TSUKUBA J. MATH.
Vol. 25 No. 2 (2001), 247–259

AN EXPLICIT FORMULA FOR THE SQUARE OF THE RIEMANN ZETA-FUNCTION ON THE CRITICAL LINE

By

A. MIYAI

1. Introduction and Statement of the Results

Let as usual $s=\sigma+it$ be a complex variable, $d(n)$ the number of positive divisors of the integer n, and $\zeta(s)$ the Riemann zeta-function. For a positive integer k, the fundamental explicit formulas for $|\zeta(1/2+iT)|^{2k}$ or its averaged form with Gaussian weight $(\Delta\sqrt{\pi})^{-1}\exp(-(t/\Delta)^2)$ are known for $k=1,2$: namely, Jutila's explicit formula for $|\zeta(1/2+iT)|^2$ with the Atkinson function $f(T,n)$ ((1.3), (1.4) below), and in the fourth power case, Motohashi's explicit formula for $(\Delta\sqrt{\pi})^{-1}\int_{-\infty}^{\infty}|\zeta(1/2+i(T+t))|^4\exp(-(t/\Delta)^2)\,dt$ with spectral analytic quantities ([7], [8]).

One of the most important features of these formulas consists in the fact that one can derive non-trivial information on the size of $|\zeta(1/2+iT)|$ from them without appealing to any general theory of exponential sums. Indeed, the Hardy-Littlewood classical bounds $\zeta(1/2+iT)\ll T^{1/6+\varepsilon}$ follows immediately from the formulas. The aim of the present paper is to give an alternative explicit formula for $|\zeta(1/2+iT)|^2$ endowed with such a nature:

THEOREM. *Let θ be a constant with $0<\theta<1$ and α a number satisfying $\theta\le\alpha<1$. Then one has*

$$(1.1)\qquad \left|\zeta\left(\frac{1}{2}+iT\right)\right|^2=\sqrt{2}\sum_{2T/\pi+1\le n\le T_C(\alpha)}\frac{(-1)^n d(n)}{\sqrt{n}(1/4-T/(2\pi n))^{1/4}}\cos(f_C(T,n))$$
$$+2\sum_{1\le n\le T\alpha/(2\pi)}\frac{d(n)}{\sqrt{n}}\cos(T\log(T/(2\pi ne))-\pi/4)+O(T^{\varepsilon})$$

Received March 6, 2000.
Revised January 24, 2001.

where $T_C(\alpha) = (1+\alpha)^2 T/(2\pi\alpha)$ *and*

$$(1.2) \qquad f_C(T,n) = 2T \operatorname{arcosh}\sqrt{\pi n/(2T)} - 2\pi n\sqrt{1/4 - T/(2\pi n)} + \pi/4.$$

As for the order of $|\zeta(1/2+iT)|$, one has

COROLLARY. *The estimate* $\zeta(1/2+iT) \ll T^{1/6+\varepsilon}$ *follows from* (1.1).

The formula (1.1) should be compared with Jutila's formula ([5, Theorem 2], see also Chapter 15 in Ivic [3]): under the same assumption for Theorem, one has

$$(1.3) \qquad \left|\zeta\left(\frac{1}{2}+iT\right)\right|^2 = \sqrt{2}\sum_{1\le n\le T(\alpha)} \frac{(-1)^n d(n)}{\sqrt{n}(1/4+T/(2\pi n))^{1/4}}\cos(f(T,n))$$
$$+ 2\sum_{1\le n\le T\alpha/(2\pi)} \frac{d(n)}{\sqrt{n}}\cos(T\log(T/(2\pi ne)) - \pi/4)$$
$$+ O(\log T),$$

where $T(\alpha) = (1-\alpha)^2 T/(2\pi\alpha)$ and

$$(1.4) \qquad f(T,n) = 2T \operatorname{arsinh}\sqrt{\pi n/(2T)} + 2\pi n\sqrt{1/4 + T/(2\pi n)} + \pi/4.$$

The function $f(T,n)$ appeared for the first time in Atkinson's now famous formula ([1]) and plays important roles in the quadratic theory of $\zeta(s)$ (see, e.g., Ivic [3], [4]). Through many applications of the Atkinson formula, it turned out that, as far as one is concerned with mean values in short intervals, its "differentiated form" (1.3) suffices for most purposes. Our formula (1.1) with the function $f_C(T,n)$ gives an alternative form for Jutila's formula with the Atkinson function $f(T,n)$.

From the fact that the formula (1.1) has the factor $(1/4 - T/(2\pi n))^{-1/4}$ and that $(d/dT)f_C(T,n) = 2\operatorname{arcosh}(\sqrt{\pi n/(2T)})$ holds, one can observe that the size of $|\zeta(1/2+iT)|^2$ depends heavily on the behavior of the divisor function $d(n)$ with n near $2T/\pi$.

The bulk of the present paper is detailed analysis of applications of the Voronoï formula to an expression for $|\zeta(1/2+iT)|^2$ ((2.1) below). It is closely related to the transformation theory of Dirichlet polynomials developed by Jutila ([5], [6]). In applying saddle point method, as is described in section 4 and 5, somewhat a delicate analysis around the saddle points is required.

