quad \in a^{\perp}.\end{aligned}$$

From this and (4.1), we have

$$(\nabla_{X_a^*}h)(Y_a^*, Z_a^*) \neq 0. \qquad \text{(Q.E.D.)}$$

THEOREM 4.4. *Let G/K be a Hermitian symmetric space and a be a point in $\mathfrak{a} \cap S$. Then an R-space M_a is almost parallel but not parallel if and only if the corresponding $\tilde{R}$-space $\tilde{M}_a$ is holomorphic.*

PROOF. Let M_a be almost parallel but not parallel. By Theorem 3.2(ii) and its proof, it suffices to prove that there exists an only index i such that $2\lambda_i \in \Delta - \Delta_a$. For this, we put

$$C_2 = \{i \mid \lambda_i(a) \neq 0\}.$$

It suffices to show that $\sharp C_2 = 1$, where $\sharp C_2$ denotes the cardinal number of C_2.

The case where Δ is of type C. Suppose that $\sharp C_2 = p$. Then for any index i we have $2\lambda_i \in \Delta - \Delta_a$. We assert that there exist indices i and j such that $\lambda_i + \lambda_j, \lambda_i - \lambda_j \in \Delta - \Delta_a$. If not so, then for any indices i and j with $1 \leq i < j \leq p$, we have

$$\lambda_i + \lambda_j \in \Delta - \Delta_a, \quad \lambda_i - \lambda_j \in \Delta_a \quad \text{or} \quad \lambda_i - \lambda_j \in \Delta - \Delta_a, \quad \lambda_i + \lambda_j \in \Delta_a.$$

Then for a suitable reordering in $\mathfrak{a}^*$, Δ_a can be expressed as

$$\{\lambda_i - \lambda_j \mid 1 \leq i < j \leq p\},$$

which contradicts Theorem 3.2(ii) and Lemma 4.2. Thus our assertion was proved. This and Lemma 4.3 imply that M_a is not almost parallel, which is a contradiction. Hence we have $\sharp C_2 < p$. Suppose that $2 \leq \sharp C_2$. Then there exist indices i and j such that $2\lambda_i, 2\lambda_j \in \Delta - \Delta_a$. Since $\sharp C_2 < p$, there exists an index k such that $2\lambda_k \in \Delta_a$. Since $2\lambda_i, 2\lambda_j \in \Delta - \Delta_a$, we choose $0 \neq X \in \mathfrak{k}_{2\lambda_i} + \mathfrak{k}_{2\lambda_j}$ of $X_a^* \in \hat{\mathfrak{D}}_a^{\perp}$. Let $0 \neq Y \in \mathfrak{k}_{\lambda_i+\lambda_k}$. Then from (1.6) and (1.8) we have $0 \neq Y_a^* \in \hat{\mathfrak{D}}_a$. Then we have from (4.1) and (1.3)

$$\begin{aligned}(\nabla_{Y_a^*}h)(Y_a^*, X_a^*) &= -2[Y, [X, [Y, a]]_M]^N \\ &\in [\mathfrak{k}_{\lambda_i+\lambda_k}, \mathfrak{p}_{\lambda_i-\lambda_k}]^N.\end{aligned}$$

Since $\lambda_i + \lambda_k - (\lambda_i - \lambda_k) = 2\lambda_k \in \Delta_a$, from (2.3) we have $(\nabla_{Y_a^*}h)(Y_a^*, X_a^*) \neq 0$, which is a contradiction. Thus we have $\sharp C_2 = 1$.

The case where Δ is of type BC. Suppose that $\sharp C_2 \geq 2$. Then there exist two indices i and j such that $2\lambda_i, 2\lambda_j \in \Delta - \Delta_a$. If both $\lambda_i + \lambda_j$ and $\lambda_i - \lambda_j$ belong to $\Delta - \Delta_a$, we see from Lemma 4.3 that M_a is not almost parallel, which is a contradiction. Hence we may assume that

$$\lambda_i + \lambda_j \in \Delta - \Delta_a \quad \text{and} \quad \lambda_i - \lambda_j \in \Delta_a.$$

Let $X \in \mathfrak{k}_{\lambda_i}$ and $Y \in \mathfrak{k}_{\lambda_i+\lambda_j}$. Then by (2.4) and (2.5), both X_a^* and Y_a^* belong to $\hat{\mathfrak{D}}_a + \hat{\mathfrak{D}}_a^{\perp}$. For these X and Y it follows from (4.1) and (1.3) that

$$\begin{aligned}(\nabla_{X_a^*}h)(X_a^*, Y_a^*) &= -2[X, [Y, [X, a]]_M]^N \\ &\in [\mathfrak{k}_{\lambda_i}, \mathfrak{p}_{\lambda_j}]^N.\end{aligned}$$

On the other hand, since $\lambda_i - \lambda_j \in \Delta_a$, we have from (2.3)

$$(\nabla_{X_a^*}h)(X_a^*, Y_a^*) \neq 0,$$

which is a contradiction. From the facts above, we have $\sharp C_2 = 1$.

