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LIFE SPAN FOR SOLUTIONS OF THE HEAT EQUATION WITH A NONLINEAR BOUNDARY CONDITION

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Abstract. In this note we obtain estimates in terms of the size of the initial data for the blow-up time of positive solutions of the heat equation in $\boldsymbol{R}_+$ with a nonlinear boundary condition $-u_x(0,t) = u^p(0,t)$.

Introduction

In this note we obtain estimates for the blow-up time of positive solutions of the following parabolic problem

$$
(1) \qquad \begin{cases} u_t = u_{xx} & \text{in } \boldsymbol{R}_+(0,T_\lambda), \\ -u_x(0,t) = u^p(0,t) & \text{in } (0,T_\lambda), \\ u(x,0) = \lambda\phi(x) > 0 & \text{in } \boldsymbol{R}_+. \end{cases}
$$

where $p > 1$ is fixed and $\lambda > 0$ is a parameter.

Throughout this note we assume that the initial datum ϕ is continuous, positive and bounded.

Existence, uniqueness, regularity and continuous dependence on the initial data for this problem were proved, for instance, in [2].

For problem (1), it is well known that if λ is large enough the solution blows up in finite time T_λ (T_λ depends on λ) if and only if $p > 1$, see for example, [1], [3], [4], [8], [10]. This means that there exists a finite time T_λ with

$$\lim_{t \nearrow T_\lambda} \|u(\cdot,t)\|_\infty = +\infty.$$

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Here we are interested in the asymptotic behaviour of T_λ when λ goes to infinity. We prove the following Theorem,

THEOREM 1. *Under the above assumptions on ϕ, the function $\lambda \mapsto T_\lambda$ is decreasing and continuous with the following asymptotic behaviour at infinity,*

$$\lim_{\lambda \to \infty} \lambda^{2(p-1)} T_\lambda = T_0.$$

Here T_0 is the blow-up time of the solution of (1) *with initial datum $u(x,0) \equiv \phi(0)$.*

Some related papers that deal with the heat equation with a nonlinear source in the entire space are [7] and [9].

Under further assumptions on the initial datum, $u(x,0) = \psi(x)$ (a compatibility condition and $\psi_{xx} \geq 0$, that guarantee $u_t \geq 0$) it was proved in [6] and [8] that the following blow up rate holds,

$$c \leq (T-t)^{1/2(p-1)} \|u(\cdot,t)\|_\infty \leq C \tag{2}$$

We observe that the exponent that appears in Theorem 1 is related to the one in the blow-up rate (2). This is a consequence of the natural scaling in the equation (1).

Proof of Theorem 1

The fact that $\lambda \mapsto T_\lambda$ is decreasing is an immediate consequence of the maximum principle. To see this, let us call u the solution of (1) with initial datum $\lambda\phi$ and v the solution of (1) with initial datum $\mu\phi$. If $\lambda \leq \mu$ then, by a comparison argument, $u(x,t) \leq v(x,t)$ for all $x > 0$ and $0 < t < \min\{T_\lambda, T_\mu\}$. As T_λ is the blow-up time for u, $\lim_{t \nearrow T_\lambda} \|u(\cdot,t)\|_\infty = +\infty$ and hence v can not be defined beyond T_λ, proving that $T_\mu \leq T_\lambda$.

To see that is continuous we can assume that $\lambda \leq \mu$, hence $T_\lambda \geq T_\mu$. Now, given $\varepsilon > 0$ we have to show that $T_\lambda - \varepsilon < T_\mu$ if $\mu - \lambda < \delta$, but this follows by the continuous dependence with respect to the initial data (see [2]). In fact,

$$\|u(\cdot, T_\lambda - \varepsilon)\|_\infty \leq C = C(\varepsilon)$$

If we replace the power by a globally Lipchitz function $g(u)$ that agrees with u^p for every $u \leq 2C$ we deal with a regular problem, and hence there exists $\delta = \delta(\varepsilon)$ such that

$$\|v(\cdot, T_\lambda - \varepsilon)\|_\infty \leq 2C < +\infty, \quad \text{if } \mu - \lambda < \delta.$$

We observe that as long as $v \le 2C$ it is a solution of the problem with u^p as nonlinear flux at $x = 0$. By uniqueness, we can conclude that v is bounded up to $T_\lambda - \varepsilon$. Therefore, $T_\mu > T_\lambda - \varepsilon$ as we wanted to prove.

Finally, let us study the asymptotic behaviour at infinity. This is the main point of the paper.

