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ON THE SMOOTHING PROBLEM

By

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In this paper we sharpen the result established in [11], see also [12], to obtain a criterion for the smoothability of any topological n-manifold (not neccessarily compact). Our criterion is the existence of a Lipschitz atlas with Lipschitz size sufficiently close to 1 (definitions below). We do this by refining proposition 1 in [11] to get a smoothing theorem (theorem 1.10) by making all computations in a suitable tubular neighbourhood of a given compact subset of a smooth manifold in some Euclidean space in which the manifold is properly and smoothly embedded. This smoothing theorem will allow an inductive construction of a smooth atlas on any given topological n-manifold that satisfy the smoothability criterion. Our final result is given in theorem 2.5.

Throughout this paper we shall adopt the following notations, conventions and definitions.

$\boldsymbol{R}^N$ is the N-dimensional real vector space consisting of N-tuples of real numbers, $\langle\cdot,\cdot\rangle$ is the canonical scalar product on $\boldsymbol{R}^N$ given by $\langle x, y\rangle = \sum_{i=1}^N x_i y_i$ and $\|\cdot\|$ is the corresponding norm [2, p. 118]. $\{e_i : 1 \leq i \leq N\}$ is the canonical basis of $\boldsymbol{R}^N$ where $(e_i)_j = \delta_{ij}$, $1 \leq i, j \leq N$. If $n < N$ we identify $\boldsymbol{R}^n$ as $\boldsymbol{R}^n \times \{0\} \subset \boldsymbol{R}^N$ and $\boldsymbol{R}^{N-n}$ as $\{0\} \times \boldsymbol{R}^{N-n} \subset \boldsymbol{R}^N$ and we let $\|\cdot\|_n$ and $\|\cdot\|_{N-n}$ be the corresponding norms on $\boldsymbol{R}^N$ and $\boldsymbol{R}^{N-n}$ respectively so that $\boldsymbol{R}^N$ is identified as $\boldsymbol{R}^n \times \boldsymbol{R}^{N-n}$. Let $p_1 : \boldsymbol{R}^N \to \boldsymbol{R}^n$ and $p_2 : \boldsymbol{R}^N \to \boldsymbol{R}^{N-n}$ be the projection maps. For all $x \in \boldsymbol{R}^N$ and $r > 0$ we let $B_r^N(x) = \{y \in \boldsymbol{R}^N : \|x - y\| < r\}$ and $B^N = B_1^N(0)$. For $\{a, b\} \subset \boldsymbol{R}^N - \{0\}$ we define $A(a, b) =$ the angle between a and b by $A(a, b) = \arccos(\langle a, b\rangle/(\|a\|\|b\|))$, $0 \leq A(a, b) \leq \pi$ and if $a \in \boldsymbol{R}^N - \{0\}$ and if L is a non-trivial vector subspace of $\boldsymbol{R}^N$ we define $A(a, L) =$ the angle between a and L by $A(a, L) = A(a, P_L(a))$ where $P_L(a)$ is the orthogonal projection of a on L [2, p. 121] (note that since $\langle a - P_L(a), P_L(a)\rangle = 0$ we have $A(a, L) = \arccos(\|P_L(a)\|/\|a\|)$.

If (X_i, d_i), $i = 1, 2$, are metric spaces, then $f : X_1 \to X_2$ is bilipschitz (respectively locally bilipschitz) if there exists some $1 \leq L < \infty$ such that: (*) For all

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x_1, $y_1 \in X_1$ (respectively for all $x \in X_1$ there exists some neighbourhood $V(x)$ of x in X_1 such that for all x_1, $y_1 \in V(x)$) we have $1/Ld_1(x_1, y_1) \le d_2(f(x_1), f(y_1)) \le Ld_1(x_1, y_1)$, and we define the Lipschitz size $L(f) = L(f, d_1, d_2)$ of f (respectively the local Lipschitz size $L_c(f) = L_c(f, d_1, d_2)$ of f) by: $L(f)$ (respectively $L_c(f)) = \inf\{1 \le L < \infty : f$ satisfies condition $(^*)\}$. Note that for two consectutive bilipschitz (respectively locally bilipschitz) maps $X \xrightarrow{f} Y \xrightarrow{g} Z$, the composite map is bilipschitz (respectively locally bilipschitz) and we have $L(g \circ f) \le L(g) \cdot L(f)$ (respectively $L_c(g \circ f) \le L_c(g) \cdot L_c(f)$).