Conversely, assume that $\tilde{M}_a$ be holomorphic. First let us prove

$$\nabla h = 0 \quad \text{on } \hat{\mathfrak{D}}_a + \hat{\mathfrak{D}}_a^{\perp}.$$

By Theorem 3.2(iii) and (3.1) we have

$$\hat{\mathfrak{D}}_a + \hat{\mathfrak{D}}_a^{\perp} = \begin{cases} \sum_{i=2}^{p} \mathfrak{p}_{\lambda_1 \pm \lambda_i} & \text{if } \Delta \text{ is of type } C \\ \sum_{i=2}^{p} \mathfrak{p}_{\lambda_1 \pm \lambda_i} + \mathfrak{p}_{\lambda_1} & \text{if } \Delta \text{ is of type } BC. \end{cases}$$

Hence it suffices to prove that:

(a) If Δ is of type C, then for any $X, Y, Z \in \sum_{i=2}^{p} \mathfrak{k}_{\lambda_1 \pm \lambda_i}$

$$(\nabla_{X_a^*} h)(Y_a^*, Z_a^*) = 0.$$

(b) If Δ is of type BC, then for any $X, Y, Z \in \mathfrak{k}_{\lambda_1} + \sum_{i=2}^{p} \mathfrak{k}_{\lambda_1 \pm \lambda_i}$,

$$(\nabla_{X_a^*} h)(Y_a^*, Z_a^*) = 0.$$

To prove (a), it suffices to prove that

$$(\nabla_{X_a^*} h)(Y_a^*, Z_a^*) = 0 \quad \text{for } X \in \mathfrak{k}_{\lambda_1 \pm \lambda_i}, \quad Y \in \mathfrak{k}_{\lambda_1 \pm \lambda_j}, \quad Z \in \mathfrak{k}_{\lambda_1 \pm \lambda_k},$$

where $i, j, k \in \{2, \dots, p\}$. From (4.1) and (1.3) we have

$$\begin{aligned}(\nabla_{X_a^*} h)(Y_a^*, Z_a^*) &= -[X, [Z, [Y, a]]_M]^N - [Y, [Z, [X, a]]_M]^N \\ &\in \mathfrak{p}^N_{\lambda_1 \pm \lambda_i \pm (\lambda_1 \pm \lambda_j) \pm (\lambda_1 \pm \lambda_k)}.\end{aligned}$$

On the other hand, if $\lambda_1 \pm \lambda_i \pm (\lambda_1 \pm \lambda_j) \pm (\lambda_1 \pm \lambda_k)$ is a root, then this root is expressed as $\lambda_1 \pm \lambda_l$, where $l \in \{2, \dots p\}$. Since $\lambda_1 \pm \lambda_l \in \Delta - \Delta_a$, it follows from (1.6) that

$$(\nabla_{X_a^*} h)(Y_a^*, Z_a^*) = 0.$$

Using the same method as in the proof of (a), we see that (b) holds. It is immediate from Theorem 3.2(ii) and Lemma 4.2 that M_a is not parallel. (Q.E.D.)

REMARK. It is well-known that a parallel submanifold in $P_n(\boldsymbol{C})$ is either holomorphic or totally real. Holomorphic parallel ones were classified by Nakagawa and Takagi ([26]) and the totally real ones by H. Naitoh ([24]).

On the other hand, S. Maeda proposed the following problem in [20]:

PROBLEM. Is there a submanifold $\tilde{L}$ in $P_n(\boldsymbol{C})$ such that $\tilde{L}$ is cyclic parallel but not parallel?

We can give a partial answer to the problem above as the following.

COROLLARY 4.5. *Let $\tilde{M}_a$ be an $\tilde{R}$-space. If $\tilde{M}_a$ is cyclic parallel, then $\tilde{M}_a$ is parallel.*

PROOF. By (1.11), we see that an R-space M_a is almost parallel if and only if the corresponding $\tilde{R}$-space $\tilde{M}_a$ is cyclic parallel. Lemma 4.2 and Theorem 4.4 imply that if M_a is almost parallel, then either $\tilde{M}_a$ is totally real or $\tilde{M}_a$ is holomorphic. Applying (1.10) to the both cases above, we see that $\tilde{M}_a$ is parallel. (Q.E.D.)

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