In the last section, together with the proof of Corollary, averaged forms with Gaussian weight are discussed in comparison with the existing formulas.

NOTATION. Throughout the paper, T stands for a large parameter and the abbreviation $L = \log T$ is frequently used. It will be convenient in the proofs to use the letter c to denote certain positive numerical constants and, ε positive constants which may be arbitrarily small, but are not necessarily the same ones at each occurrence. For complex numbers z_1 and z_2, the symbol $[z_1, z_2]$ stands for the oriented segment from the point z_1 to z_2. We reserve the letter η for $\exp(\pi i/4)$. Symbols $T(\alpha)$ and $T_C(\alpha)$ are defined in (1.1) and (1.3). I_α is the interval $[\alpha T/(2\pi), T/(2\pi)]$. Also recall that $\operatorname{arsinh}(z) = \log(z + (z^2+1)^{1/2})$ for $|z| < 1$ and $\operatorname{arcosh}(z) = \log(z + (z^2-1)^{1/2})$ for $|z| > 1$.

Acknowledgment

The author is grateful to Prof. Yoichi Motohashi for careful reading of the original manuscript and useful comments. He also thanks the referee for his/her kind comments.

2. An Application of the Voronoï Formula

The following expression for $|\zeta(1/2+iT)|^2$ is the starting point of our proof:

$$\left|\zeta\left(\frac{1}{2}+iT\right)\right|^2 = 2\sum_{1\le n\le T/(2\pi)} \frac{d(n)}{\sqrt{n}}\cos(T\log T/(2\pi n e) - \pi/4) + O(L) \tag{2.1}$$

where $L = \log T$. This follows from the approximate functional equation for $\zeta(1/2+iT)^2$:

$$\zeta\left(\frac{1}{2}+iT\right)^2 = \sum_{1\le n\le T/(2\pi)} d(n)n^{-1/2-iT} + \chi^2\left(\frac{1}{2}+iT\right)\sum_{1\le n\le T/(2\pi)} d(n)n^{-1/2+iT} + O(L)$$

combined with the functional equation $\zeta(1/2-iT) = \chi(1/2-iT)\zeta(1/2+iT)$ where $\chi(s) = \pi^{s-1/2}\Gamma(1/2-s/2)\Gamma(s/2)^{-1}$ and the formula

$$\chi\left(\frac{1}{2}-iT\right) = \exp(iT\log(T/2\pi e) - \pi/4)(1+O(T^{-1})),$$

which is obtained by Stirling's formula.

Putting α a number satisfying $0 < \theta \le \alpha < 1$, we split the sum (2.1) into two sums:

$$\left|\zeta\left(\frac{1}{2}+iT\right)\right|^2 = \sum\nolimits_1 + \sum\nolimits_2 + O(L) \tag{2.2}$$

where $\sum_1$ is the sum of the terms with $\alpha T/(2\pi) \le n \le T/(2\pi)$ and $\sum_2$ the others.

The first sum $\sum_1$ is to be transformed by the Voronoï formula for $\Delta(X)$, the error term in the Dirichlet divisor problem:

$$D(X) = \sum_{1\le n\le X}{}' d(n) = X(\log X + 2\gamma - 1) + 1/4 + \Delta(X) \tag{2.3}$$

where γ is the Euler constant and the symbol $\sum'_{1\le n\le X}$ denotes that the last term in the sum is halved if X is an integer. Voronoï's classical formula for $\Delta(X)$ is

$$\Delta(X) = -\sqrt{X}\sum_{n=1}^{\infty}\frac{d(n)}{\sqrt{n}}\left(Y_1(4\pi\sqrt{nX}) + \frac{2}{\pi}K_1(4\pi\sqrt{nX})\right) \tag{2.4}$$

where Y_ν is the ordinary Bessel function of the second kind and K_ν is the modified Bessel functions in usual notation and the series is boundedly convergent in any closed finite subinterval of the interval $(0, \infty)$, and uniformly convergent in any such interval free from integers. By using the well-known asymptotic approximations for Y_ν- and K_ν-Bessel functions (see, e.g., Ivic [3, (3.12), (3.13)]), one can describe the series in (2.4) as the sum of the series with terms containing trigonometric functions: namely, for a given positive integer K, one has

$$\Delta(X) = \sum_{k=1}^{K} a_k X^{3/4-k/2}\sum_{n=1}^{\infty}\frac{d(n)}{n^{1/4+k/2}}\sin(4\pi\sqrt{nX} - (-1)^k\pi/4) + O(X^{1/4-K/2}), \tag{2.5}$$

where a_k's are computable absolute constants. We use the first two of them;

$$a_1 = 1/(\sqrt{2}\pi) \quad\text{and}\quad a_2 = -3/(32\sqrt{2}\pi^2). \tag{2.6}$$

Here, before applying the Voronoï formula, we multiply every term in the relevant sum $\sum_1$ by a trivial factor $1 = \exp(-2\pi i n)$, which regulates the distribution of the saddle points which appear in the exponential integrals.