A topological n-manifold is a separable metric space such that every point of which has an open neighbourhood homeomorphic to an open set in $\boldsymbol{R}^n$. Let X be a topological n-manifold. A chart of X is a couple (X_i, φ_i) where X_i open $\subset X$ and φ_i a homeomorphism of X_i onto an open set in $\boldsymbol{R}^n$. An atlas $\mathscr{A}$ of X is a family of charts $\mathscr{A} = \{((X_i, \varphi_i) : i \ge 1\}$ such that $X = \bigcup_{i \ge 1} X_i$. It is a Lipschitz (respectively locally Lipschitz) atlas if for all $i, j \ge 1$ such that $X_i \cap X_j \ne \varnothing$, the map $\varphi_j \circ \varphi_i^{-1} : \varphi_i(X_i \cap X_j) \to \varphi_j(X_i \cap X_j)$ is bilipschitz (respectively locally bilipschitz) and we define its Lipschitz (respectively locally Lipschitz) size $L(\mathscr{A})$ (respectively $L_c(\mathscr{A})$) by:

$$L(\mathscr{A})(\text{respectively } L_c(\mathscr{A})) = \sup\{L(\varphi_j \circ \varphi_i^{-1})(\text{respectively } L_c(\varphi_j \circ \varphi_i^{-1})) : i, j \ge 1, X_i \cap X_j \ne \varnothing\}.$$

A smooth ($= C^\infty$) structure on a topological manifold X is an atlas $\mathscr{A} = \{(X_i, \varphi_i) : i \ge 1\}$ such that for all $i, j \ge 1$ such that $X_i \cap X_j \ne \varnothing$ the transition homeomorphisms $\varphi_j \circ \varphi_i^{-1} : \varphi_i(X_i \cap X_j) \to \varphi_j(X_i \cap X_j)$ are C^∞ diffeomorphisms. $\mathscr{A}$ is then a C^∞ atlas of X and X is a C^∞ manifold. If X is a C^∞ n-manifold, a C^∞ chart of X is a couple (X_i, φ_i) where X_i open $\subset X$ and φ_i, a C^∞ diffeomorphism of X_i onto an open set in $\boldsymbol{R}^n$. If $a \in X$ we define $T_a(X)$, the tangent space of X at a whose elements are the tangent vectors at a, as the quotient set of the set $\{(c, \boldsymbol{h}) : c = (Y, \varphi) C^\infty \text{ chart of } X, a \in Y, \boldsymbol{h} \in \boldsymbol{R}^n\}$ under the equivalence relation $(c_1, \boldsymbol{h}_1) R (c_2, \boldsymbol{h}_2)$ iff $D(\varphi_2 \circ \varphi_1^{-1}) \cdot (\varphi_1(a)) \cdot \boldsymbol{h}_1 = \boldsymbol{h}_2$ and for any C^∞ chart $c = (Y, \varphi)$, $a \in Y$, we define θ_c (also denoted by $\theta_{c,a}$) as the bijection $\theta_c : \boldsymbol{R}^n \to T_a(X)$ given by $\theta_c(\boldsymbol{h}) =$ the tangent vector represented by $(c, \boldsymbol{h})$ [1, p. 41]. We let $T(X) = \bigcup_{a \in X} T_a(X)$ be the tangent bundle of X and $o_X : T(X) \to X$ be the canonical map defined by $o_X(T_a(X)) = a$. If X is a C^∞ Riemannian n-manifold we let exp be the exponential map defined by the goedesic field of its Levi-Civita connection, and we let $\Omega \subset T_a(X)$ be its domain of definition and $\exp_a$ be its restriction to $\Omega \cap T_a(X)$.

