

NONLINEAR WEAKLY HYPERBOLIC EQUATIONS WITH LEVI CONDITION IN GEVREY CLASSES

By

Massimo CICOGNANI and Luisa ZANGHIRATI

Introduction

In [9] K. Kajitani proved that the Cauchy problem for nonlinear weakly hyperbolic equations is locally well posed in Gevrey classes G^σ with $1 < \sigma \leq r/(r-1)$, r the largest multiplicity of the characteristic roots, without any further condition as in the linear case.

On the other hand, D. Gourdin in [6] and the authors in [3] obtained the well posedness in C^∞ for some classes of nonlinear hyperbolic equations with constant multiplicity under Levi conditions on the linearized operator.

Here we treat the same topics in classes G^σ with $\sigma > r/(r-1)$ considering the generalized Levi condition (L_σ) introduced by V. Ya. Ivrii in [8] for linear equations and imposing a further Levi condition (\tilde{L}_σ) of nonlinear type.

Conditions (L_σ) and (\tilde{L}_σ) are empty for $1 < \sigma \leq r/(r-1)$ whereas (L_σ) reduces to the usual Levi condition as $\sigma \rightarrow +\infty$, so the results of this paper fit the ones of [9] and [3].

Concerning well posedness in Gevrey classes for nonlinear equations we quote also Leray-Ohya [15].

1. Main Results

We shall study the well posedness in Gevrey classes for the quasilinear problem:

$$(1.1) \quad \sum_{|\alpha| \leq m} a_\alpha(t, x, D^{m'} u) D_{t,x}^\alpha u = f(t, x, D^{m'} u)$$

$$D_t^j u|_{t=0} = g_j, \quad 0 \leq j < m$$

where $(t, x) \in [-T_0, T_0] \times \mathbf{R}^n$ and $D^{m'} u := (D_{t,x}^\alpha u; |\alpha| \leq m')$, $m' < m$. The functions $a_\alpha(t, x, w)$ and $f(t, x, w)$ are defined in $[-T_0, T_0] \times \mathbf{R}^n \times W_0$, where W_0 is a neighborhood in \mathbf{R}^ℓ of the set $\{D_x^\beta g_j(x); x \in \mathbf{R}^n, j + |\beta| \leq m'\}$, ℓ the number of the multi-indices $\alpha \in \mathbf{Z}_+^{n+1}$ with length $|\alpha| \leq m'$.

Since one can expect only local regular solutions, we can assume that g_j and $f(t, x, D_x^\beta g_j)$ have compact supports.

To state our main result we need to introduce the following notations: for $A > 0$ and $\sigma > 1$ let us denote by $G_A^\sigma := G_A^\sigma(\mathbf{R}^n)$ the space of all functions f satisfying

$$\|f\|_{G_A^\sigma} := \sup_{\substack{x \in \mathbf{R}^n \\ \alpha \in \mathbf{Z}_+^n}} |D^\alpha f(x)| A^{-|\alpha|} |\alpha|!^{-\sigma} < \infty$$

and by $G_A^{\sigma,1} := G_A^{\sigma,1}(\mathbf{R}^n \times W_0)$, W_0 an open set in \mathbf{R}^ℓ , the space of all functions $f(x, w)$ such that

$$\|f\|_{G_A^{\sigma,1}} := \sup_{\substack{(x,w) \in \mathbf{R}^n \times W_0 \\ (\alpha,\beta) \in \mathbf{Z}_+^n \times \mathbf{Z}_+^\ell}} |D_x^\alpha D_w^\beta f(x, w)| A^{-|\alpha|-|\beta|} |\alpha|!^{-\sigma} |\beta|!^{-1} < \infty;$$

so $G^\sigma := \bigcup_{A>0} G_A^\sigma$ and $G^{\sigma,1} := \bigcup_{A>0} G_A^{\sigma,1}$ are Gevrey spaces.

Finally, we write $C^k(-T, T; G_A^\sigma)$ for the space of all functions of class C^k in $[-T, T]$ with values in G_A^σ . We define $C^k(-T, T; G_A^{\sigma,1})$ in a similar way.

Concerning the regularity of the data in problem (1.1), we assume:

$$(R) \quad g_j \in G_{A_0}^\sigma, \quad 0 \leq j < m; \quad a_\alpha(t, x, w), \quad |\alpha| \leq m, \quad \text{and}$$

$$f(t, x, w) \quad \text{are in } C^{k_0}(-T_0, T_0; G_{A_0}^{\sigma,1}).$$

Moreover, we assume weak hyperbolicity with characteristic roots of constant multiplicity, that is, writing

$$(1.2) \quad \begin{aligned} P(t, x, w; D_t, D_x) &:= \sum_{j=0}^m P_j(t, x, w; D_t, D_x), \\ P_j(t, x, w; D_t, D_x) &:= \sum_{|\alpha|=j} a_\alpha(t, x, w) D_t^{\alpha_0} D_x^{\alpha'}, \\ \alpha &= (\alpha_0, \alpha'), \end{aligned}$$

the principal symbol P_m admits the factorization

$$(H) \quad P_m(t, x, w; \tau, \xi) = \prod_{j=1}^s (\tau - \lambda_j(t, x, w, \xi))^{m_j}$$

where the roots λ_j are real and satisfy $\lambda_h(t, x, w, \xi) \neq \lambda_k(t, x, w, \xi)$ for every (t, x, w, ξ) if $h \neq k$. The number s and the multiplicity m_j are independent of (t, x, w, ξ) , $\sum_{j=1}^s m_j = m$.

Let $r := \max\{m_j; 1 \leq j \leq s\}$ denote the largest multiplicity.

We assume:

$$(\tilde{L}_\sigma) \quad m' \leq m - r(1 - 1/\sigma).$$

We are interested in the case $r > 1$, $\sigma > r/(r - 1)$, since local existence and uniqueness of solutions in G^σ is well known for every σ if $r = 1$ (strict hyperbolicity) whereas for $r > 1$, $1 < \sigma \leq r/(r - 1)$ it has been proved by Kajitani [9] for any fully nonlinear hyperbolic equation.

Nevertheless, for $r > 1$, $\sigma > r/(r - 1)$ condition (H) is not sufficient as one can see already in the linear case, i.e. a_α and f independent of w [4].

