TSUKUBA J. MATH.
Vol. 25 No. 1 (2001), 203–213

UNSTABLE HARMONIC MAPS INTO REAL HYPERSURFACES OF A COMPLEX HOPF MANIFOLD

By

Sorin DRAGOMIR and Maria Rosaria ENEA

Abstract. Let $\phi : M \to N$ be a pseudohermitian immersion ([6]) of a compact strictly pseudoconvex CR manifold M into a totally umbilical real hypersurface N, of nonzero mean curvature ($\|H\| \neq 0$), of a complex Hopf manifold CH^n, tangent to the Lee field B_0 of CH^n. If B_0 is orthogonal to the CR structure of N and $E(\phi) > Vol(M)/[(1+\|H\|^2)\|H\|^2]$ then ϕ is an unstable harmonic map.

1. Introduction

By a well known result of P. F. Leung (cf. [12]) any nonconstant harmonic map from a compact Riemannian manifold into a sphere S^n, $n \geq 3$, is unstable. This carries over easily to totally umbilical real hypersurfaces N of a real space form $M^{n+1}(c)$. Precisely, if $(n-2)\|H\|^2 + (n-1)c > 0$ then any nonconstant harmonic map of a compact Riemannian manifold into N is unstable. The proof is a *verbatim* transcription of the proof of Theorem 4 in [3]. Cf. also Theorem 7.1 in [1]. Here $\|H\|$ is the mean curvature of $N \subset M^{n+1}(c)$ (a constant *a posteriori*, cf. Prop. 3.1 in [5], p. 49, i.e. $N = M^n(c + \|H\|^2)$).

In the present paper we take up the following complex analogue of the problem above: *given a compact Riemannian manifold M, study the stability of harmonic maps of M into a totally umbilical CR submanifold of a Hermitian manifold* N_0.

By a result of *A*. Bejancu, [4], if N_0 is a Kähler manifold then totally umbilical CR submanifolds may only occur in real codimension one. Even worse, by a result of Y. Tashiro & S. Tachibana, [13], neither elliptic nor hyperbolic complex space forms possess totally umbilical real hypersurfaces. Umbilical submanifolds are however abundant in locally conformal Kähler ambient spaces (cf. [10] and [7] for a partial classification). We obtain the following

Received January 31, 2000.
Revised November 13, 2000.

THEOREM 1. *Let N be a totally umbilical real hypersurface of the complex Hopf manifold CH^{n+1} with the Boothby metric g_0. Let $\phi : M \to N$ be a non-constant harmonic map of a compact Riemannian manifold (M, g) into $(N, j^* g_0)$, where $j : N \subset CH^{n+1}$. If*

$$\int_M \left(2n + (2n-1)\|H\|^2 - \frac{1}{4}\|B\|^2\right)\|d\phi\|^2 v_g > \frac{2n-1}{4}\int_M \|\phi^*\omega\|^2 v_g \tag{1}$$

then ϕ is unstable. In particular, if N is tangent to the Lee field of CH^{n+1} and

$$(1 + \|H\|^2)E(\phi) > \frac{1}{8}\int_M \|\phi^*\omega\|^2 v_g \tag{2}$$

then ϕ is unstable.

In section 2 we recall the facts of locally conformal geometry needed throughout the paper (cf. also [8]). Theorem 1 is proved in section 3. In section 4 we discuss the case of pseudohermitian immersions of a compact *CR* manifold into a real hypersurface of a complex Hopf manifold (cf. our Theorem 4). The Authors are grateful to the referee, whose suggestions improved the original version of the manuscript.