Finally we recall the following patching argument for construction of topological spaces [3, p. 4].

THEOREM AND DEFINITION. *Let $\{A_\alpha : \alpha \in I\}$ be a family of topological spaces. Suppose that for all $\alpha, \beta \in I$ we are given $A_{\beta\alpha}$ open $\subset A_\alpha$ and $h_{\beta\alpha} : A_{\beta\alpha} \to A_{\alpha\beta}$ a homeomorphism onto such that:*

1) $A_{\alpha\alpha} = A_\alpha, h_{\alpha\alpha} = 1$ *for all* $\alpha \in I$.

2) *for all $\alpha, \beta, \gamma \in I$ we have a commutative diagram of homeomorphisms*

$$\begin{tikzcd}[column sep=small] A_{\gamma\alpha} \cap A_{\beta\alpha} \arrow[rr, "h^{\beta}_{\gamma\alpha}"] \arrow[dr, "h^{\gamma}_{\beta\alpha}"'] & & A_{\alpha\gamma} \cap A_{\beta\gamma} \\ & A_{\alpha\beta} \cap A_{\gamma\beta} \arrow[ur, "h^{\alpha}_{\gamma\beta}"'] & \end{tikzcd}$$

where for all $i, j, k \in I$, $h^k_{ji} = h_{ji} | A_{ji} \cap A_{ki}$, then there exists a topological space, unique up to homeomorphism, satisfying:

1. *for each $\alpha \in I$, there exists a continuous map $p_a : A_a \to A$ such that for all $\alpha, \beta \in I$, $p_\alpha | A_{\beta\alpha} = p_\beta \circ h_{\beta\alpha}$ and $\bigcup_{\alpha \in I} p_\alpha(A_\alpha) = A$.*

2. *for any topological space A' and any family of continuous maps $p'_\alpha : A_\alpha \to A'$, $\alpha \in I$ such that $p'_\alpha | A_{\beta\alpha} = p'_\beta \circ h_{\beta\alpha}$ there exists a unique continuous map $\varphi : A \to A'$ such that $\varphi \circ p_\alpha = p'_\alpha$ for all $\alpha \in I$.*

We denote this topological space by $A = (\sum_{\alpha \in I} A_\alpha) \operatorname{mod}(\{A_{\beta\alpha}\}, \{h_{\beta\alpha}\})$. The corresponding p_α's are then open embeddings. Note that if A_α is an open set in $\boldsymbol{R}^n$ for all $\alpha \in I$, then A is a topological n-manifold and if, in addition, $h_{\beta\alpha}$ is a C^∞ diffeomorphism for all $\alpha, \beta \in I$ then A is a C^∞ n-manifold.

Here we make two remarks about this theorem.

REMARK A. Suppose A'_α open $\subset A_\alpha$ for all $\alpha \in I$ and

$$A'_{\beta\alpha} = A_{\beta\alpha} \cap A'_a, \quad h'_{\beta\alpha} = h_{\beta\alpha} | A'_{\beta\alpha}, \quad h'_{\beta\alpha}(A'_{\beta\alpha}) = A'_{\alpha\beta} \quad \text{for all } \alpha, \beta \in I,$$

then:

1. $A'_{\alpha\alpha} \doteq A'_\alpha$, $h'_{\alpha\alpha} = 1$ for all $\alpha \in I$.
2. for all $\alpha, \beta, \gamma \in I$ we have a commutative diagram of honeomorphisms

$$\begin{tikzcd}[column sep=small] A'_{\beta\alpha} \cap A'_{\gamma\alpha} \arrow[rr, "h'^{\beta}_{\alpha\gamma}"] \arrow[dr, "h'^{\gamma}_{\beta\alpha}"'] & & A'_{\alpha\gamma} \cap A'_{\beta\gamma} \\ & A'_{\alpha\beta} \cap A'_{\gamma\beta} \arrow[ur, "h'^{\alpha}_{\gamma\beta}"'] & \end{tikzcd}$$