In addition to (H), for every $u \in C^{k_0+m'}(-T_0, T_0; G^\sigma)$ such that $D^{m'}u$ takes values in W_0 , we require that the linear operator

$$(1.3) \quad P^{(u)}(t, x, ; D_t, D_x) := P(t, x, w; D_t, D_x)|_{w=D^{m'}u(t,x)},$$

with characteristic roots

$$\lambda_j^{(u)}(t, x, \xi) := \lambda_j(t, x, w, \xi)|_{w=D^{m'}u(t,x)},$$

satisfies the generalized Levi condition stated by Ivrii [8], that is: for every solution φ of

$$(1.4) \quad \partial_t \varphi = \lambda_j^{(u)}(t, x, \nabla_x \varphi),$$

for every real $\psi \in C^\infty([-T_0, T_0] \times \mathbf{R}^n)$ and every $h \in C_0^\infty([-T_0, T_0] \times \mathbf{R}^n)$ one has

$$(L_\sigma) \quad P^{(u)}(he^{i\rho\varphi+i\rho^{1/\sigma}\psi}) = O(\rho^{m-m_j(1-1/\sigma)}) \quad \rho \rightarrow +\infty, \quad j = 1, \dots, s.$$

Under hypotheses (H), (R) and (L_σ) the linear Cauchy problem:

$$(1.5) \quad \begin{aligned} P^{(u)}v &= 0 \\ D_t^j v|_{t=0} &= g_j \quad 0 \leq j < m \end{aligned}$$

with data $g_j \in G_A^\sigma$ has a unique global solution $v \in C^{k_0+m}(-T_0, T_0; G_B^\sigma)$ for some $B \geq A$, [8], [16], provided that $k_0 > m$ is large enough to perform the Mizohata “good factorization” of $P^{(u)}$, [16]. Hereafter we take such k_0 in condition (R).

Condition (L_σ) becomes also necessary for the well posedness of problem (1.5) in G^σ when $P^{(u)}$ has analytic coefficients, [12], [13], [8].

The asymptotic behavior requested in (L_σ) is clearly independent of the terms $P_j(t, x, w; D_t, D_x)$ of $P(t, x, w; D_t, D_x)$ with degree $j \leq m - r(1 - 1/\sigma)$, thus assumption (L_σ) and (\tilde{L}_σ) are strictly connected. In fact, (\tilde{L}_σ) is the appropriate

nonlinear Levi condition as it will be clear at the end of the proof of our main result. Leray-Ohya [15] impose the same condition for the particular class of operators they consider, that is for $P(t, x, D_{t,x}^{m'}u, D_t, D_x) = \prod_{j=1}^p A_j(t, x, D^{m_j}u, D_t, D_x)$, A_j strictly hyperbolic. Such a P is not necessarily of constant multiplicity but the completely different techniques we use here can easily adapted to cover this situation.

We shall prove the following result of local existence and uniqueness for the solution of problem (1.1).

THEOREM 1.1. *Assume that conditions (R), (H), (L_σ) and (\tilde{L}_σ) are fulfilled. Then there exist $T < T_0$, $A > A_0$ such that problem (1.1) has a unique solution $u \in C^{k_0+m}(-T, T; G_A^\sigma)$.*

REMARK 1.2. Condition (L_σ) concerns only the terms P_j of $P(t, x, w; D_t, D_x)$ with degree $j \in]m - r(1 - 1/\sigma), m[$, therefore it is empty for $1 < \sigma \leq r/(r - 1)$. Also hypothesis (\tilde{L}_σ) gives no restriction since it becomes $m' \leq m - 1$ (any quasilinear equation), so from Theorem 1.1 we reobtain the already quoted result by Kajitani [9]. In the opposite direction, we recognize (L_∞) to be the usual Levi condition, whereas (\tilde{L}_∞) is $m' \leq m - r$, so that Theorem 1.1 for $\sigma = \infty$, $G^\infty := C^\infty$, coincides with Theorem 1.1 in [3].

Ivrii [8] and Mizohata [17] show that assumptions (H) and (L_σ) mean that the terms P_j of P can be factorized in suitable ways, see also Komatsu [13]. In the remaining part of this section we use the differential factorization by Ivrii to give examples of equations satisfying the hypotheses of Theorem 1.1 while we shall use the pseudodifferential “good factorization” by Mizohata during the proof of our main result throughout Section 3 and Section 4.

Precisely, if P satisfies condition (H), then we have [16]:

$$(1.6) \quad P_m = (A_1)^{r_1} \cdot \dots \cdot (A_{s'})^{r_{s'}} + (\text{terms of order } \leq m - 1)$$

where A_j , $j = 1, \dots, s'$, and $A_1 \cdot \dots \cdot A_{s'}$ are strictly hyperbolic differential operators.

It is $\{r_1, \dots, r_{s'}\} = \{m_1, \dots, m_s\}$, hence $\max_{1 \leq i \leq s'} r_i = \max_{1 \leq j \leq s} m_j = r$.

For $1 \leq k \leq r$, let us define

$$Q_k = (A_1)^{(r_1-k)_+} \cdot \dots \cdot (A_{s'})^{(r_{s'}-k)_+}, \quad (z)_+ = \max\{z, 0\}.$$

P satisfies condition (L_σ) if and only if in the case of analytic coefficients (see [18] for a detailed proof):

$$(1.7) \quad P = \sum_{j=0}^m L_j, \quad \text{ord } L_j \leq j, \quad L_j = B_j Q_k$$

where $k_j = [(m - j)/(1 - 1/\sigma)]$, $0 < j \leq m$, B_j is an arbitrary differential operator of order $(j - \text{ord } Q_{k_j})$, $j < m$, $B_m = 1$ and $[z] = \max\{\ell \in \mathbf{Z}; \ell \leq z\}$.

Note that for $j \leq m - r(1 - 1/\sigma)$ it is $k_j \geq r$ hence $Q_{k_j} = I$ and the factorization of L_j in (1.7) becomes trivial: L_j any operator of order j . In particular (1.7) is always fulfilled for $\sigma \leq r/(r - 1)$ as (L_σ) .

Now, let $\sigma_1 < \sigma_2 < \dots < \sigma_\mu$ be a numbering of the set $\{\ell/d; \ell, d \in \mathbf{Z}, 1 \leq d < \ell \leq r\}$. Then, given P_m in the factorized form (1.6), we are able to write, for every $\sigma > 1$, the structure of all operators P satisfying (1.7), such structure remaining the same for $\sigma \in]\sigma_h, \sigma_{h+1}[$, $h = 0, \dots, \mu$, $\sigma_0 := 1$, $\sigma_{\mu+1} := \infty$.

For example, let A_1, A_2 and A_1A_2 be strictly hyperbolic operators of orders μ_1, μ_2 and $\mu_1 + \mu_2$ respectively. Then, for

$$P_m = A_1^3 A_2^2 + (\text{terms of order } \leq m - 1), \quad m = 3\mu_1 + 2\mu_2$$

we have $r = 3$, $\sigma_1 = 3/2$, $\sigma_2 = 3$, $\sigma_3 = 3$. In (1.7) it is

$$k_{m-1} = \begin{cases} 2, & 3/2 < \sigma \leq 2 \\ 1, & 2 \leq \sigma \leq \infty \end{cases}$$

$$k_{m-2} = \begin{cases} k \geq 3, & 3/2 < \sigma \leq 3 \\ 2, & 3 < \sigma \leq \infty \end{cases}$$

whereas for $j \geq 3$ we have $k_{m-j} \geq 3$ for every $\sigma > 1$.