Denote by I_α the interval $[\alpha T/(2\pi), T/(2\pi)]$ and write the first sum $\sum_1$ as

$$\textstyle\sum_1 = \mathrm{Re}\, 2\eta^{-1}\sum_0 \tag{2.7}$$

where the letter η stands for $\exp(\pi i/4)$. Then the sum $\sum_0$ is transformed by using (2.3), up to a possible error term $O(1)$, into

$$\int_{I_\alpha} X^{-1/2}\exp(iT\log(T/(2\pi Xe)) - 2\pi iX)(\log X + 2\gamma)\, dX \tag{2.9}$$

$$+ \int_{I_\alpha} X^{-1/2}\exp(iT\log(T/(2\pi Xe)) - 2\pi iX)\, d\Delta(X).$$

By using the first derivative test, the first integral in (2.9) is estimated by $O(1)$. The main contribution comes from the second integral term. This we integrate by parts to give, coupled with the classical estimation for $\Delta(X)$,

$$(2.10)\qquad -\int_{I_\alpha} \Delta(X)\frac{d}{dX}\{X^{-1/2}\exp(iT\log(T/(2\pi Xe)) - 2\pi iX)\}\,dX + O(1).$$

Thus, using the Voronoï formula (2.5) for $K = 2$, we have

$$(2.11)\quad \sum_1 = -(1/(\sqrt{2}\pi))\,\mathrm{Re}(V_1^{(+)} + V_1^{(-)}) + (3/(32\sqrt{2}\pi^2))\,\mathrm{Re}(V_2^{(+)} - V_2^{(-)}) + O(1)$$

where

$$(2.12)\quad V_1^{(\pm)} = \eta^{\mp 1-1}\exp(iT\log T/(2\pi e))\sum_{n=1}^{\infty}\frac{d(n)}{n^{3/4}}$$
$$\times\int_{I_\alpha} X^{1/4}\exp(\pm 4\pi i\sqrt{nX})\frac{d}{dX}\{X^{-1/2}\exp(-iT\log X - 2\pi iX)\}\,dX$$

and

$$(2.13)\quad V_2^{(\pm)} = i^{-1}\eta^{\mp 1-1}\exp(iT\log T/(2\pi e))\sum_{n=1}^{\infty}\frac{d(n)}{n^{5/4}}$$
$$\times\int_{I_\alpha} X^{-1/4}\exp(\pm 4\pi i\sqrt{nX})\frac{d}{dX}\{X^{-1/2}\exp(-iT\log X - 2\pi iX)\}\,dX.$$

The termwise integration is legitimate from the bounded convergence of the series in the Voronoï formula.

3. Integral Terms Without Saddle Points

In the series $V_j^{(-)}$ $(j = 1, 2)$ in (2.12) and (2.13), since the derivatives of the functions $-4\pi\sqrt{nX} - T\log X - 2\pi X$ are monotone and smaller than $-cT^{-1/2}(\sqrt{n} + \sqrt{T})$, the integrals involved are estimated by $cT^{1/4}(\sqrt{n} + \sqrt{T})^{-1}$. These contribute to the series $V_j^{(-)}$ $(j = 1, 2)$ an amount $O(L)$.

As for the integrals involved in the series $V_j^{(+)}$ $(j = 1, 2)$, saddle points can occur. Denote the functions in the exponential integrals in the series $V_j^{(+)}$ $(j = 1, 2)$ by

$$F(X) = F(T, X, n) = 4\pi\sqrt{nX} - T\log X - 2\pi X.$$

For the terms with $n \geq 2T/\pi$, the saddle points, the roots of the equation $F'(X) = 0$;

$$x - \sqrt{nx} + T/(2\pi) = 0 \tag{3.1}$$

are given by

$$x_n^{\pm} = n/2 - T/(2\pi) \pm n\sqrt{1/4 - T/(2\pi n)}. \tag{3.2}$$

The saddle points occur for the terms with $n \geq 2T/\pi$ and, since the points x_n^+ always exceed the bounds $T/(2\pi)$, only the points x_n^- comes into question. We denote x_n^- by x_n for simplicity. According to the occurrence and the location of x_n in the interval $I_\alpha = [\alpha T/(2\pi), T/(2\pi)]$, we have the following three cases;

[I]: $1 \leq n < 2T/\pi$, the case with no saddle point,

[II]: $2T/\pi \leq n \leq T_C(\alpha)$ where $T_C(\alpha) = (1+\alpha)^2 T/(2\pi\alpha)$, the case with saddle point coming into the interval I_α, and

[III]: $T_C(\alpha) < n$, the case with the saddle points being outside the interval I_α.

Main contribution comes from the series $V_1^{(+)}$ and the computation of this part is rather complicated. The series $V_2^{(+)}$ can be treated in much the same way as the series $V_1^{(+)}$, and in fact, easier than that. The series $V_2^{(+)}$ contributes to $\sum_1$ an amount $O(T^\varepsilon)$. Hence we shall dwell on the computation on the series $V_1^{(+)}$ only.

In view of $F'(X) = (2\pi/\sqrt{X})(\sqrt{n} - T/(2\pi\sqrt{X}) - \sqrt{X})$, the integrals in the cases [I] and [III] are estimated as follows by using the first derivative test.