and the canonical map $\varphi : (\sum_{\alpha \in I} A'_\alpha) \operatorname{mod}(\{A'_{\beta\alpha}\}, \{h'_{\beta\alpha}\}) \to (\sum_{\alpha \in I} A_\alpha) \cdot \operatorname{mod}(\{A_{\beta\alpha}\}, \{h_{\beta\alpha}\})$ induced by the injections $A'_\alpha \to A_\alpha$, $\alpha \in I$, is an open embedding.

REMARK B. Suppose $I = \bigcup_{j \in J} I_j$ is a partition of I and for all $j \in J$ let $A_j = (\sum_{\alpha_j \in I_j} A_{\alpha_j}) \bmod(\{A_{\beta_j\alpha_j}\}, \{h_{\beta_j\alpha_j}\})$ and $p^j_{\alpha_j} : A_{\alpha_j} \to A_j$ the corresponding open embeddings and let $A' = (\sum_{j \in J} A_j) \bmod(\{p^j_{\alpha_j}(A_{\alpha_k\alpha_j}) : j, k \in J, j \neq k\}, \{p^k_{\alpha_k} \circ h_{\alpha_k\alpha_j} \circ (p^j_{\alpha_j})^{-1} : j, k \in J, j \neq k\})$ and $p_j : A_j \to A'$ the corresponding open embeddings, then the canonical map $\varphi : A \to A'$ induced by the open embeddings $p_j \circ p^j_{\alpha_j} : A_{\alpha_j} \to A'$, $\alpha_j \in I_j$ and $j \in J$, is a homeomorphism by virtue of which we identify these two spaces.

The paper is divided into two sections. In section 1 we establish the smoothing theorem (theorem 1.10). In section 2 we prove our smoothing criterion (theorem 2.5).

1. We shall need the following sequence of Lemmas to establish the smoothing theorem.

LEMMA 1.1. *There exists $\varphi : \boldsymbol{R} \to \boldsymbol{R}$, a C^∞ function such that*:
1. $0 \leq \varphi \leq 1$, $-10 \leq \varphi' \leq 0$
2. $\varphi^{-1}(0) = \{x \in \boldsymbol{R} : |x| \geq 1\}$
3. $\varphi^{-1}(1) = \{x \in \boldsymbol{R} : |x| \leq 1/2\}$.

PROOF. Let $h : \boldsymbol{R} \to \boldsymbol{R}$ be defined by

$$h(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

and define $\varphi(x) = h(1 - x^2)/(h(1 - x^2) + h(x^2 - 1/4))$ then one can easily show that φ has all the stated properties. //.

LEMMA 1.2 [11, Lemma 1]. *Let V open $\subset \overline{V} \subset V_0$ open $\subset \overline{V_0} \subset V_1$ open $\subset \overline{V_1} \subset V_2$ open $\subset \overline{V_2} \subset U$ open $\subset B^n$, then there exists $t : \boldsymbol{R}^n \to \boldsymbol{R}$, C^∞ function such that*:
1. $0 \leq t \leq 1$.
2. $t|_{R^n - V_2} \equiv 0$ *and* $t|_V \equiv 1$.
3. *If* $\delta = \min(1, d(\overline{V}, V_0^c), d(\overline{V_0}, V_1^c), d(\overline{V_1}, V_2^c), d(\overline{V_2}, U^c))$ *then* $\delta t(y) \leq d(y, t^{-1}(0))$.
4. $|t'(y) \cdot z| \leq \dfrac{10 \cdot 2^n}{\delta} \|z\|$.

PROOF. Let $u(x) = \min(1, d(x, V_1^c)/d(\overline{V_0}, V_1))$ and let φ be the function defined in Lemma 1.1 and define

$$t(y) = \frac{1}{a_n \delta^n} \int \varphi\left(\frac{\|x - y\|}{\delta}\right) u(x) \, d\lambda(x)$$

where $a_n = \int \varphi(\|x\|) \, d\lambda(x)$ and λ is the Lebesgue measure on $\boldsymbol{R}^n$.