Hence, the operators P satisfying (1.7), with principal part $P_m = A_1^3 A_2^2 +$ (terms of order $\leq m - 1$) are of the form:

$$A_1^3 A_2^2 + B_{m-\mu_1-1} A_1 + B_{m-2}, \quad 3/2 < \sigma \leq 2$$

$$A_1^3 A_2^2 + B_{m-2\mu_1-\mu_2-1} A_1^2 A_2 + B_{m-2}, \quad 2 < \sigma \leq 3$$

$$A_1^3 A_2^2 + B_{m-2\mu_1-\mu_2-1} A_1^2 A_2 + B_{m-\mu_1-2} A_1 + B_{m-3}, \quad 3 < \sigma \leq \infty$$

where B_ν denotes an arbitrary operator of order ν .

From this, we can give examples of equations satisfying the hypotheses of Theorem 1.1 with double and triple characteristic roots:

$$A_1^3(t, x, D^{\mu_1-1}u, D_t, D_x)A_2^2(t, x, D^{\mu_2-2}u, D_t, D_x)u$$

$$+ B_{m-\mu_1-1}(t, x, D^{m-2}u, D_t, D_x)A_1(t, x, D^{\mu_1-1}u, D_t, D_x)u = f(t, x, D^{m-2}u)$$

if $3/2 < \sigma \leq 2$;

$$A_1^3(t, x, D^{\mu_1-1}u, D_t, D_x)A_2^2(t, x, D^{\mu_2-2}u, D_t, D_x)u$$

$$+ B_{\mu_1+\mu_2-1}(t, x, D^{m-2}u, D_t, D_x)A_1^2(t, x, D^{\mu_1-1}u, D_t, D_x)A_2(t, x, D^{\mu_2-2}u, D_t, D_x)u$$

$$= f(t, x, D^{m-2}u)$$

if $2 < \sigma \leq 3$;

$$\begin{aligned} & A_1^3(t, x, D^{\mu_1-1}u, D_t, D_x)A_2^2(t, x, D^{\mu_2-3}u, D_t, D_x)u \\ & + B_{\mu_1+\mu_2-1}(t, x, D^{m-3}u, D_t, D_x)A_1^2(t, x, D^{\mu_1-1}u, D_t, D_x)A_2(t, x, D^{\mu_2-3}u, D_t, D_x)u \\ & + B_{m-\mu_1-2}(t, x, D^{m-3}u, D_t, D_x)A_1(t, x, D^{\mu_1-1}u, D_t, D_x)u = f(t, x, D^{m-3}u) \end{aligned}$$

if $3 < \sigma \leq \infty$.

2. Pseudodifferential Operators in Gevrey Classes

In section 1 we introduced the spaces of functions $G_A^\sigma, G_A^{\sigma,1}, C^k(-T, T; S_A^{\sigma,1})$. Correspondently we denote by $S_{A,\ell}^{m;\sigma}$, $m \in \mathbf{R}$, $\sigma > 1$, $A \in \mathbf{R}_+$, $\ell \in \mathbf{Z}_+$, the space of the symbols $a(x, \xi)$ of order m such that

$$\|a\|_{S_{A,\ell}^{m;\sigma}} := \sup_{\substack{\alpha \in \mathbf{Z}_+^n \\ |\beta| \leq \ell}} \sup_{\mathbf{R}^{2n}} \frac{|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)|}{A^{|\alpha|} \alpha!^\sigma \langle \xi \rangle^{m-|\beta|}} < \infty,$$

where $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$, and define:

$$S_A^{m;\sigma} := \varprojlim_{\ell \rightarrow +\infty} S_{A,\ell}^{m;\sigma}; \quad S^{m;\sigma} := \varinjlim_{A \rightarrow +\infty} S_A^{m;\sigma}.$$

For a symbol $a(x, \xi)$ in $S_{A,\ell}^{m;\sigma}$ we define the operator $a_\Lambda(x, D)$ by

$$a_\Lambda(x, D) = e^\Lambda a(x, D) e^{-\Lambda},$$

where $\Lambda = \tau \langle D \rangle^{1/\sigma}$, $\tau \in \mathbf{R}$.

For $k \in \mathbf{Z}_+$, $T > 0$, denote by $C^k(-T, T; S_{A,\ell}^{m;\sigma})$ the space of all functions $a(t, x, \xi)$ of class C^k in $[-T, T]$ with values in $S_{A,\ell}^{m;\sigma}$ and define $C^k(-T, T; S^{m;\sigma})$ analogously. Finally, let S_ℓ^m denote the class of symbols $a(x, \xi)$ of usual pseudodifferential operators in \mathbf{R}^n with norm

$$\|a\|_{S_\ell^m} = \sup_{|\alpha|+|\beta| \leq \ell} \sup_{\mathbf{R}^{2n}} |\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \langle \xi \rangle^{|\beta|-m}.$$

The following proposition has been proved by Kajitani [9].

PROPOSITION 2.1. *Let $a(t, x, \xi)$ be in $C^k(-T, T; S_{A,\ell}^{m;\sigma})$ and $\Lambda = \Lambda(t) = \lambda(2T - t) \langle D_x \rangle^{1/\sigma}$, $\lambda \in \mathbf{R}$. Then, if $2|\lambda|T \leq (24^\sigma n A)^{-1/\sigma}$ and $[\ell'\sigma/(\sigma - 1)] + \ell' + [n/2] + 2 \leq \ell$, $a_{\Lambda(t)}(t, x, \xi)$ is in $C^k(-T, T; S_{\ell'}^m)$ and satisfies:*

$$\|a_\Lambda\|_{C^k(-T, T; S_{\ell'}^m)} \leq c_{\ell'} \|a\|_{C^k(-T, T; S_{A,\ell}^{m;\sigma})},$$

where $c_{\ell'}$ is independent of $a(t, x, \xi)$.

We shall use symbols depending on a parameter w .

For $M \in \mathbf{Z}_+$, let $d_M = \text{card}\{\alpha \in \mathbf{Z}^{n+1}; \alpha = (\alpha_0, \alpha'), \alpha_0 \leq k_0, |\alpha| \leq M\}$ and $W = W_M = W_0 \times \mathbf{R}^{d_M}$, where W_0 is an open given set in \mathbf{R}^ℓ . We denote by $S_{A,\ell}^{m;\sigma,1}(\mathbf{R}^n \times W \times \mathbf{R}^n)$ the space of the symbols $a(x, w, \xi)$ of order m such that:

$$\|a\|_{S_{A,\ell}^{m;\sigma,1}} := \sup_{\substack{\alpha, \gamma \in \mathbf{Z}_+^n \\ |\beta| \leq \ell}} \sup_{\substack{x, \xi \in \mathbf{R}^n \\ w \in W}} \frac{|\partial_x^\alpha \partial_w^\gamma \partial_\xi^\beta a(x, w, \xi)|}{A^{|\alpha|+|\gamma|} \alpha! \sigma \gamma! \langle \xi \rangle^{m-|\beta|}} < \infty,$$

and by $C^k(-T, T; S_{A,\ell}^{m;\sigma,1})$ the space of all functions of class C^k in $[-T, T]$ with values in $S_{A,\ell}^{m;\sigma,1}(\mathbf{R}^n \times W \times \mathbf{R}^n)$.