Let u be the solution of (1) and inspired by the natural scaling of the problem we define

$$v_\lambda(x,t) = \frac{1}{\lambda} u(\lambda^{1-p}x, \lambda^{2(1-p)}t). \tag{3}$$

As u satisfies (1), v_λ verifies

$$\begin{cases} (v_\lambda)_t = (v_\lambda)_{xx} & \text{in } \boldsymbol{R}_+ \times (0, \tilde{T}_\lambda), \\ -(v_\lambda)_x(0,t) = v_\lambda^p(0,t) & \text{in } (0, \tilde{T}_\lambda), \\ v_\lambda(x,0) = \phi(\lambda^{1-p}x) \equiv \phi_\lambda(x) & \text{in } \boldsymbol{R}_+. \end{cases} \tag{4}$$

where $\tilde{T}_\lambda = \lambda^{2(p-1)} T_\lambda$.

We want to compute $\lim_{\lambda\to\infty} \tilde{T}_\lambda$. For that purpose, let us define w as the solution of

$$\begin{cases} w_t = w_{xx} & \text{in } \boldsymbol{R}_+ \times (0, T_0), \\ -w_x(0,t) = w^p(0,t) & \text{in } (0, T_0), \\ w(x,0) = \phi(0) & \text{in } \boldsymbol{R}_+, \end{cases} \tag{5}$$

which is the natural "limit" equation as $\phi_\lambda \to \phi(0)$ uniformly over compact sets of $[0, +\infty)$.

As $\phi(0) > 0$, w blows up in finite time, T_0 (see [4]).

The Theorem will follows if we prove that

$$\tilde{T}_\lambda \to T_0, \quad \text{as } \lambda \to \infty.$$

To prove this claim, let $\varepsilon > 0$ and take $T' = T_0 - \varepsilon$. Let $M = \sup_{0<t<T'} \|w(\cdot,t)\|_\infty$.

As before, we take $g \in Lip(\boldsymbol{R})$ such that $g(s) = s^p$ for $s < 2M$. With this g, we define φ the solution of the following problem,

$$\begin{cases} \varphi_t = \varphi_{xx} & \text{in } \boldsymbol{R}_+ \times (0, T'), \\ -\varphi_x(0,t) = g(\varphi)(0,t) & \text{in } (0, T'), \\ \varphi(x,0) = \phi_\lambda(x) & \text{in } \boldsymbol{R}_+. \end{cases} \tag{6}$$

Observe that $\varphi = v_\lambda$ if $v_\lambda < 2M$.

Let us see that $|w(0,t) - \varphi(0,t)| < \delta$ if $\lambda > \lambda_0(\delta)$ for all $t < T'$. For this purpose, let us define $z = w - \varphi$. As $g \in Lip(\boldsymbol{R})$, z verifies

$$
(7) \qquad \begin{cases} z_t = z_{xx} & \text{in } \boldsymbol{R}_+ \times (0, T'), \\ -z_x(0,t) = g(w)(0,t) - g(\varphi)(0,t) & \text{in } (0, T'), \\ z(x,0) = \phi(0) - \phi_\lambda(x) & \text{in } \boldsymbol{R}_+. \end{cases}
$$

Then we have

$$
(8) \qquad |z_x(0,t)| \le K|z(0,t)|,
$$

where K depends only on M.

Let $\Gamma(x,t)$ be the fundamental solution of the heat equation, namely

$$
\Gamma(x,t) = \frac{1}{(4\pi t)^{1/2}} \exp\left(-\frac{x^2}{4t}\right).
$$

For $x \in \boldsymbol{R}_+$, by (8) we have (see [5])

$$
(9) \qquad z(x,t) = \int_{\boldsymbol{R}_+} \Gamma(x-y,t) z(y,0)\, dy - \int_0^t \frac{\partial z}{\partial x}(0,\tau)\Gamma(x,t-\tau)\, d\tau + \int_0^t \frac{\partial \Gamma}{\partial x}(x,t-\tau) z(0,\tau)\, d\tau.
$$

Now we observe that Γ satisfies

$$
\frac{\partial \Gamma}{\partial x}(0,t-\tau) = 0, \quad \Gamma(0,t-\tau) = \frac{1}{2\sqrt{\pi}(t-\tau)^{1/2}}.
$$

Hence, using the initial and boundary conditions we get that

$$
|z(0,t)| \le \int_{\boldsymbol{R}_+} \Gamma(-y,t)|z(y,0)|\, dy + \frac{K}{2\sqrt{\pi}} \int_0^t \frac{|z(0,\tau)|}{(t-\tau)^{1/2}}\, d\tau.
$$