In the case [I], since $F'(X) \leq -cT^{-1/2}(\sqrt{2T/\pi} - \sqrt{n})$ holds from $F''(X) > 0$, the integrals are estimated by $cT^{3/4}(2T/\pi - n)^{-1}$. These contribute to $\sum_1$ an amount $O(T^\varepsilon)$.

In the case [III], $F''(X)$ changes sign at the point $\tilde{x}_n$ with

$$\tilde{x}_n = \frac{1}{n}\left(\frac{T}{\pi}\right)^2. \tag{3.3}$$

Since $\tilde{x}_n \leq T/(2\pi)$ holds in this case, $F''(X)$ may change sign in the interval I_α. We divide the case further; the case $[\mathrm{III}]_1$: $T_C(\alpha) < n < 2T/(\pi\alpha)$ and the case $[\mathrm{III}]_2$: $n \geq 2T/(\pi\alpha)$.

In the case $[\mathrm{III}]_1$, $\tilde{x}_n$ comes into the interval I_α. We split I_α into $I_0 + I_1$ where $I_0 = [\alpha T/(2\pi), \tilde{x}_n]$ and $I_1 = [\tilde{x}_n, T/(2\pi)]$. In the interval I_0, since $F'(X) \geq cT^{-1/2}(\sqrt{n} - \sqrt{T_C(\alpha)})$ holds from $F''(X) \geq 0$, the integrals are estimated by $cT^{3/4}(n - T_C(\alpha))^{-1}$. In the interval $I_1 = [\tilde{x}_n, T/(2\pi)]$, since $F'(X) \geq cT^{-1/2}(\sqrt{n} - \sqrt{2T/\pi})$ holds from $F''(X) \leq 0$, the integrals are estimated by $cT^{3/4}(n - 2T/\pi)^{-1}$.

In the case $[\mathrm{III}]_2$, since $F'(X) \geq cT^{-1/2}(\sqrt{n} - \sqrt{2T/\pi})$ holds from $F''(X) \leq 0$, the integrals are estimated by $cT^{1/4}n^{-1/2}$ or $cT^{3/4}(n - 2T/\pi)^{-1}$. Thus the terms in the case [III] contribute to $\sum_1$ an amount $O(T^{\varepsilon})$.

4. Integrals Round the Saddle Points (1)

It remains for us to compute the integral terms in the case [II]. One may suppose $n \geq 2T/\pi + 1$ with an admissible error. To evaluate the integrals by changing the contour I_α, we give an approximation of the exponential part $iF(X)$ of the integrands in somewhat a general setting: let us put $X = x(1 + \omega u)$ with $\alpha T/(2\pi) \leq x \leq T/(2\pi)$ for small u and a complex number ω with $|\omega| = 1$ which will be given later in each time the contour is changed. Then, by Taylor's theorem, we have an approximation;

$$(4.1)\qquad \begin{aligned} iF(X) = {} & 4\pi i\sqrt{nx} - iT\log x - 2\pi ix + 2\pi i\omega(\sqrt{nx} - T/(2\pi) - x)u \\ & + i\omega^2(T/2 - (\pi/2)\sqrt{nx})u^2 - i\omega^3(T/3 - (\pi/4)\sqrt{nx})u^3 \\ & + O((T + \sqrt{nx})u^4). \end{aligned}$$

To calculate the saddle-point terms, we give some facts on the saddle points $x_n = n/2 - T/(2\pi) - n\sqrt{1/4 - T/(2\pi n)}$ with $2T/\pi + 1 \leq n \leq T_C(\alpha)$. Note that

$$(4.2)\qquad 2\pi x_n/T = (\sqrt{\pi n/(2T)} + \sqrt{\pi n/(2T) - 1})^{-2}.$$

In the second order approximation in (4.1) for $x = x_n$, one has

$$(4.3)\qquad T/2 - (\pi/2)\sqrt{nx_n} = \pi\sqrt{nx_n}\sqrt{1/4 - T/(2\pi n)}.$$

This follows, combined with (4.2), from that the left hand side is equal to $\pi\sqrt{nx_n}(\sqrt{T/(2\pi n)}\sqrt{T/(2\pi x_n)} - 1/2)$.

Since $T/2 - (\pi/2)\sqrt{nx_n} = \pi\sqrt{nx_n}\sqrt{1/4 - T/(2\pi n)} = T/4 - \pi x_n/2$ holds by the equation (3.1) satisfied by x_n, combining this with (4.3), we have

$$4\pi\sqrt{nx_n} = (T - 2\pi x_n)/\sqrt{1/4 - T/(2\pi n)},$$

which is, by the definition of x_n,

$$= -4\pi n\sqrt{1/4 - T/(2\pi n)} + 2\pi n.$$

Also from (4.2) one has

$$\log(T/(2\pi x_n)) = 2\,\mathrm{arcosh}(\sqrt{\pi n/(2T)}).$$

From these and

$$-2\pi x_n = -\pi n + T + 2\pi n\sqrt{1/4 - T/(2\pi n)},$$

we are led to

$$(4.4)\qquad T\log(T/(2\pi e)) + 4\pi\sqrt{nx_n} - T\log x_n - 2\pi x_n$$
$$= 2T\operatorname{arcosh}\sqrt{\pi n/(2T)} - 2\pi n\sqrt{1/4 - T/(2\pi n)} + \pi n.$$

This gives the function denoted by $f_C(T,n)$ in (1.2) in Theorem.