2. A Reminder of Locally Conformal Kähler Geometry

Let $\lambda \in C$, $0 < |\lambda| < 1$, and G_λ the discrete group of analytic transformations of $C^n \backslash \{0\}$ generated by $z \mapsto \lambda z$. It is well known (cf. e.g. [11], p. 137) that G_λ acts freely on $C^n \backslash \{0\}$, as a properly discontinuous group of analytic transformations, hence the quotient $CH^n = (C^n \backslash \{0\})/G_\lambda$ is a complex manifold (the *complex Hopf manifold*). The complex Hopf manifold is compact (as $CH^n \approx S^{2n-1} \times S^1$, a diffeomorphism) and its first Betti number is $b_1(CH^n) = 1$, hence admits no global Kähler metrics. It is known however (cf. [8], p. 22) to possess a natural Hermitian metric, i.e. $g_0 = |z|^{-2}\delta_{jk}\, dz^j \otimes d\bar{z}^k$, $|z|^2 = \delta_{jk} z^j \bar{z}^k$ (the *Boothby metric*). Moreover g_0 is a *locally conformal Kähler* metric, i.e. CH^n admits an open cover $\{U_\alpha\}_{\alpha \in \Gamma}$ and a family of C^∞ functions $f_\alpha : U_\alpha \to \boldsymbol{R}$, so that each (local) metric $g_\alpha = \exp(-f_\alpha) g_0|_{U_\alpha}$ is Kählerian, $\alpha \in \Gamma$. The (local) 1-forms df_α glue up to a (global) 1-form ω_0 (the *Lee form* of (CH^n, g_0)) expressed locally as $\omega_0 = d\log|z|^2$. The Lee form is parallel (with respect to the Levi-Civita connection of g_0) and the local Kähler metrics g_α are flat. Viceversa, by a well known result of I. Vaisman (cf. [14]) any generalized Hopf manifold (i.e. locally conformal Kähler manifold with a parallel Lee form) with flat local Kähler metrics is locally analytically homothetic to $(\mathbf{C}H^n, g_0)$. Cf. also [8], p. 56.

The *Lee field* is $B_0 = \omega_0^\sharp$ (where $\sharp$ denotes raising of indices with respect to g_0). Note that, on a Hopf manifold, $\|B_0\| = 2$.

A study of submanifolds of $(\mathbf{C}H^n, g_0)$, regarding both the geometry of their second fundamental form and their position with respect to the 'preferential direction' B_0 is in act (cf. [8], p. 147–298, for an account of the research over the last decade). If N is an orientable real hypersurface in $(\mathbf{C}H^{n+1}, g_0)$, we shall need the Gauss and Weingarten formulae

$$\nabla^0_X Y = \nabla^N_X Y + b(X, Y) \tag{3}$$

$$\nabla^0_X \eta = -A_\eta X + \nabla^\perp_X \eta \tag{4}$$

Here ∇^N, b, A_η and $\nabla^\perp$ are respectively the induced connection, the second fundamental form (of the given immersion $j : N \subset \mathbf{C}H^{n+1}$), the Weingarten operator (associated with the normal section η), and the normal connection. Let ξ be a global unit normal field on N and set $A = A_\xi$. The Gauss and Codazzi equations are

$$\begin{aligned} R^N(X, Y)Z = {} & (X \wedge Y)Z + g_N(AY, Z)AX - g_N(AX, Z)AY \\ & + \tfrac{1}{4}\{[\omega(X)Y - \omega(Y)X]\omega(Z) \\ & + [g_N(X, Z)\omega(Y) - g_N(Y, Z)\omega(X)]B\} \end{aligned} \tag{5}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \tfrac{1}{4}\{\omega(Y)X - \omega(X)Y\}\omega_0(\xi) \tag{6}$$

These may be obtained from (3)–(4) and an explicit calculation of the curvature of the Boothby metric (or as a consequence of (12.19)–(12.20) in [8], p. 152, the Gauss and Codazzi equations of a submanifold in an arbitrary l.c.K. manifold with flat local Kähler metrics). Here $\omega = j^*\omega_0$ and $B = tan(B_0)$ is the tangential component of the Lee field. As a straightforward consequence of (6) one has

THEOREM 2. *Let N be an orientable totally umbilical $(b = H \otimes g_N)$ real hypersurface of $(\mathbf{C}H^{n+1}, g_0)$. Then N has a parallel mean curvature vector $(\nabla^\perp H = 0)$ if and only if for any $x \in N$ either $\omega_x = 0$ or N is tangent to the Lee field B_0 at x.*