Note that t is C^∞ on $\boldsymbol{R}^n$ [3, p. 125] and

$$|t'(y)\cdot z| = \left|\frac{1}{a_n\delta^n}\int \varphi'\left(\frac{\|x-y\|}{\delta}\right)\frac{(y-x)\cdot z}{\delta\|y-x\|}u(x)\,d\lambda(x)\right|$$
$$\le \frac{\|z\|}{\delta a_n}\left|\int \varphi'\left(\frac{\|x-y\|}{\delta}\right)\frac{1}{\delta^n}\,d\lambda(x)\right|$$
$$= \frac{\|z\|}{\delta a_n}\left|\int \varphi'(\|x\|)\,d\lambda(x)\right| \quad \text{by [4, p. 155]}$$
$$\le \frac{10\|z\|}{\delta a_n}\lambda(B^n).$$

Observe that $a_n \ge \lambda(1/2B^n) = 1/2^n\lambda(B^n)$ by [3, p. 247] so that $|t'(y)\cdot z| \le 10\cdot 2^n\|z\|/\delta$.

All the other stated properties of t follow easily from the definition as shown in [11, lemma 1]. //.

LEMMA 1.3. *Let Y be a C^∞ n-submanifold of $\boldsymbol{R}^N$ and let $j : Y \to \boldsymbol{R}^N$ be the canonical injection and Let K compact $\subset Y$, then for all $\varepsilon > 0$ there exists coverings of Y by open N-balls $\{w_j : j \ge 1\}$ and $\{w_j^0 : j \ge 1\}$ satisfying*:

1- *For all $j \ge 1$, $w_j^0 = a_j + \varepsilon_j B^N$, $w_j = a_j + 1/2\varepsilon_j B^N$, $2\varepsilon_j < \varepsilon$, $a_j \in Y$ such that $w_j^0 \cap Y = \{\zeta \in \varepsilon_j B^N : \zeta_{n+k} = f_k^{(j)}(\zeta_1,\ldots,\zeta_n), 1 \le k \le N-n\}$ modulo an affine transformation of $\boldsymbol{R}^N$, where $f_k^{(j)} : \varepsilon_j B^n \to \boldsymbol{R}$ are C^∞ functions and $f_k^{(j)}(0) = 0$, $f_k^{(j)'}(0) = 0$ for $1 \le k \le N-n$; $\sup\{|D_i f_k^{(j)}(\zeta)| : 1 \le i \le n, 1 \le k \le N-n, \|\zeta\| < \varepsilon_j\} < \varepsilon$, and $\sup\{\varepsilon_j|D_pD_q f_k^{(j)}(\zeta)| : 1 \le p,q \le n, 1 \le k \le N-n, \|\zeta\| < \varepsilon_j\} < \varepsilon$. Also $K \subset \bigcup_{j=1}^{s} w_j$, $K \cap w_j^0 = \varnothing$ for all $j > s$.*

2- *For all $j \ge 1$ there exists T_j open $\subset w_j$ such $T_j \cap Y = w_j \cap Y$ and, up to the affine congruence of* 1., *we have a C^∞ diffeomorphism*

$$(w_j \cap Y) \times r_j B^{N-n} \to T_j$$

$$(x,t) \mapsto x + \sum_{k=1}^{N-n} t_k\left(e_{n+k} - \sum_{L=1}^{n} D_i f_k^{(j)}(p_1(x))e_i\right)$$

whose inverse is $z \to (\pi(z), \theta(z))$ such that:

i- *$\pi(z)$ is the unique point of Y satisfying $\|\pi(z) - z\| = d(z, Y)$.*

ii- *For all $z \in T_j$, $\pi^{-1}(\pi(z)) = T_j \cap (\pi(z) + N_{\pi(z)}(Y))$ where*