Let δ be an arbitrarily small positive number, fixed throughout in this and next sections. We divide the case [II] into two cases, $[\mathrm{II}]_1$: $2T/\pi + 1 \le n \le 2T/\pi + T^{1/3+\delta}$ and $[\mathrm{II}]_2$: $2T/\pi + T^{1/3+\delta} < n \le T_C(\alpha)$.

In the case $[\mathrm{II}]_1$, note that $T/3 - (\pi/4)\sqrt{nx} \ge T/12 - cT^{1/3+\delta}$ holds. Putting

$$u_1 = T^{-1/3}L,$$

we change the contour I_α to $C_1 + C_2 + C_0 + C_0' + C_3 + C_4$ where $C_1 = [\alpha T/(2\pi), (\alpha T/(2\pi))(1 - \eta u_1)]$, $C_2 = [(\alpha T/(2\pi))(1 - \eta u_1), x_n(1 - \eta u_1)]$, $C_0 = [x_n(1 - \eta u_1), x_n]$, $C_0' = [x_n, x_n(1 + iu_1)]$, $C_3 = [x_n(1 + iu_1), (T/(2\pi))(1 + iu_1)]$ and $C_4 = [(T/(2\pi))(1 + iu_1), T/(2\pi)]$. Here, in the approximation (4.1) of $iF(X)$, ω is chosen as $\omega = -\eta$ on C_1, C_2 and C_0, and as $\omega = i$ on C_0', C_3 and C_4. The variable X is changed by $X = x(1 + \omega u)$ into u with $0 \le u \le u_1$ on C_1, C_0, C_0' and C_4, where $x = \alpha T/(2\pi)$, x_n or $T/(2\pi)$. In view of $x_n \le \tilde{x}_n$, the conditions $\mathrm{Re}(i\omega(\sqrt{nx} - T/(2\pi) - x)u) \le 0$, $\mathrm{Re}(i\omega^2(T/2 - (\pi/2)\sqrt{nx})u^2) \le 0$ and $\mathrm{Re}(-i\omega^3(T/3 - (\pi/4)\sqrt{nx})u^3) \le 0$ are satisfied on each of the contours C's. Note that, on the contours C_0', C_3 and C_4, $\mathrm{Re}(i\omega^2(T/2 - (\pi/2)\sqrt{nx})u^2) = 0$ holds. The error term $O((T + \sqrt{nx})u^4)$ in (4.1) is estimated by $cT^{-1/3}L^4$. Thereby, on the contour C_2 and C_3, the integrands are estimated by small factors $\exp(-cL^3)$ and these contribute to $\sum_1$ an amount $O(1)$. The integrals on C_1, C_0, C_0' and C_4 contribute to $\sum_1$ an amount $O(T^{\delta+\varepsilon})$, for the number of terms in the case $[\mathrm{II}]_1$ is $O(T^{1/3+\delta})$: this follows from the estimate

$$\sum_{2T/\pi+1 \le n \le 2T/\pi+T^{1/3+\delta}} \frac{d(n)}{n^{3/4}} x^{3/4} \int_0^{u_1} (1 + \omega u)^{-1/4} \exp(iF(x(1 + \omega u)))\, du \ll T^{\delta+\varepsilon}$$

for $\omega = i$ or η and $x = \alpha T/(2\pi)$, x_n or $T/(2\pi)$.

5. Integrals Round the Saddle Points (2)

To evaluate the integrals in the case $[\mathrm{II}]_2$: $2T/\pi + T^{1/3+\delta} < n \le T_C(\alpha)$, we split the interval I_α into $I_0 + I_1$ where $I_0 = [\alpha T/(2\pi), \tilde{x}_n]$ and $I_1 = [\tilde{x}_n, T/(2\pi)]$, $\tilde{x}_n$

being defined in (3.3). Since $x_n \leq \tilde{x}_n$ holds, the saddle point does not come into I_1 and since $F''(X) \leq 0$ holds there, the integrals on I_1 are estimated by the way similar to that on the interval I_1 in the case $[\mathrm{III}]_1$ in section 3. These contribute to $\sum_1$ an amount $O(T^{\varepsilon})$.

As for the contour $I_0 = [\alpha T/(2\pi), \tilde{x}_n]$, note first that $T/3 - (\pi/4)\sqrt{nx} \geq T/12$ holds for x in the interval I_0. We change the interval I_0 into $C_1 + C_2 + C_0 + C_0' + C_3 + C_4$, where C's are indicated in the following together with evaluating the integrals on each of them.