3. Proof of Theorem 1

Let (M, g) and (N, g_N) be Riemannian manifolds. Assume M to be m-dimensional, compact, and orientable. A C^∞ map $\phi : M \to N$ is said to be *harmonic* if it is a critical point of the energy functional

$$E(\phi) = \frac{1}{2}\int_M \|d\phi\|^2 v_g$$

where $\|d\phi\|$ is the Hilbert-Schmidt norm, i.e. $\|d\phi\|^2 = trace_g(\phi^* g_N)$, and v_g the canonical volume form on (M, g). Let $\{\phi_{s,t}\}_{-\varepsilon<s,t<\varepsilon}$ be a 2-parameter variation of a harmonic map ϕ ($\phi_{0,0} = \phi$) and set

$$I(V, W) = \frac{\partial^2}{\partial s \partial t} E(\phi_{s,t})_{s=t=0}$$

where $V = \partial\phi_{s,t}/\partial t|_{s=t=0}$ and $W = \partial\phi_{s,t}/\partial s|_{s=t=0}$. Then ϕ is said to be *stable* if $I(V, V) \geq 0$ for any $V \in \Gamma^\infty(\phi^{-1}TN)$.

Let $j : N \subset \boldsymbol{C}^{n+1}H$ be a real hypersurface, under the hypothesis of Theorem 1. Let $N(j) \to N$ be the normal bundle of the immersion j, and let $X = tan(X) + X^\perp$ be the decomposition of $X \in T(\boldsymbol{C}H^{n+1})$ with respect to

$$T_x(\boldsymbol{C}H^{n+1}) = T_x(N) \oplus N(j)_x, \quad x \in N$$

Let $\{X_i : 1 \leq i \leq m\}$ be a (local) g-orthonormal frame on an open set $U \subseteq M$ and $\{V_a : 1 \leq a \leq 2n+2\}$ a (local) g_0-orthonormal parallel (i.e. $\nabla^0 V_a = 0$) frame on an open set $V \subseteq \boldsymbol{C}H^{n+1}$, so that $\phi(U) \subset V$. The frame $\{V_a\}$ may be obtained by parallel translation of a $g_{0,x}$-orthonormal basis in $T_x(\boldsymbol{C}H^{n+1})$ along geodesics issueing at x, in a simple and convex neighborhood V of x.

Let $\tilde{\nabla} = \phi^{-1}\nabla^N$ be the connection in $\phi^{-1}TN \to M$, induced by ∇^N. Then

$$\tilde{\nabla}_{X_i} tan(V_a) = A_{V_a^\perp}(d\phi)X_i \tag{7}$$

Indeed (by (3)–(4))

$$\begin{aligned}\tilde{\nabla}_{X_i} tan(V_a) &= \nabla^N_{(d\phi)X_i} tan(V_a) = tan(\nabla^0_{(d\phi)X_i} tan(V_a)) \\ &= tan(\nabla^0_{(d\phi)X_i}(V_a - V_a^\perp)) = -tan(\nabla^0_{(d\phi)X_i} V_a^\perp) = A_{V_a^\perp}(d\phi)X_i\end{aligned}$$

Moreover

$$\sum_{a=1}^{2n+2}\sum_{i=1}^{m} \|\tilde{\nabla}_{X_i} tan(V_a)\|^2 = \|d\phi\|^2\|H\|^2 \tag{8}$$

To prove (8) one uses (7) and $\|X\|^2 = \sum_{a=1}^{2n+2} g_0(X, V_a)^2$, for any $X \in T(CH^{n+1})$, and conducts the following calculation

$$\|\tilde{\nabla}_{X_i} tan(V_a)\|^2 = \|A_{V_a^\perp}(d\phi)X_i\|^2 = \sum_{b=1}^{2n+2} g_0(A_{V_a^\perp}(d\phi)X_i, V_b)^2$$

$$= \sum_{b=1}^{2n+2} g_N(A_{V_a^\perp}(d\phi)X_i, tan(V_b))^2 = \sum_{b=1}^{2n+2} g_0(b((d\phi)X_i, tan(V_b)), V_a^\perp)^2$$

Next, as N is totally umbilical

$$\|\tilde{\nabla}_{X_i} tan(V_a)\|^2 = \sum_{b=1}^{2n+2} g_N((d\phi)X_i, tan(V_b))^2 g_0(H, V_a^\perp)^2$$

$$= \|(d\phi)X_i\|^2 g_0(H, V_a^\perp)^2$$

which leads to (8). Again by the umbilicity assumption, the Gauss equation (5) becomes

$$R^N(X, Y)Z = (1 + \|H\|^2)\{g_N(Y,Z)X - g_N(X,Z)Y\}$$
$$+ \frac{1}{4}\{[\omega(X)Y - \omega(Y)X]\omega(Z) + [g_N(X,Z)\omega(Y) - g_N(Y,Z)\omega(X)]B\}$$