Next, we need to introduce a class of weighted Sobolev spaces. We denote by $H^{\tau,\sigma,\mu}(\mathbf{R}^n)$, $\tau, \mu \geq 0, \sigma > 1$, the spaces of all functions $u \in \mathcal{S}(\mathbf{R}^n)$ such that

$$\|u\|_{H^{\tau,\sigma,\mu}} := \|e^{\tau \langle D \rangle^{1/\sigma}} u\|_{H^\mu} < \infty,$$

$H^\mu = H^\mu(\mathbf{R}^n)$, the usual Sobolev space.

Gevrey-Sobolev spaces of similar and even more general type, have been studied by several authors, see [10], [11], [1], [7], [2].

We recall that

$$(2.1) \quad G_A^{(\sigma)} \cap C_0^\infty \subset H^{\tau,\sigma,\mu},$$

for some positive τ and every μ , whereas

$$(2.2) \quad H^{\tau,\sigma,\mu} \subset G_A^{(\sigma)}$$

for every $\mu > n/2$, $\tau > 0$, $A \geq c_\sigma \tau^{-\sigma}$. More precisely, there exists a constant $c = c(n, \mu)$ such that

$$\|u\|_{G_A^{(\sigma)}} \leq c \|u\|_{H^{\tau,\sigma,\mu}}.$$

For $\mu > n/2$ and arbitrary τ, σ , $H^{\tau,\sigma,\mu}$ is a Banach algebra:

$$(2.3) \quad \|uv\|_{H^{\tau,\sigma,\mu}} \leq C(n, \mu) \|u\|_{H^{\tau,\sigma,\mu}} \cdot \|v\|_{H^{\tau,\sigma,\mu}}.$$

We shall use classes of functions and symbols depending also on a parameter t . Define $C_T^k(H^{\lambda,\sigma,\mu})$, $k \in \mathbf{Z}$, $\mu > 0$, $\lambda > 0$, the space of the functions $u(t, x)$ such that $e^{\lambda(2T-t)\langle D_x \rangle^{1/\sigma}} \partial_t^j u \in C(-T, T; H^{\mu-j})$, $0 \leq j \leq k$, with norm:

$$\|u\|_{C_T^k(H^{\lambda,\sigma,\mu})} := \sup_{0 \leq j \leq k} \sup_{|t| \leq T} \|\partial_t^j u(t, \cdot)\|_{H^{\lambda(2T-t), \sigma, \mu-j}}$$

and denote by $C_T^k(S_\ell^m; H^{\lambda,\sigma,\mu})$, $\ell \in \mathbf{Z}_+$, the space of symbols $a(t, x, \xi)$ of order m such that:

$$\|a\|_{C_T^k(S_{\ell'}^m; H^{\lambda, \sigma, \mu})} := \sup_{|\alpha| \leq \ell} \sup_{\xi \in \mathbf{R}^n} \frac{\|\partial_{\xi}^{\alpha} a(\cdot, \cdot, \xi)\|_{C_T^k(H^{\lambda, \sigma, \mu})}}{\langle \xi \rangle^{m-|\alpha|}} < \infty.$$

Note that for $\mu > n/2 + k$, we have $C_T^k(H^{\lambda, \sigma, \mu}) \subset C^k(-T, T; G_A^{\sigma})$, $A > c_{\sigma}(\lambda T)^{-\sigma}$, since it is $2T - t \geq T$ for $|t| \leq T$.

Similar spaces of functions and symbols have been considered by Kajitani [10] (see also [7]) who gave a detailed treatment of them.

We refer to these authors for the proofs of the following two propositions.

PROPOSITION 2.2. *Let $a(t, x, \xi)$ be in $C_T^k(S_{\ell'}^m; H^{\lambda, \sigma, \mu})$ and $\Lambda = \Lambda(t) = e^{\lambda(T-t)\langle D_x \rangle^{1/\sigma}}$. Then $a_{\pm\Lambda(t)}(t, x, \xi)$ is in $C^k(-T, T; S_{\ell'}^m)$ and satisfies:*

$$\|a_{\pm\Lambda(t)}\|_{C^k(-T, T; S_{\ell'}^m)} \leq c_{\ell'} \|a\|_{C_T^k(S_{\ell'}^m; H^{\lambda, \sigma, \mu})}$$

for $[\ell'\sigma/(\sigma - 1)] + 2\ell' + [n/2] + 2 \leq \ell$, where $c_{\ell'}$ is independent of $a(t, x, \xi)$.

Moreover $a_{\Lambda(t)}(t, x, \xi)$ can be expanded as follows:

$$a_{\Lambda(t)}(t, x, \xi) = \sum_{|\gamma| < N} (\gamma!)^{-1} \partial_x^{\gamma} a(t, x, \xi) \omega_{\gamma}(t, \xi) + R_N(a)(t, x, \xi)$$

where:

$$\omega_{\gamma}(t, \xi) = e^{-\lambda(2T-t)\langle \xi \rangle^{1/\sigma}} \partial_{\xi}^{\gamma} e^{\lambda(2T-t)\langle \xi \rangle^{1/\sigma}}$$

and $R_N(a)(t, x, \xi)$ is in $C^k(-T, T; S_{\ell'}^{m-(1-1/\sigma)N})$ for $[\ell'\sigma/(\sigma - 1)] + 2\ell' + [n/2] + N \leq \ell$.

PROPOSITION 2.3. *Let $f(t, x, w)$ be in $C^k(-T, T; G_A^{(\sigma, 1)})$, f satisfying $f(t, x, 0) = 0$, and let $u \in C_T^k(H^{\lambda, \sigma, \mu})$, $\mu \geq \mu_0(n, \sigma)$.*

If $\|u\|_{C_T^k(H^{\lambda, \sigma, \mu})} \leq r$, with $rA \leq c := c(n, k, \mu)$, $T \leq T_1$, $A^{1/\sigma} T_1 \leq c_0(\sigma, n)$, then:

$$f(t, x, u(t, x)) \in C_T^k(H^{\lambda, \sigma, \mu})$$

and

$$\|f(t, x, u(t, x))\|_{C_T^k(H^{\lambda, \sigma, \mu})} \leq c \|u\|_{C_T^k(H^{\lambda, \sigma, \mu})}.$$

3. Factorization and System Form

In this section we consider the equation:

$$(3.1) \quad \begin{cases} Pv = f \\ D_t^j v|_{t=0} = g_j \end{cases}$$

for the linear operator:

$$(3.2) \quad P = \sum_{|\alpha| \leq m} a_\alpha(t, x, D^{m'} u(t, x)) D_{t,x}^\alpha$$

assuming:

- i) $P(t, x, w, \xi)$ satisfies (R), (H), (L_σ) , (\tilde{L}_σ)
- ii) $u \in C_T^{k_0+m'}(H^{\lambda, \sigma, \mu})$ and $D^{m'} u(t, x) \in W_0$ for every $(t, x) \in [-T, T] \times \mathbf{R}^n$.

Our aim is to prove that if λ and μ are sufficiently large, T and $\|u\|_{C_T^{k_0+m'}(H^{\lambda, \sigma, \mu})}$ sufficiently small, then problem (3.1) can be reduced to an equivalent Cauchy problem for a suitable system for which we are able to state an energy estimate in Gevrey-Sobolev spaces.