Let us put

$$(5.1)\quad u_1(x) = u_1(x,n) = \min\{(T/3 - (\pi/4)\sqrt{nx})^{-1/3}, (T/2 - (\pi/2)\sqrt{nx})^{-1/2}\}L.$$

We define the contour C_1 by $[\alpha T/(2\pi), (\alpha T/(2\pi))(1 - \eta u_1(\alpha T/(2\pi)))]$. In the approximation (4.1), ω is given by $\omega = -\eta$ and the variable X is changed into u with $0 \leq u \leq u_1$ by $X = x_1(1 - \eta u)$ with $x_1 = \alpha T/(2\pi)$. On this contour, the conditions $\mathrm{Re}(i\omega^2(T/2 - (\pi/2)\sqrt{nx_1})u^2) \leq 0$ and $\mathrm{Re}(-i\omega^3(T/3 - (\pi/4)\sqrt{nx_1})u^3) \leq 0$ are satisfied and the error terms $O((T + \sqrt{nx})u^4)$ in (4.1) are estimated by $cT^{-1/3}L^4$. Since $T/(2\pi) + x_1 - \sqrt{nx_1} = \sqrt{\alpha T/2\pi}(\sqrt{T_C(\alpha)} - \sqrt{n})$ holds, the integrals on this contour are estimated by $cT^{3/4}(T_C(\alpha) - n)^{-1}$ and contribute to $\sum_1$ an amount $O(T^{\varepsilon})$, here one may suppose that $n < T_C(\alpha) - 1$.

The contour C_2 is defined by the curve $X = x(1 - \eta u_1(x,n))$ with $\alpha T/(2\pi) \leq x \leq x_n$. Here in the approximation (4.1), ω is chosen to be $-\eta$. On this curve, the conditions $\mathrm{Re}(i\omega^2(T/2 - (\pi/2)\sqrt{nx})u_1(x,n)^2) < 0$ and $\mathrm{Re}(-i\omega^3(T/3 - (\pi/4)\sqrt{nx}) \cdot u_1(x,n)^3) < 0$ are satisfied and $O((T + \sqrt{nx})u^4) \ll T^{-1/3}L^4$ holds. Thereby, by the definition (5.1) of $u_1(x,n)$, the integrals are estimated by small factors $\exp(-cL^2)$.

At the end point $x = x_n$ of the curve C_2, since $T/2 - (\pi/2)\sqrt{nx_n} \geq cT^{2/3+\delta/2}$ holds from (4.3), note that one has

$$(5.2)\qquad u_1(x_n,n) = (T/2 - (\pi/2)\sqrt{nx_n})^{-1/2}L.$$

The segment C_0 passing through the saddle point x_n is defined by $C_0 = [x_n(1 - \eta u_1(x_n,n)), x_n(1 + \eta u_0)]$ where, using (4.3),

$$(5.3)\qquad \begin{aligned} u_0 &= (T/2 - (\pi/2)\sqrt{nx_n})^{-1/2}T^{\delta/5} \\ &= cx_n^{-1/4}(n - 2T/\pi)^{-1/4}T^{\delta/5}. \end{aligned}$$

On the segment C_0, in the approximation (4.1), ω is chosen to be η and the variable X is changed into u with $-u_1(x_n,n) \leq u \leq u_0$ by $X = x_n(1 + \eta u)$. The main contribution of the series V_1^+ comes from the integrals on C_0, this is to be evaluated later.

Since $u_0 \leq cT^{-1/3-\delta/20}$ holds from (5.3), if one puts

$$u_2 = (T/3 - (\pi/4)\sqrt{nx_n})^{-1/3}L, \tag{5.4}$$

the inequality $u_0 < u_2$ holds. We define the segment C_0' by $C_0' = [x_n(1+\eta u_0), x_n(1+\eta u_2)]$, where ω is chosen to be η in the approximation (4.1). On this contour, since

$$\begin{aligned}
&|\exp(i\eta^2(T/2 - (\pi/2)\sqrt{nx_n})u^2 - i\eta^3(T/3 - (\pi/4)\sqrt{nx_n})u^3)| \\
&\quad \leq \exp(-(T/2 - (\pi/2)\sqrt{nx_n})u_0^2 + 2^{-1/2}(T/3 - (\pi/4)\sqrt{nx_n})u_2^3) \\
&\quad \leq \exp(-cT^{2\delta/5})
\end{aligned}$$

holds and the error terms $O((T+\sqrt{nx})u^4)$ are estimated by $cT^{-1/3}L^4$, the integrals on C_0' are very small. Also note that, under the condition $n \geq 2T/\pi + T^{1/3+\delta}$, since

$$\tilde{x}_n - x_n \geq cT^{2/3+\delta/2} \tag{5.5}$$

holds, the end point $x_n(1+\eta u_2)$ of the segment C_0' is contained in the half plane $\sigma < \tilde{x}_n$. The assertion (5.5) follows, in view of $\sqrt{x_n} = 2^{-1}(\sqrt{n} - \sqrt{n-2T/\pi})$, from

$$\begin{aligned}
\tilde{x}_n - x_n &= (\sqrt{\tilde{x}_n} + \sqrt{x_n})(n^{-1/2}T/\pi - 2^{-1}(\sqrt{n} - \sqrt{n-2T/\pi})) \\
&= 2^{-1}n^{-1/2}(\sqrt{\tilde{x}_n} + \sqrt{x_n})(\sqrt{n}\sqrt{n-2T/\pi} - (n-2T/\pi)) \\
&= n^{-1/2}(\sqrt{x_n\tilde{x}_n} + x_n)\sqrt{n-2T/\pi}.
\end{aligned}$$