Therefore

$$g_N(R^N(X, Y)Y, X) = (1 + \|H\|^2)\{\|X\|^2\|Y\|^2 - g_N(X, Y)^2\}$$
$$- \frac{1}{4}\{\omega(X)^2\|Y\|^2 - 2\omega(X)_\omega(Y)g_N(X, Y) + \omega(Y)^2\|X\|^2\}$$

for any $X, Y \in T(N)$. Using

$$\sum_{a=1}^{2n+2} \|V_a^\perp\|^2 = 1, \quad \sum_{a=1}^{2n+2} \|tan(V_a)\|^2 = 2n + 1$$

$$\sum_{a=1}^{2n+2} \omega(tan(V_a))V_a = B$$

we may conduct the following calculation

$$\sum_{a=1}^{2n+2}\sum_{i=1}^{m} g_N(R^N(tan(V_a),(d\phi)X_i)(d\phi)X_i, tan(V_a))$$

$$= (1+\|H\|^2)\sum_{a=1}^{2n+2}\left\{\|tan(V_a)\|^2\|d\phi\|^2 - \sum_{i=1}^{m} g_0((d\phi)X_i, V_a)^2\right\}$$

$$-\frac{1}{4}\sum_{a=1}^{2n+2}\{\omega(tan(V_a))^2\|d\phi\|^2 + \|\phi^*\omega\|^2\|tan(V_a)\|^2\}$$

$$+\frac{1}{2}\sum_{a=1}^{2n+2}\omega(tan(V_a))g_0\left(V_a, \sum_{i=1}^{m}\omega((d\phi)X_i)(d\phi)X_i\right)$$

$$= 2n(1+\|H\|^2)\|d\phi\|^2 - \frac{1}{4}\{\|B\|^2\|d\phi\|^2 + (2n+1)\|\phi^*\omega\|^2\}$$

$$+\frac{1}{2}g_N(B,(d\phi)(\phi^*\omega)^\sharp)$$

where $\sharp$ denotes raising of indices with respect to g. Next

$$g_N(B,(d\phi)(\phi^*\omega)^\sharp) = \sum_{i=1}^{m}\omega((d\phi)X_i)^2 = \|\phi^*\omega\|^2$$

hence

$$\sum_{a=1}^{2n+2}\sum_{i=1}^{m} g_N(R^N(tan(V_a),(d\phi)X_i)(d\phi)X_i, tan(V_a))$$
$$= \left\{2n(1+\|H\|^2) - \frac{1}{4}\|B\|^2\right\}\|d\phi\|^2 - \frac{2n-1}{4}\|\phi^*\omega\|^2 \tag{9}$$

By the second variation formula (cf. e.g. [9]), for any harmonic map ϕ of (M,g) into (N,g_N)

$$I(V,W) = \sum_{i=1}^{m}\int_M\{g_N(\tilde{\nabla}_{X_i}V, \tilde{\nabla}_{X_i}W) - g_N(R^N(V,(d\phi)X_i)(d\phi)X_i, W)\}v_g$$

Then (by (8)–(9) and our assumption (1))

$$\sum_{a=1}^{2n+2} I(tan(V_a), tan(V_a))$$

$$= -\int_M\left\{2n + (2n-1)\|H\|^2 - \frac{1}{4}\|B\|^2\right\}\|d\phi\|^2 v_g + \frac{2n-1}{4}\int_M\|\phi^*\omega\|^2 v_g < 0$$

hence ϕ is unstable. If $B^{\perp} = 0$ then (by Theorem 2) N has constant mean curvature; also, if this is the case, then $\|B\| = 2$ hence

$$\int_M \left(2n + (2n-1)\|H\|^2 - \frac{1}{4}\|B\|^2\right)\|d\phi\|^2 v_g = 2(2n-1)(1 + \|H\|^2)E(\phi),$$

i.e. (1) assumes the simpler form (2).