We start by remarking that from (R) and the well known composition rule in Gevrey spaces, it follows that the coefficients of P are in $C^{k_0}(-T, T; G_A^\sigma)$, $A \geq \max\{A_0, c_\sigma(\lambda T)^{-\sigma}\}$.

Hence we can perform for the operator P the “good factorization” of Mizohata [17] (see also [14], where the case of C^∞ coefficients is considered), so obtaining:

PROPOSITION 3.1. *Assume that the operator (3.2) satisfies conditions i) and ii). Then we have:*

$$(3.3) \quad P = P_s \circ \dots \circ P_1 + R;$$

where:

$$(3.4) \quad P_j = (D_t - \lambda_j(t, x, D^{m'} u(t, x), D_x))^{m_j} + a_1^{(j)}(t, x, D^{M_1} u(t, x), D_x) \cdot (D_t - \lambda_j(t, x, D^{m'} u(t, x), D_x))^{m_j-1} + \dots + a_{m_j}^{(j)}(t, x, D^{M_1} u(t, x), D_x),$$

$$(3.5) \quad \lambda_j(t, x, w, \xi) \in C^{k_0}(-T, T; S^{1; \sigma, 1})$$

$$(3.6) \quad a_h^{(j)}(t, x, w, \xi) \in C^{m-1}(-T, T; S^{h/\sigma; \sigma, 1})$$

$$(3.7) \quad R = \sum_{h=1}^m r_h(t, x, D^{M_1} u(t, x), D_x) D_t^{m-h}, \quad r_h(t, x, w, \xi) \in C(-T, T; S^{h-m; \sigma, 1})$$

where $D^{M_1} u := \{\partial_{t,x}^\alpha u; \alpha = (\alpha_0, \alpha') \in \mathbf{Z}_+ \times \mathbf{Z}_+^n, \alpha_0 \leq k_0, |\alpha| \leq M_1\}$, M_1 an integer depending only on n and m .

In (3.4), the λ_j 's are the characteristic roots of P after a modification for small $|\xi|$, say $|\xi| < R$, hence they have the regularity as in (3.5) from the hypotheses (H)

and (R). The construction of the $a_h^{(j)}$ can be performed as in [14] hence they are analytic functions of some derivatives of the coefficients a_α and of u depending only on m and n .

The factorization (3.3) holds under the only hypotheses (H) with $\text{ord } a_h^{(j)} \leq h-1$; then condition (L_σ) implies $\text{ord } a_h^{(j)} \leq h/\sigma$. We remark that the order of $a_h^{(j)}$ is invariant under permutations of the indices $(1, \dots, s)$ in (3.3).

By using Proposition 3.1, we are able to reduce the equation in (3.1) to a suitable system form.

Without loss of generality, but only to have simpler notation, hereafter we consider an operator P with two characteristic roots (i.e. $s = 2$). We have:

$$(3.8) \quad \begin{aligned} P &= P_2 \circ P_1 + R \\ P &= \tilde{P}_1 \circ \tilde{P}_2 + \tilde{R} \end{aligned}$$

with P_j and \tilde{P}_j defined by (3.3) with respectively coefficients $a_h^{(j)}$ and $\tilde{a}_h^{(j)}$ of order $\leq h/\sigma$, while R and \tilde{R} are of the type (3.7).

Let us set:

$$(3.9) \quad \begin{cases} v_0 = v \\ v_1 = (D_t - \lambda_1)v \\ \vdots \\ v_{m_1-1} = (D_t - \lambda_1)^{m_1-1}v \\ v_{m_1} = P_1v \\ v_{m_1+1} = (D_t - \lambda_2)P_1v \\ \vdots \\ v_{m-1} = (D_t - \lambda_2)^{m_2-1}P_1v \end{cases} \quad \begin{cases} v_m = v \\ v_{m+1} = (D_t - \lambda_2)v \\ \vdots \\ v_{m+m_2-1} = (D_t - \lambda_2)^{m_2-1}v \\ v_{m+m_2} = \tilde{P}_2v \\ v_{m+m_2+1} = (D_t - \lambda_1)\tilde{P}_2v \\ \vdots \\ v_{2m-1} = (D_t - \lambda_1)^{m_1-1}\tilde{P}_2v. \end{cases}$$

The equation (3.1) is equivalent to:

$$(3.10) \quad \begin{cases} (D_t - \lambda_1)v_j = v_{j+1} & (0 \leq j \leq m_1 - 2) \\ (D_t - \lambda_1)v_{m_1-1} = -a_{m_1}^{(1)}v_0 - a_{m_1-1}^{(1)}v_1 - \dots - a_1^{(1)}v_{m_1-1} + v_{m_1} \\ (D_t - \lambda_2)v_j = v_{j+1} & (m_1 \leq j \leq m - 2) \\ (D_t - \lambda_2)v_{m-1} = -a_{m_2}^{(2)}v_{m_1} - a_{m_2-1}^{(2)}v_{m_1+1} - \dots - a_1^{(2)}v_{m-1} - Rv + f \\ (D_t - \lambda_2)v_{m+j} = v_{m+j+1} & (0 \leq j \leq m_2 - 2) \\ (D_t - \lambda_2)v_{m+m_2-1} = -\tilde{a}_{m_2}^{(2)}v_m - \tilde{a}_{m_2-1}^{(2)}v_{m+1} - \dots - \tilde{a}_1^{(2)}v_{m+m_2-1} + v_{m+m_2} \\ (D_t - \lambda_1)v_{m+j} = v_{m+j+1} & (m_2 \leq j \leq m - 2) \\ (D_t - \lambda_1)v_{2m-1} = -\tilde{a}_{m_1}^{(1)}v_{m+m_2} - \tilde{a}_{m_1-1}^{(1)}v_{m+m_2+1} - \dots - \tilde{a}_1^{(1)}v_{2m-1} - \tilde{R}v + f \end{cases}$$

We can handle the terms Rv and $\tilde{R}v$ by means of the following lemma which can be proved as Lemma 5 pag. 77 in [17]; in facts only finitely many products are involved, so we can argue as in proving Prop. 3.1.