On account of this, the point $x_n(1+\eta u_2)$ can be written also as $x_n(1+\eta u_2) = x_n'(1+iu_3)$ for some x_n' with $x_n \leq x_n' \leq \tilde{x}_n$ and u_3 with $T^{-1/3}L \ll u_3 \ll T^{-1/3}L$. In fact one may take $x_n' = x_n(1+2^{-1/2}u_2)$ and

$$u_3 = u_2(\sqrt{2}+u_2)^{-1}.$$

By putting

$$u_4 = (T/3 - (\pi/4)\sqrt{n\tilde{x}_n})^{-1/3}L = (T/12)^{-1/3}L,$$

the segment C_3 with the starting point $x_n(1+\eta u_2) = x_n'(1+iu_3)$ is defined by $C_3 = [x_n'(1+iu_3), \tilde{x}_n(1+iu_4)]$ and C_4 by $C_4 = [\tilde{x}_n(1+iu_4), \tilde{x}_n]$. Note that on these contours C_3 and C_4, in the approximation (4.1), the conditions $\mathrm{Re}(i\omega^2(T/2 - (\pi/2)\sqrt{nx})) = 0$ and $\mathrm{Re}(-i\omega^3(T/3 - (\pi/4)\sqrt{nx})) < 0$ with $\omega = i$ are satisfied, and the error terms $O((T+\sqrt{nx})u^4)$ are estimated by $cT^{-1/3}L^4$. Since $u_4 > u_3 \geq cT^{-1/3}L$ holds, the integrals on C_3 are estimated by $\exp(-cL^3)$. On the segment C_4, since $\sqrt{n\tilde{x}_n} - T/(2\pi) - \tilde{x}_n = (T/(2\pi n))(n - 2T/\pi)$ holds, the integrals are estimated by $cT^{3/4}(n - 2T/\pi)^{-1}$, which contribute to $\sum_1$ an amount $O(T^\varepsilon)$.

Thus we are left with the integrals on the contour C_0 passing through the saddle point x_n:

$$\int_{C_0} X^{1/4} \exp(4\pi i\sqrt{nX}) \frac{d}{dX}\{X^{-1/2} \exp(-iT \log X - 2\pi iX)\}\, dX.$$

These we integrate by parts to give

$$(5.6) \quad -2\pi i\sqrt{n} \int_{C_0} X^{-3/4}\left(1 + \left(8\pi i\sqrt{nX}\right)^{-1}\right) \exp\left(4\pi i\sqrt{nX} - iT \log X - 2\pi iX\right) dX,$$

by using (5.2) and (5.3), with an admissible error term $O(\exp(-cL^2))$. In the approximation (4.1) of $iF(x_n(1 + \eta u))$ on the contour C_0, the terms $-i\eta^3(T/3 - (\pi/4)\sqrt{nx_n})u^3 + O((T + \sqrt{nx_n})u^4)$ are negligible. This follows, by using (4.3), from the estimate

$$\sum_{2T/\pi + T^{1/3+\delta} < n \le T c(\alpha)} d(n) n^{-3/4} x_n^{3/4} (T + \sqrt{nx_n})(x_n(n - 2T/\pi))^{-1} \ll T^\varepsilon.$$

In view of (4.3), (4.1), (5.2) and (5.3), the integral (5.6) is equal to

$$(5.7) \quad -2\pi i\eta\sqrt{n} x_n^{1/4} \exp(4\pi i\sqrt{nx_n} - iT \log x_n - 2\pi i x_n)$$
$$\times \int_{-\infty}^{\infty} (1 + O(u + (nx_n)^{-1/2})) \exp(-(T/2 - (\pi/2)\sqrt{nx_n})u^2)\, du.$$

The error term in (5.7) contributes to $\sum_1$ an amount $O(T^\varepsilon)$. Combining these with (2.2), (2.11), (2.12), (4.3) and (4.4), we are led to the formula (1.1) in Theorem.

6. An Exponential Sum Bounding $|\zeta(1/2 + iT)|^2$

The proof of Corollary is carried out in a familiar way by means of the well-known inequality

$$(6.1) \quad \zeta\left(\frac{1}{2} + iT\right)^2 \ll L \int_{-L^2}^{L^2} \left|\zeta\left(\frac{1}{2} + i(T + t)\right)\right|^2 dt + L$$

due to Heath-Brown [2, Lemma 3], and an exponential integral ([2, (A.38)]); for $\operatorname{Re} B > 0$,

$$(6.2) \quad \int_{-\infty}^{\infty} \exp(At - Bt^2)\, dt = \sqrt{\pi/B} \exp(A^2/(4B)).$$