4. Unstable Pseudohermitian Immersions

Let $(M, T_{1,0}(M))$ be a CR manifold (of hypersurface type), of CR dimension p, and $H(M) = Re\{T_{1,0}(M) \oplus \overline{T_{1,0}(M)}\}$ its *Levi* (or maximally complex) *distribution*. A *pseudohermitian structure* on M is a nonzero global section θ_M in the conormal bundle $H(M)^{\perp} \subset T^*(M)$. Given a pseudohermitian structure θ_M, the *Levi form* is given by

$$G_{\theta_M}(X, Y) = (d\theta_M)(X, J_M Y), \quad X, Y \in H(M),$$

where $J_M(Z + \bar{Z}) = \sqrt{-1}(Z - \bar{Z})$, $Z \in T_{1,0}(M)$, is the complex structure in $H(M)$. The CR manifold M is *nondegenerate* if the Levi form G_{θ_M} is nondegenerate for some pseudohermitian structure θ_M (and thus for all). If this is the case then θ_M is a contact form on M, i.e. $\theta_M \wedge (d\theta_M)^p$ is a volume form on M. A CR manifold $(M, T_{1,0}(M))$ is *strictly pseudoconvex* if the Levi form G_{θ_M} is positive definite, for some pseudohermitian structure θ_M on M.

Let $(M, T_{1,0}(M))$ be a nondegenerate CR manifold and θ_M a contact form on M. Under the mild additional assumption that M be orientable, there is a nonzero tangent vector field T on M (the *characteristic direction* of (M, θ_M)), uniquely determined by

$$\theta_M(T) = 1, \quad T \rfloor d\theta_M = 0.$$

As $T(M) = H(M) \oplus \boldsymbol{R}T$, this may be used to extend the Levi form G_{θ_M} to a (semi-Riemannian, in general) metric on the whole of $T(M)$, by requesting that T be orthogonal to $H(M)$ and assigning to T a fixed length, i.e. let g_{θ_M} be defined by setting

$$g_{\theta_M}(X, Y) = G_{\theta_M}(X, Y),$$
$$g_{\theta_M}(X, T) = 0, \quad g_{\theta_M}(T, T) = 1,$$

for any $X, Y \in H(M)$. This is referred to as *the Webster metric* of (M, θ_M) (compare to (2.18) in [15], p. 34). If M is strictly pseudoconvex and a contact form θ_M is chosen so that G_{θ_M} be positive definite, then g_{θ_M} is a Riemannian metric on M. Note that $g_{\theta_M}(X,T)=\theta_M(X)$, for any $X\in T(M)$. In particular $\|\theta_M\|=1$.

Let $(M,T_{1,0}(M))$ and $(A,T_{1,0}(A))$ be strictly pseudoconvex CR manifolds. Let $\phi: M\to A$ be a *CR immersion*, i.e. a C^∞ immersion and a CR map (i.e. $(d_x\phi)T_{1,0}(M)_x\subseteq T_{1,0}(A)_{\phi(x)}$, $x\in M$). If θ_M and θ_A are contact forms, on M and A respectively, so that G_{θ_M} and G_{θ_A} be positive-definite, then $\phi^*\theta_A=\lambda\theta_M$, for some C^∞ function $\lambda: M\to(0,+\infty)$. If $\lambda\equiv 1$ then ϕ is said to be *isopseudohermitian*. An isopseudohermitian CR immersion $\phi: M\to A$ is said to be a *pseudohermitian immersion* if $\phi(M)$ is tangent to the characteristic direction of (A,θ_A). A theory of pseudohermitian immersions has been started in [6] and [2]. We recall (cf. Theorem 7 in [6], p. 189)

THEOREM 3. *Any pseudohermitian immersion between two strictly pseudoconvex CR manifolds is a minimal isometric* (*with respect to the Webster metrics*) *immersion.*

Set $U=-J\xi$ and $\theta(X)=g_N(X,U)$, for any $X\in T(N)$. We establish

THEOREM 4. *Let N be an orientable real hypersurface of the complex Hopf manifold $\boldsymbol{C}H^{n+1}$, tangent to the Lee field B_0. Assume that N is totally umbilical of nonzero mean curvature ($\|H\|\neq 0$). Let $\phi: M\to N$ be a pseudohermitian immersion of a compact strictly pseudoconvex CR manifold M into N, thought of as a map of (M,θ_M) into $(N,\hat{\theta})$, where $\hat{\theta}=g_0(H,\xi)\theta$. If the Lee field of $\boldsymbol{C}H^{n+1}$ is orthogonal to the CR structure of N and*

$$E(\phi)>\frac{Vol(M)}{2(1+\|H\|^2)\|H\|^2} \tag{10}$$

then ϕ is an unstable harmonic map.