LEMMA 3.2. *Assume that the operator (3.2) satisfies i), ii). There exists $M_2 = M_2(m, n) \geq M_1$, such that:*

$$(3.11) \quad \begin{aligned} D_t^j v &= \sum_{h=1}^j c_h^{(j)}(t, x, D^{M_2} u, D_x) v_{j-h} + v_j \\ D_t^j v &= \sum_{h=1}^j \tilde{c}_h^{(j)}(t, x, D^{M_2} u, D_x) v_{m+j-h} + v_{m+j}, \quad 0 \leq j \leq m-1 \end{aligned}$$

where:

$$c_h^{(j)}(t, x, w, \xi), \quad \tilde{c}_h^{(j)}(t, x, w, \xi) \in C(-T, T; S^{h; \sigma, 1}).$$

From Lemma 3.2 and (3.7) it follows that

$$(3.12) \quad \begin{aligned} Rv &= \sum_{j=0}^{m-1} b_j(t, x, D^{M_3} u, D_x) v_j \\ \tilde{R}v &= \sum_{j=0}^{m-1} \tilde{b}_j(t, x, D^{M_3} u, D_x) v_{m+j}, \end{aligned}$$

where $M_3 = M_3(m, n) \geq M_2$ and:

$$b_j(t, x, w, \xi), \quad \tilde{b}_j(t, x, w, \xi) \in C(-T, T; S^{0; \sigma, 1}).$$

If now we define:

$$(3.13) \quad \begin{aligned} \tilde{v}_j &= \langle D_x \rangle^{(m-1-j)/\sigma} v_j \\ \tilde{v}_{m+j} &= \langle D_x \rangle^{(m-1-j)/\sigma} v_{m+j}, \quad j = 0, \dots, m-1 \\ \tilde{V} &= (\tilde{v}_0, \dots, \tilde{v}_{2m-1}) \end{aligned}$$

the equations (3.10), taking into account (3.12), show that $Pv = f$ is equivalent to a hyperbolic system of size $2m \times 2m$:

$$(3.14) \quad (D_t - \Lambda(t, x, D^{m'} u(t, x), D_x) + A(t, x, D^{M_4} u(t, x), D_x)) \tilde{V} = F,$$

$M_4 = M_4(m, n) \geq M_3$, where

$$F = (0, \dots, f(t, x, D^{m'} u(t, x)), \dots, f(t, x, D^{m'} u(t, x)))^t,$$

Λ is diagonal with elements the λ_j 's each one repeated $2m_j$ times, whereas A has order $1/\sigma$; more precisely the entries of $A(t, x, w, \xi)$ are in $C(-T, T; S^{1/\sigma; \sigma, 1})$.

Under the hypotheses of Proposition 3.1, we obtain:

PROPOSITION 3.3. *Let \tilde{V} be defined by (3.13) and (3.9). Then there is $M_5 \geq M_4$, depending only on m, n, σ , such that*

$$(3.15) \quad (\partial_{t,x}^\alpha v; |\alpha| \leq m') = Q(t, x, D^{M_5} u, D_x) \tilde{V}$$

with Q a $d_{m'} \times 2m$ matrix, $Q(t, x, w, \xi) \in C(-T, T; S^{m'-m+r(1-1/\sigma)+1/\sigma; \sigma, 1})$.

The above proposition is similar to Proposition 2.6 in [3] where we considered the C^∞ case (formally, $\sigma = \infty$, $1/\sigma = 0$). The same proof holds, taking now into account the positive order h/σ of $a_h^{(j)}$ in (3.4) and the assumption (R) of Gevrey analytic regularity.

For a fixed integer $M \geq k_0$, let us consider the derivatives $\partial_{t,x}^\alpha v = \partial_t^{\alpha_0} \partial_x^{\alpha'} v$, $\alpha_0 \leq k_0$, $|\alpha| \leq M$, and let us denote by $\tilde{v}_i^{(\alpha)}$ the functions defined by (3.13) and (3.9) with $\partial_{t,x}^\alpha v$ in place of v . Then set

$$(3.16) \quad \tilde{\tilde{V}} = (\tilde{v}_i^{(\alpha)}; |\alpha| \leq M, \alpha_0 \leq k_0, 0 \leq j \leq 2m - 1).$$

The following proposition, which gives a representation of the commutators $[\partial_{t,x}^\alpha, P]v$, can be proved as Proposition 2.8 in [3]:

PROPOSITION 3.4. *Let $\tilde{\tilde{V}}$ be defined by (3.16). Then there exists $M_6 \geq M_5$ depending only on m, n, σ and not depending on M , such that:*

$$(3.17) \quad ([\partial_{t,x}^\alpha, P]v; |\alpha| \leq M, \alpha_0 \leq k_0) = B(t, x, D^{M+m'} u) H(t, x, D^{M_6} u, D_x) \tilde{\tilde{V}}$$

where B, H are $d_M \times d_M$, $d_M \times 2md_M$ matrices respectively,

$$B(t, x, w) \in C(-T, T; G_A^{\sigma, 1}),$$

$$H(t, x, w, \xi) \in C(-T, T; S^{1/\sigma; \sigma, 1})$$

and

$$D^{M+m'} u = \{\partial_t^j \partial_x^\beta u; j \leq k_0 + m', j + |\beta| \leq M + m'\}.$$

Since M_6 does not depend on M , now we can fix

$$(3.18) \quad M > M_6 + \frac{3}{2}n + 6.$$

In next section we shall use the fact that, under the hypotheses of Theorem 1.1, from Proposition 3.1, Lemma 3.2 and Proposition 3.4, the equation $Pv = f$ is equivalent to a first order system $L\tilde{V} = \tilde{F}$ with L of the form:

$$(3.19) \quad L = \partial_t - iK(t, x, D^{M_6}u, D_x) + B(t, x, D^{M+m'}u)H(t, x, D^{M_6}u, D_x)$$

where:

- K is a $2md_M \times 2md_M$ matrix, $K(t, x, w, \xi) \in C(-T, T; S_A^{1; \sigma, 1})$;
- $K(t, x, w, \xi) - K^*(t, x, w, \xi) \in C(-T, T; S_A^{1/\sigma; \sigma, 1})$;
- B is a $d_M \times d_M$ matrix, $B(t, x, w) \in C(-T, T; G_A^{\sigma, 1})$;
- H is a $d_M \times 2Md_M$ matrix, $H(t, x, w, \xi) \in C(-T, T; S_A^{1/\sigma; \sigma, 1})$.

PROPOSITION 3.5. *Let L be given by (3.19) with M satisfying (3.18) and $u \in C_T^{k_0+m'}(H^{\lambda, \sigma, \mu})$, $\mu = M + s + m'$, $s = n/2 + 1$. There are $r, h, \ell > 0$ such that if $\lambda \geq h$, $2\lambda T \leq \ell$ and*

$$(3.20) \quad \|u\|_{C_T^{k_0+m'}(H^{\lambda, \sigma, \mu})} \leq r$$

then:

$$(3.21) \quad \|V(t)\|_{H^{\lambda(2T-t), \sigma, s}} \leq \|V(0)\|_{H^{2\lambda T, \sigma, s}} + 2 \left| \int_0^t \|LV(\tau)\|_{H^{\lambda(2T-\tau), \sigma, s}} d\tau \right|, \quad |t| \leq T$$

for every $V \in C_T^1(H^{\lambda, \sigma, s+1})$.

The numbers r, ℓ depend only on A, n, m, σ , whereas h depends only on $K(t, x, w, \xi)$, $B(t, x, w)$, $H(t, x, w, \xi)$, n, σ .