We substitute the formula (1.1) for the integrand in (6.1) multiplied by $\exp(-(t/\Delta)^2)$ with $\Delta = T^{1/3}L$. If we choose $\alpha = 1 - T^{-1/3}$ in (1.1), the first sum in the formula may be degenerated to $O(T^{\varepsilon})$. As for the second sum in (1.1), denoting $g(T,n) = T\log(T/(2\pi ne)) - \pi/4$, by Taylor's theorem one has, for $|t| \le \Delta L^3$,

$$g(T+t) = g(T,n) + (\log T/(2\pi n))t + t^2/(2T) + O(|t|^3 T^{-2}),$$

the error term $O(|t|^3 T^{-2})$ being negligible. Hence, by using (6.2) with $A = \log(T/(2\pi n))i$ and $B = (2T)^{-1}i + \Delta^{-2}$, we can see that, in view of $\log(T/(2\pi n)) \ge T^{-1/3}$ for $1 \le n \le \alpha T/(2\pi)$, the integral terms from the second sum are very small. From this, Corollary follows. Or, taking θ small, if we choose $\alpha = 1/2$ in (1.1), we have that, uniformly in Δ with $T^{\varepsilon} \le \Delta \le T^{1/3}$, $|\zeta(1/2+iT)|^2$ is surpassed by an exponential sum

$$(6.3)\qquad \sqrt{2\pi}\Delta \sum_{2T/\pi+1 \le n \le 9T/(4\pi)} \frac{(-1)^n d(n)}{\sqrt{n}(1/4 - T/(2\pi n))^{1/4}} \cos(f_C(T,n)) \times \exp(-(\Delta \operatorname{arcosh}\sqrt{\pi n/(2T)})^2) + O(\Delta T^{\varepsilon}).$$

From this, Corollary also follows by choosing $\Delta = T^{1/3}$. This is obtained by using, combined with (6.1) and (6.2), for $|t| \le \Delta L^2$,

$$(6.4)\qquad (1/4-(T+t)/(2\pi n))^{-1/4} = (1/4-T/(2\pi n))^{-1/4} + O(n^{1/4}(n-2T/\pi)^{-5/4}|t|)$$

and

$$(6.5)\qquad f_C(T+t,n) = f_C(T,n) + 2\operatorname{arcosh}(\sqrt{\pi n/(2T)})t - \sqrt{n}(2T\sqrt{n-2T/\pi})^{-1}t^2 + O(\sqrt{n}(n-2T/\pi)^{-3/2}|t|^3 T^{-1}).$$

The argument to obtain exponential sums of the type (6.3) is closely related to the averaged form with Gaussian weight; if we denote

$$I(T,\Delta) = (\Delta\sqrt{\pi})^{-1}\int_{-\infty}^{\infty}\left|\zeta\left(\frac{1}{2} + i(T+t)\right)\right|^2 \exp(-(t/\Delta)^2)\,dt,$$

we are led to the expression, from Theorem,

$$(6.6)\qquad I(T,\Delta) = \frac{2^{3/4}\pi^{1/4}}{T^{1/4}} \sum_{n=2T/\pi+1}^{\infty} \frac{(-1)^n d(n)}{(n-2T/\pi)^{1/4}} \cos(f_C(T,n)) \exp\left(-\frac{\pi n - 2T}{2T}\Delta^2\right) + O(T^{\varepsilon})$$

for Δ with $T^{1/7} \le \Delta \le T^{1/3}$. This should also be compared with the expression with the Atkinson function $f(T,n)$ in the form given by Motohashi ([7, (1.18)]); namely, one has

$$(6.7) \qquad I(T,\Delta) = \frac{2^{3/4}\pi^{1/4}}{T^{1/4}} \sum_{n=1}^{\infty} \frac{(-1)^n d(n)}{n^{1/4}} \cos(f(T,n)) \exp\Big(-\frac{\pi n}{2T}\Delta^2\Big) + O(L)$$

for Δ with $T^{1/4} < \Delta < TL^{-1}$. For a range of Δ with $T^{1/7} \le \Delta \le T^{1/2}$, this follows from Jutila's formula (1.3).

Remark. Among various expressions obtainable by applying the Voronoï formula to the sum (2.1), the formulas (1.3) and (1.1) seem to be the only two formulas that bear the exponential sum of the type (6.3) or (6.7) which bring the bounds $\zeta(1/2 + iT) \ll T^{1/6+\varepsilon}$. To an intimate relationship between these two explicit formulas and other formulas with the function $f_C(T,n)$, we hope to return elsewhere.

References

[1] Atkinson, F. V., The mean value of the Riemann zeta-function. Acta Math. **81** (1949), 353–376.
[2] Heath-Brown, D. R., The twelfth power moment of the Riemann zeta-function, Quart. J. Math. (Oxford) **29** (1979), 443–462.
[3] Ivic, A., The Riemann Zeta-Function, The Theory of Riemann Zeta-Function with Applications. Wiley and Sons, 1985.
[4] Ivic, A., Lectures on the Mean Values of the Riemann Zeta-Function. Tata Inst. Fund. Res. Lectures on Math. and Phys, vol. 82, Springer Verlag, Berlin, 1991.
[5] Jutila, M., Transformation formulae for Dirichlet polynomials, J. Number Theory **18** (1984), 135–156.
[6] Jutila, M., A method in the theory of exponential sums. Tata Inst. Fund. Res. Lectures on Math. and Phys, vol. 80, Springer-Verlag, Berlin, 1987.
[7] Motohashi, Y., An explicit formula for fourth power mean of the Riemann zeta-function. Acta Math. **170** (1993), 181–220.
[8] Motohashi, Y., Spectral Theory of the Riemann Zeta-Function. Cambridge Univ. Press. Cambridge, 1997.

Department of Mathematics
Faculty of Education
Iwate University
Morioka, 020-8550 (Japan)
miyai@iwate-u.ac.jp