$$N_{\pi(z)}(Y) = \theta_{id}^{-1} \circ T_{\pi(z)}(j)(T_{\pi(z)}(Y))^{\perp}.$$

PROOF. Let $y \in Y$ and assume after translation that $y = 0$ then, after a suitable permutation of coordinates, there exists some $r > 0$ and a smooth submersion $g : B_r^N(0) \to \boldsymbol{R}^{N-n}$ such that $g^{-1}(0) = B_r^N(0) \cap Y$. By the implicit function theorem [2, p. 270] we may further assume that there exists a smooth function $f : B_r^n(0) \to \boldsymbol{R}^{N-n}$ such that $f(0) = 0$, $B_r^N(0) \cap Y$ is the graph of f and

$$B_r^n(0) \to B_r^N(0) \cap Y$$
$$x \mapsto (x, f(x))$$

is a bijective submersion, hence a C^∞ diffeomorphism and it defines a C^∞ chart of Y about y. Now let R be the vector subspace of $\boldsymbol{R}^N$ generated by the $N - n$ row vectors of $g'(0)$ so that $\dim R = N - n$ and let

$$A = \left[\begin{array}{c|c} \text{orthonormal} & \text{orthonormal} \\ \text{basis of} & \text{basis of} \\ R^\perp & R \end{array}\right] \in \boldsymbol{O}_N(\boldsymbol{R})$$

then $(g \circ A)'(0) = g'(0)$. $A = [O|^n \ *]$ and $D_1(g \circ A)(0) = 0$. Hence replacing g by $g \circ A$, and passing to $A^{-1}(Y)$, we may assume, since $f'(x) = -(D_2 g(x, f(x)))^{-1} \circ D_1 g(x, f(x))$, that $f'(0) = 0$. Composing by a non-singular Linear transformation we may further assume that $g'(0) = [O|^n \ -I_{N-n}]$. Note that

$$\begin{aligned} f''(x) &= (D_2 g(x, f(x)))^{-1} \cdot (D_2 g(x, f(x)))' \cdot (D_2 g(x, f(x)))^{-1} \cdot D_1 g(x, f(x)) \\ &\quad - (D_2 g(x, f(x)))^{-1} \cdot (D_1 g(x, f(x))' \\ &= -(D_2 g(x, f(x)))^{-1} \cdot [(D_2 g(x, f(x)))' f'(x) + (D_1 g(x, f(x)))'] \quad \text{and} \end{aligned}$$

$D_p D_q f(0) = D_p D_q g(0)$ for $1 \leq p$, $q \leq n$. We have for $1 \leq k \leq N - n$

$$|g_k(t) - g_k(0) - g_k'(0) \cdot t| \leq \frac{\varepsilon}{12} \|t\| \quad \text{and}$$

$$\left| g_k(t) - g_k(0) - g_k'(0) \cdot t - \frac{1}{2} g_k''(0) \cdot t^{(2)} \right| \leq \frac{\varepsilon}{12} \|t\|^2 \quad \text{for } \|t\| < 2r_0 < 1 \quad \text{by}$$

Taylor formula [2, p. 190], hence for $t \in \overline{B_{2r_0}^n(0)}$

$$|g_k''(0) \cdot t^{(2)}| \leq \frac{\varepsilon}{6} (\|t\| + \|t\|^2), \quad |D_p^2 g_k(0)| r_0 < \frac{\varepsilon}{3} \quad \text{for } 1 \leq p \leq n \quad \text{and}$$

$$|g_k''(0) \cdot (e_p + e_q, e_p + e_q)| r_0 < \frac{2\varepsilon}{3} \quad \text{for } 1 \leq p, q \leq n, p \neq q \quad \text{so that}$$

$$|D_pD_qg_k(0)|r_0 = \frac{1}{2}|g_k''(0)\cdot(e_p+e_q,e_p+e_q) - D_p^2g_k(0)$$

$$- D_q^2g_k(0)|r_0 < \frac{2\varepsilon}{3} \quad \text{for } 1 \le p,q \le n, p \ne q.$$

Then there exists some $0 < r < r_0$ such that $\sup\{r|D_pD_qf_k(\zeta)| : 1 \le p,\ q \le n,$ $1 \le k \le N-n, \|\zeta\|| < r\} < \varepsilon$ and we have the open covers $\{w_j^0 : j \ge 1\}$ and $\{w_j : j \ge 1\}$ satisfying 1) as desired.