The source manifold M carries the Webster metric $g=g_{\theta_M}$, while N is endowed with the induced metric $g_N=j^*g_0$. Also N carries the induced CR structure

$$T_{1,0}(N)=T^{1,0}(\boldsymbol{C}H^{n+1})\cap[T(N)\otimes\boldsymbol{C}]$$

($T^{1,0}(\boldsymbol{C}H^{n+1})$ is the holomorphic tangent bundle over $\boldsymbol{C}H^{n+1}$). The Levi form of N is

$$G_\theta(X,Y)=(d\theta)(X,JY)$$

for any $X,Y\in H(N)=Ker(\theta)$. We need

LEMMA 1. *Let N be a totally umbilical real hypersurface of the complex Hopf manifold. If $H + \frac{1}{2}B^{\perp} \neq 0$ everywhere on N then $(N, T_{1,0}(N))$ is a strictly pseudo-convex CR manifold.*

We recall (cf. Corollary 1.1 in [8], p. 4) that

$$\nabla^0_X JY = J\nabla^0_X Y + \tfrac{1}{2}\{\omega_0(JY)X - \omega_0(Y)JX + g_0(X, Y)JB_0 - g_0(X, JY)B_0\}$$

for any $X, Y \in T(CH^{n+1})$. Then (as $\nabla^{\perp}\xi = 0$)

$$\begin{aligned}(\nabla^N_X \theta)Y &= g_N(Y, \nabla^N_X U) = -g_0(Y, \nabla^0_X J\xi)\\ &= g_N(PAX, Y) + \tfrac{1}{2}\{\omega(U)g_N(X, Y) + \omega_0(\xi)g_N(PX, Y) - \theta(X)\omega(Y)\}\end{aligned}$$

for any $X, Y \in T(N)$. Here $PX = tan(JX)$. Next, using

$$2(d\theta)(X, Y) = (\nabla^N_X \theta)Y - (\nabla^N_Y \theta)X$$

we get

$$2(d\theta)(X, Y) = g_N((PA + AP)X, Y) + g_N(PX, Y)\omega_0(\xi) - (\theta \wedge \omega)(X, Y) \tag{11}$$

hence the Levi form of N is expressed by

$$G_\theta(X, Y) = \tfrac{1}{2}\{g_0(b(X, Y) + b(JX, JY), \xi) + g_N(X, Y)\omega_0(\xi)\} \tag{12}$$

for any $X, Y \in H(N)$. Assume from now on that $b = H \otimes g_N$. Then (12) becomes

$$G_\theta(X, Y) = g_0(H + \tfrac{1}{2}B^{\perp}, \xi)g_N(X, Y)$$

hence either G_θ or $G_{-\theta}$ is positive definite.

LEMMA 2. *Let N be a real hypersurface of CH^{n+1}, under the hypothesis of Lemma 1. If additionally $B^{\perp} = 0$ then $f := g_0(H, \xi) = const.$ and (by replacing θ by $-\theta$ if necessary) one may assume $f > 0$. Moreover, if B is orthogonal to $T_{1,0}(N)$ then the induced metric $g_N = j^* g_0$ and the Webster metric $g_{f\theta}$ are homothetic.*

To prove Lemma 2 assume that $B^{\perp} = 0$. Set $f = g_0(H, \xi) \in C^{\infty}(N)$. By Theorem 2, $f = const.$ (indeed, $\nabla^{\perp}H = 0$ yields $f^2 = \|H\|^2 = const.$).