PROOF. From Proposition 2.3 it follows that if $T \leq T_1$, $T_1 = T_1(A, \sigma, n)$, and

$$\|u\|_{C_T^{\alpha_0}(H^{\lambda, \sigma, \mu})} \leq r, \quad r = r(A, \sigma, m, n)$$

then

$$K_1(t, x, \xi) := K(t, x, D^{M_6}u(t, x), \xi) - K(t, x, 0, \xi) \in C_T(S^1; H^{\lambda, \sigma, s}),$$

$$B_1(t, x) := B(t, x, D^{M+m'}u(t, x)) - B(t, x, 0) \in C_T(H^{\lambda, \sigma, s}),$$

$$H_1(t, x, \xi) := H(t, x, D^{M_6}u(t, x), \xi) - H(t, x, 0, \xi) \in C_T(S^{1/\sigma}; H^{\lambda, \sigma, s}),$$

whereas

$$K_2(t, x, \xi) := K(t, x, 0, \xi) \in C(-T, T; S_A^{1; \sigma}),$$

$$B_2(t, x) := B(t, x, 0) \in C(-T, T; G_A^{\sigma}),$$

$$H_2(t, x, \xi) := H(t, x, 0, \xi) \in C(-T, T; S_A^{1/\sigma, \sigma}).$$

We can apply Proposition 2.2 to K_1 and H_1 , so obtaining:

$$e^{\pm\lambda(2T-t)\langle D_x \rangle^{1/\sigma}} K_1 e^{\mp\lambda(2T-t)\langle D_x \rangle^{1/\sigma}} = K_1 + R_1^\pm$$

with $R_1^\pm(t, x, \xi) = R_1^\pm(t, x, D^{M_6}u(t, x), \xi) \in C(-T, T; S^{1/\sigma})$ and

$$e^{\pm\lambda(2T-t)\langle D_x \rangle^{1/\sigma}} H_1 e^{\mp\lambda(2T-t)\langle D_x \rangle^{1/\sigma}} = H_1 + N_1^\pm,$$

with $N_1^\pm(t, x, \xi) = N_1^\pm(t, x, D^{M_6}u(t, x), \xi) \in C(-T, T; S^{-1+2/\sigma})$.

On the other side, under the hypothesis $2\lambda T \leq (24^\sigma n A)^{-1/\sigma}$ we can apply Proposition 2.1 to K_2 and H_2 to have:

$$e^{\pm\lambda(2T-t)\langle D_x \rangle^{1/\sigma}} K_2 e^{\mp\lambda(2T-t)\langle D_x \rangle^{1/\sigma}} = K_2 + R_2^\pm,$$

with $R_2^\pm(t, x, \xi) \in C(-T, T; S^{1/\sigma})$ and

$$e^{\pm\lambda(2T-t)\langle D_x \rangle^{1/\sigma}} H_2 e^{\mp\lambda(2T-t)\langle D_x \rangle^{1/\sigma}} = H_2 + N_2^\pm,$$

with $N_2^\pm(t, x, \xi) \in C(-T, T; S^{-1+2/\sigma})$.

Now we are ready to estimate $d/dt \|V(t)\|_{H^{\lambda(2T-t), \sigma, s}}^2$,

$$\|V(t)\|_{H^{\lambda(2T-t), \sigma, s}} = \|e^{\Lambda(t)} V(t)\|_{H^s}, \quad \Lambda(t) = \lambda(2T-t)\langle D_x \rangle^{1/\sigma}.$$

We have:

$$\begin{aligned} (3.22) \quad \frac{d}{dt} \|V(t)\|_{H^{\lambda(2T-t), \sigma, s}}^2 &= 2 \operatorname{Re}(e^{\Lambda(t)}(V'(t) - \lambda\langle D_x \rangle^{1/\sigma} V(t)), e^{\Lambda(t)} V(t))_{H^s} \\ &= -2\lambda \|V(t)\|_{H^{\lambda(2T-t), \sigma, s+1/2\sigma}}^2 + 2 \operatorname{Re}(LV(t), V(t))_{H^{\lambda(2T-t), \sigma, s}} \\ &\quad + 2 \operatorname{Re}(i(K(t) - B(t)H(t))V(t), V(t))_{H^{\lambda(2T-t), \sigma, s}}. \end{aligned}$$

Let us denote by $S(t) := K(t) - B_2(t)H(t)$ the part of $K(t) - B(t)H(t)$ which contains derivatives of u of order not exceeding M_6 . $S - S^*$ has order $1/\sigma$ and for $S_\Lambda = e^\Lambda S e^{-\Lambda}$ one has $S_\Lambda = S + R$, $R(t, x, \xi) = R(t, x, D^{M_6}u(t, x), \xi) \in C(-T, T; S^{1/\sigma})$.

Thus, with $E := i\langle D_x \rangle^{-1/\sigma}(S - S^* + R - R^*)$, $E(t, x, \xi) = E(t, x, D^{M_6}u(t, x), \xi) \in C(-T, T; S^0)$, at a fixed t we have:

$$2 \operatorname{Re}(iSV, V)_{H^{\lambda(2T-t), \sigma, s}} = (i(S_\Lambda - S_\Lambda^*)e^\Lambda V, e^\Lambda V)_{H^s} = (Ee^\Lambda V, e^\Lambda V)_{H^{s+1/2\sigma}}.$$

Hence, taking Proposition 2.3 into account, the usual continuity of pseudo-differential operators in Sobolev spaces gives:

$$\begin{aligned}
 |2 \operatorname{Re}(iS(t)V(t), V(t))_{H^{\lambda(2T-t), \sigma, s}}| &\leq 2C(n) \sum_{|\alpha|, |\beta| \leq (3/2)n+6} \sup |\partial_\xi^\alpha \partial_x^\beta E(t, x, \xi)| \\
 &\cdot \|V(t)\|_{H^{\lambda(2T-t), \sigma, s+1/2\sigma}}^2 \\
 &\leq h(\|u\|_{C_T^{k_0+m'}(H^{\lambda, \sigma, \mu})} + 1) \|V(t)\|_{H^{\lambda(2T-t), \sigma, s+1/2\sigma}}^2
 \end{aligned}$$

taking $M_6 + (3/2)n + 6 + s < \mu$ into account. The constant h depends only on $K(t, x, w, \xi)$, $B(t, x, w)$, $H(t, x, w, \xi)$, n and σ .