Now for all $x \in B_r^n(0)$ we have

$$M_{(x,f(x))} = (\theta_{id})^{-1} \circ T_{(x,f(x))}(j)(T_{(x,f(x))}(Y))$$

$$= \left\{z \in \boldsymbol{R}^N : z_{n+k} - \sum_{i=1}^{n} D_if_k(x)z_i = 0 \text{ for } 1 \le k \le N-n\right\} \quad \text{hence}$$

$$M_{(x,f(x))}^{\perp} = \sum_{k=1}^{N-n} \boldsymbol{R}\left(\boldsymbol{e}_{n+k} - \sum_{i=1}^{n} D_if_k(x)\boldsymbol{e}_i\right) = \sum_{k=1}^{N-n} \boldsymbol{R}\boldsymbol{u}_k \quad \text{and}$$

$$(B_r^N(0) \cap Y) \times \boldsymbol{R}^{N-n} \to \boldsymbol{R}^N$$

$$(x,t) \mapsto x + \sum_{k=1}^{N-n} t_k\boldsymbol{u}_k$$

is a C^∞ submersion and restricts, for r sufficiently small, to a C^∞ diffeomorphism $(B_r^N(0) \cap Y) \times B_r^{N-n}(0) \to T$ open $\subset B_r^N(0)$ whose inverse is $z \mapsto (\pi(z), \theta(z))$ and the properties stated in 2) are satisfied by [4, p. 180]. //.

LEMMA 1.4. *Let U open $\subset \boldsymbol{R}^N$ and $h : U \to \boldsymbol{R}^N$ a bilipschitz embedding such that $L(h) \le \alpha$. For all $y \in U$, $d < d(y, U^c)$, Let $h_\sigma =$ the d-simplicial approximation of h at y, be defined by $h_\sigma : \boldsymbol{R}^n \to \boldsymbol{R}^N$.*

$$h_\sigma\left(y + \sum_{i=1}^{n} a_i\boldsymbol{v}_i\right) = h(y) + \sum_{i=1}^{n} a_i(h(y+\boldsymbol{v}_i) - h(y))$$

where $\boldsymbol{v}_i = d\boldsymbol{e}_i$ for $1 \le i \le n$, *then*:

1- *For all $x \in \boldsymbol{R}^N$ we have*

$$\left(\frac{1}{\alpha^2} - 2(\alpha^2-1)(n-1)\right)\|x-y\|^2 \le \|h_\sigma(x) - h(y)\|^2$$

$$\le (\alpha^2 + 2(\alpha^2-1)(n-1))\|x-y\|^2.$$

2- *For* $\dfrac{d}{2} \le \|x - y\| \le d$ *we have*

$$\|h_\sigma(x) - h(x)\|^2 \le 2(\alpha^2 - 1)(n + 5\sqrt{n})\|x - y\|^2. \qquad //$$

PROOF. 1- [7, lemma 3.2].

2- Let $x = y + \sum_{i=1}^n a_i v_i$, then since $\|x - y\| \le d$ we get $|a_i| \le 1$ and by [7, lemma 3.6.2]

$$\begin{aligned}\|h_\sigma(x) - h(x)\|^2 &\le 2(\alpha^2 - 1)\left[n + \sum_{i=1}^n |a_i|\left(\frac{\|x - y\|^2 + d^2 + |a_i| d^2}{\|x - y\|^2}\right)\right]\|x - y\|^2 \\ &\le 2(\alpha^2 - 1)\left[n + \sqrt{n}\left(\frac{\|x - y\|}{d} + \frac{2d}{\|x - y\|}\right)\right]\|x - y\|^2 \\ &\le 2(\alpha^2 - 1)(n + 5\sqrt{n})\|x - y\|^2. \qquad //.\end{aligned}$$

We shall need five more geometric lemmas.