Assume now that $B \perp T_{1,0}(N)$. In particular $B \perp H(N)$ (i.e. $Ker(\omega) = H(N)$, hence ω and θ are proportional). As N is umbilical and tangent to B_0 the equation (11) becomes

$$(d\theta)(X, Y) = fg_N(PX, Y)$$

for any $X, Y \in T(N)$. Then $PU = 0$ yields $U \rfloor d\theta = 0$, i.e. U is the characteristic direction of (N, θ). Set $\hat{\theta} = f\theta$ and let $g_{\hat{\theta}}$ be the corresponding Webster metric (of $(N, \hat{\theta})$) (a Riemannian metric on N). Then

$$g_{\hat{\theta}} = \|H\|^2 g_N$$

and Lemma 2 is proved. By Theorem 3, ϕ is harmonic, as a map of (M, g) into $(N, g_{\hat{\theta}})$, i.e. ϕ is a critical point of the energy functional

$$E_\theta(\phi) = \frac{1}{2}\int_M trace_g(\phi^* g_{\hat{\theta}}) v_g$$

Yet $E_\theta(\phi) = f^2 E(\phi)$ hence ϕ is harmonic as a map of (M, g) into (N, g_N). On the other hand

$$\omega = \theta(B)\theta \tag{13}$$

Note that (13) yields $|\theta(B)| = 2$ (indeed $\theta(B)^2 = \omega(B) = \|B\|^2 = \|B_0\|^2 = 4$) hence

$$\int_M \|\phi^*\omega\|^2 v_g = \frac{4}{f^2}\int_M \|\theta_M\|^2 v_g = 4\frac{Vol(M)}{\|H\|^2}$$

(as $\phi^*\hat{\theta} = \theta_M$). Therefore, the assumption (10) is equivalent to (2), and (by Theorem 1) ϕ must be unstable.

References

[1] M. Ara, Geometry of F-harmonic maps, Kodai Math. J., (2) 22 (1999), 243–263.

[2] E. Barletta & S. Dragomir, Pseudohermitian immersions, pseudo-Einstein structures, and the Lee class of a CR manifold, Kodai Mathem. J., 19 (1996), 62–86.

[3] E. Barletta & S. Dragomir & H. Urakawa, Pseudoharmonic maps from nondegenerate CR manifolds to Riemannian manifolds, to appear in Indiana Univ. Mathem. J., 2001.

[4] A. Bejancu, Umbilical CR submanifolds of a complex space form, Rendiconti di Matem., (3) 15 (1980), 431–446.

[5] B. Y. Chen, Geometry of submanifolds, Marcel Dekker, Inc., New York, 1973.

[6] S. Dragomir, On pseudohermitian immersions between strictly pseudoconvex CR manifolds, American J. Math., (1) 117 (1995), 169–202.

[7] S. Dragomir & R. Grimaldi, A classification of totally umbilical CR submanifolds of a generalized Hopf manifold, Boll. U.M.I., (7) 9-A (1995), 557–568.

[8] S. Dragomir & L. Ornea, Locally conformal Kähler geometry, Progress in Mathematics, Vol. 155, Birkhäuser, Boston-Basel-Berlin, 1998.

[9] J. Eells & L. Lemaire, Selected topics in harmonic maps, Regional Conf. Ser. in Math., No. 50, 1983.

[10] S. Ianus & L. Ornea & K. Matsumoto, Immersions sphèriques dans une varietè de Hopf généralisée, C. R. Acad. Sci. Paris, Sér. I Math., 316 (1993), 63–66.

[11] S. Kobayashi & K. Nomizu, Foundations of differential geometry, Vol. II, Interscience Publishers, New York, 1969.

[12] P. F. Leung, On the stability of harmonic maps, Lecture Notes in Math., Vol. 949, Springer-Verlag, 1982, 122–129.
[13] Y. Tashiro & S. Tachibana, On Fubinian and C-Fubinian manifolds, Kodai Math. Sem. Rep., 15 (1963), 176–183.
[14] I. Vaisman, Locally conformal Kähler manifolds with parallel Lee form, Rendiconti di Matem., 12 (1979), 263–284.
[15] S. Webster, Pseudohermitian structures on a real hypersurface, J. Differential Geometry, 13 (1978), 25–41.

Università degli Studi della Basilicata
Dipartimento di Matematica
Via N. Sauro 85
85100 Potenza, Italia
e-mail: DRAGOMIR@UNIBAS.IT