Now we choose $t_0 = t_0(K)$ such that

$$
\frac{K}{2\sqrt{\pi}} \int_0^{t_0} \frac{1}{(t_0-\tau)^{1/2}}\, d\tau \le \frac{1}{2}.
$$

Hence, for $t \in [0, t_0]$ we have

$$
\max_{[0,t_0]} |z(0,t)| \le 2 \max_{[0,t_0]} \int_{\boldsymbol{R}_+} \Gamma(-y,t)|z(y,0)|\, dy
$$

We observe that for every $\delta_1 > 0$ there exists $\lambda_1 > 0$ such that

$$\begin{aligned}\int_{\mathbf{R}_+} \Gamma(-y,t)|z(y,0)|\,dy &= \int_0^L \Gamma(-y,t)|z(y,0)|\,dy + \int_L^{+\infty} \Gamma(-y,t)|z(y,0)|\,dy \\ &\le \eta \int_0^L \Gamma(-y,t)\,dy + C\int_L^{+\infty} \Gamma(-y,t)\,dy \\ &\le \delta_1\end{aligned}$$

if $\lambda > \lambda_1$.

Now, choose L large so that $\int_L^{+\infty} \Gamma(x-y,t)\,dy$ is small uniformly in $(x,t) \in (0,L/2) \times (0,t_0)$, and take $\lambda_2 > 0$ such that $|z(y,0)| < \eta$ for $y \in (0,L)$ and η small.

With this bound on $|z(0,t)|$ we can control $z(x,t)$ for $(x,t) \in (0,L/2) \times (0,t_0)$, in fact, from (8) and (9) we have

$$\begin{aligned}|z(x,t)| &\le \int_{\mathbf{R}_+} \Gamma(x-y,t)|z(y,0)|\,dy + K\delta_1 \int_0^t \Gamma(x,t-\tau)\,d\tau + \delta_1 \int_0^t \frac{\partial\Gamma}{\partial x}(x,t-\tau)\,d\tau \\ &\le \int_0^L \Gamma(x-y,t)|z(y,0)|\,dy + \int_L^{+\infty} \Gamma(x-y,t)|z(y,0)|\,dy + C\delta_1 \\ &\le \eta \int_0^L \Gamma(x-y,t)\,dy + C\int_L^{+\infty} \Gamma(x-y,t)\,dy + C\delta_1 \le \delta_2\end{aligned}$$

if λ is big enough.

Now, as t_0 is independent of λ, we can repeat this procedure beginning with $z(x,t_0)$ as initial datum to find that $|z(x,t)| < \delta_3$ for $(x,t) \in (0,L/4) \times (t_0, 2t_0)$. Therefore, after a finite number of iterations we obtain that, for λ large $(\lambda > \lambda_0(\delta))$

$$|z(0,t)| < \delta \quad \text{for all } t < T',$$

as we wanted to see.

Now, as $w(0,T') \le M$ and $|w(0,t) - \varphi(0,t)| < \delta$, we have that $\varphi(0,t) < 2M$ in $[0,T']$. Therefore, by uniqueness, $\varphi = v_\lambda$ in $[0,T']$. Hence $\tilde{T}_\lambda \ge T' = T_0 - \varepsilon$.

Now, take ψ a compatible initial datum with compact support and $\psi_{xx} \ge 0$ such that $\psi(x) < \phi(0)$ and $\phi(0) - \psi(x)$ small enough in $(0,L)$. From the previous argument, it follows that the solution $\underline{w}$ of (1) with ψ as initial datum verifies

$$w(0,T') - \underline{w}(0,T') < \delta.$$

Hence $T(\psi) \geq T'$. By the assumptions on ψ, $\underline{w}$ verifies (2). Then

$$w(0,T') - \delta \leq \underline{w}(0,T') \leq \|\underline{w}(\cdot,T')\|_{\infty} \leq C(T(\psi) - T')^{-1/2(p-1)}.$$

Therefore it is easy to see that $T(\psi) - T' < \kappa$ if $\varepsilon = T_0 - T'$ is small (depending on κ). Now, choosing λ large enough, we can obtain $\phi_{\lambda}(x) > \psi(x)$, then $\tilde{T}_{\lambda} \leq T(\psi) < T' + \kappa$ and hence as $T_0 - T' = \varepsilon$, we conclude the desired result. □

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