Next, to estimate $-2 \operatorname{Re}(iB_1(t)H(t)V(t), V(t))_{H^{\lambda(2T-t), \sigma, s}}$ in (3.22), we use also the Banach algebra property of $H^{\lambda(2T-t), \sigma, s-1/2\sigma}$:

$$\begin{aligned}
 &|2 \operatorname{Re}(B_1(t)H(t)V(t), V(t))_{H^{\lambda(2T-t), \sigma, s}}| \\
 &\leq 2 \|B_1(t)H(t)V(t)\|_{H^{\lambda(2T-t), \sigma, s-1/2\sigma}} \\
 &\quad \cdot \|V(t)\|_{H^{\lambda(2T-t), \sigma, s+1/2\sigma}} \leq 2C(n) \|B_1(t, \cdot, D^{M+m'}u(t, \cdot))\|_{H^{\lambda(2T-t), \sigma, s-1/2\sigma}} \\
 &\quad \cdot \|H(t)V(t)\|_{H^{\lambda(2T-t), \sigma, s-1/2\sigma}} \cdot \|V(t)\|_{H^{\lambda(2T-t), \sigma, s+1/2\sigma}} \\
 &\leq 2C(n, B_1) \|u\|_{C_T^{k_0+m'}(H^{\lambda, \sigma, \mu})} \sum_{|\alpha|, |\beta| \leq (3/2)n+6} \sup |\partial_\xi^\alpha \partial_x^\beta H_\Lambda(t, x, D^{M_6}u(t, x), \xi)| \\
 &\quad \cdot \|V(t)\|_{H^{\lambda(2T-t), \sigma, s+1/2\sigma}}^2 \leq h(\|u\|_{C_T^{k_0+m'}(H^{\lambda, \sigma, \mu})} + 1) \|V(t)\|_{H^{\lambda(2T-t), \sigma, s+1/2\sigma}}^2
 \end{aligned}$$

with a larger h if necessary but still depending only on $K(t, x, w, \xi)$, $B(t, x, w)$, $H(t, x, w, \xi)$, n and σ .

In (3.20) we can assume $r < 1$ without losing generality, so we have

$$2 \operatorname{Re}(i(K(t) - B(t)H(t))V(t), V(t))_{H^{\lambda(2T-t), \sigma, s}}| \leq 2h \|V(t)\|_{H^{\lambda(2T-t), \sigma, s+1/2\sigma}}^2.$$

Now if we choose in (3.22) $\lambda > h$, we obtain:

$$\frac{d}{dt} \|V(t)\|_{H^{\lambda(2T-t), \sigma, s}}^2 \leq 2 \operatorname{Re}(LV(t), V(t))_{H^{\lambda(2T-t), \sigma, s}}$$

hence (3.21) is proved.

4. Proof of Theorem 1.1.

Let us consider the Cauchy problem (1.1)

$$P(t, x, D^{m'}u; D_t, D_x)u = f(t, x, D^{m'}u)$$

$$D_t^j u|_{t=0} = g_j, \quad 0 \leq j < m$$

$P(t, x, D^{m'}u; D_t, D_x) = \sum_{|\alpha| \leq m} a_\alpha(t, x, D^{m'}u) D_{t,x}^\alpha$, under the hypotheses of Theorem 1.1.

Taking $M \in \mathbf{Z}_+$ as in (3.18), if u is a solution then the derivatives $u^{(\alpha)} := \partial_t^{\alpha_0} \partial_x^{\alpha'} u$, $\alpha = (\alpha_0, \alpha')$, $\alpha_0 \leq k_0$, $|\alpha| \leq M$, satisfy:

$$(4.1) \quad Pu^{(\alpha)} + [\partial_{t,x}^\alpha, P]u = \partial_{t,x}^\alpha f(t, x, D^{m'}u).$$

Now, defining \tilde{U} as \tilde{V} in (3.16) with u in place of v , from (3.14) and Proposition 3.4 we can write equations (4.1) in the system form:

$$(4.2) \quad L\tilde{U} = F(t, x, D^{M+m'}u)$$

with $L = \partial_t - iK(t, x, D^{M_6}u, D_x) + B(t, x, D^{M+m'}u)H(t, x, D^{M_6}u, D_x)$ as in (3.19).

From Proposition 3.3, we have

$$(4.3) \quad D^{M+m'}u = Q(t, x, D^{M_6}u, D_x)\tilde{U}$$

whereas $D^{M_6}u$ is a subset of \tilde{U} , so the Cauchy problem (1.1) for u is equivalent to the first order problem

$$(4.4) \quad \begin{aligned} L(t, x, \tilde{U}, Q\tilde{U}; D_t, D_x)\tilde{U} &= F(t, x, Q\tilde{U}) \\ \tilde{U}|_{t=0} &= G \end{aligned}$$

for the vector \tilde{U} . It is not restrictive to assume $G = 0$ and $F(t, x, 0)$ of compact support.

In proving the well posedness of problem (4.4), we start by considering the case

$$m' < m - r(1 - 1/\sigma) - 1/\sigma$$

that means Q of negative order, say $-\delta$. So, for $\tilde{U} \in C_T(H^{\lambda, \sigma, s})$, $s = n/2 + 1$, $\|\tilde{U}\|_{C_T(H^{\lambda, \sigma, s})} \leq R$, R and T sufficiently small, we have $Q\tilde{U}, F(t, x, Q\tilde{U}) \in C_T(H^{\lambda, \sigma, s+\delta})$ and, moreover, u satisfies (3.20) in view of (4.3). Thus we can apply Proposition 3.5 to the linear problem for \tilde{V}

$$\begin{aligned} L(t, x, \tilde{U}, Q\tilde{U}, D_t, D_x)\tilde{V} &= F(t, x, Q\tilde{U}) \\ \tilde{V}|_{t=0} &= 0 \end{aligned}$$

obtaining a unique solution $\tilde{V} \in C_T^1(H^{\lambda, \sigma, s+\delta})$ which satisfies inequality (3.21) with $s + \delta$ in place of s . In this way, taking a smaller T if necessary, we have a compact map

$$\tilde{U} \rightarrow \tilde{V}$$

from the ball $\{\tilde{U} \in C_T(H^{\lambda, \sigma, s}); \|\tilde{U}\|_{C_T(H^{\lambda, \sigma, s})} \leq R\}$ to itself and the fixed point Schauder theorem allows us to conclude.

To prove Theorem 1.1 in the general case, i.e. with

$$(\tilde{L}_\sigma) \quad m' \leq m - r(1 - 1/\sigma)$$

we can make a linearization in (4.4) taking derivatives ∂_{x_j} , $j = 1, \dots, n$, so obtaining a system for the vector $\hat{U} := (\tilde{U}, \nabla_x \tilde{U})$:

$$(4.5) \quad \begin{aligned} \hat{L}\hat{U} &= \hat{F}(t, x, \hat{Q}\hat{U}) \\ \hat{U}|_{t=0} &= 0 \end{aligned}$$

with $\hat{L} = \partial_t - i\hat{K} - \hat{B}\hat{H}$, \hat{H} of order $m' - m + r(1 - 1/\sigma) + 1/\sigma$, \hat{Q} of order $m' - m + r(1 - 1/\sigma) + 1/\sigma - 1$. From (\tilde{L}_σ) we have \hat{H} of order $1/\sigma$ and \hat{Q} of order $1/\sigma - 1 < 0$ and we can argue as above.

Note that the order of \hat{H} corresponds to the Levi condition for \hat{L} in G^σ : it is well known that this order can not exceed $1/\sigma$ and this leads to the upper bound $m - r(1 - 1/\sigma)$ for m' . (\tilde{L}_σ) is the appropriate nonlinear Levi condition in G^σ .

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Università di Bologna, Dipartimento di Matematica
Piazza di Porta San Donato 5, Bologna, Italy
(e-mail: cicognan@dm.unibo.it)

Università di Ferrara, Dipartimento di Matematica
Via Machiavelli 35, 44100 Ferrara, Italy
(e-mail: zan@dns.unife.it)