LEMMA 1.5. *Let* $\{a, b\} \subseteq \boldsymbol{R}^N - \{0\}$ *such that* $\|a - b\| \le c\|b\|$, $c < 1$, *then the angle* $A(a, b)$ *between* a, b *satisfies* $A(a, b) \le \pi/2 - \arcsin(1 - c)$

PROOF.

$$\begin{aligned}1 - \cos A(a, b) &= 1 - \frac{\langle a, b\rangle}{\|a\|\,\|b\|} = 1 - \frac{\|a^2\| + \|b\|^2 - \|a - b\|^2}{2\|a\|\,\|b\|} \\ &\le 1 - \frac{\|a\|^2 + \|b\|^2 - c^2\|b\|^2}{2\|a\|\,\|b\|} \\ &= 1 - \sqrt{1 - c^2} - \frac{(\|a\| - \sqrt{1 - c^2}\|b\|)^2}{2\|a\|\,\|b\|} \\ &\le 1 - \sqrt{1 - c^2} \le c\end{aligned}$$

hence $A(a, b) = \pi/2 - \psi$ and $\sin\psi \ge 1 - c$ so that $\psi \ge \arcsin(1 - c)$. //.

LEMMA 1.6. *Let* $\{a, b\} \subseteq \boldsymbol{R}^N - \{0\}$ *and* L *a non-trivial vector subspace of* $\boldsymbol{R}^N$ *then* $|A(a, L) - A(b, L)| \le A(a, b)$.

PROOF. We may assume $A(a, b) \le \pi/2$. For all $x \in \boldsymbol{R}^N$ let $P_L(x) = p(x)$ and $P_{L^\perp}(x) = p'(x)$. We have

$$\begin{aligned}
\|a\|^2\|b\|^2\sin^2 A(a,b) &= \|a\|^2\|b\|^2 - \langle a,b\rangle^2 \\
&= \|a\|^2\|b\|^2 - (\langle p(a),p(b)\rangle + \langle p'(a),p'(b)\rangle)^2 \\
&\geq \|a\|^2\|b\|^2 - (\|p(a)\|\|p(b)\| + \|p'(a)\|\|p'(b)\|)^2 \\
&= \|a\|^2\|b\|^2\left(\frac{\|p(a)\|}{\|a\|}\cdot\frac{\|p'(b)\|}{\|b\|} - \frac{\|p'(a)\|}{\|a\|}\cdot\frac{\|p(b)\|}{\|b\|}\right)^2 \\
&= \|a\|^2\|b\|^2\sin^2|A(a,L) - A(b,L)|
\end{aligned}$$

and $|A(a,L) - A(b,L)| \leq A(a,b)$ as desired. //.

LEMMA 1.7. *Let $a \in \boldsymbol{R}^N$ and $L = \sum_{i=1}^n \boldsymbol{R}(\boldsymbol{e}_i + \sum_{k=1}^{N-n} A_{ik}\boldsymbol{e}_{n+k})$ and $\varepsilon > 0$ such that*:

i- $\mathrm{Max}\{|A_{ik}| : 1 \leq i \leq n, 1 \leq k \leq N-n\} < \varepsilon$
ii- $\|a\|_{N-n} < M\varepsilon\|a\|_n$
iii- $\varepsilon < \min\left(\dfrac{1}{\sqrt{2n(N-n)}}, \dfrac{\sin\psi}{4(N-n)(M+n)}\right)$
where $0 < \psi < \pi/2$, then $A(a,L) \leq \psi$.

PROOF. Note that $L^\perp = \sum_{k=1}^{N-n} \boldsymbol{R}(\boldsymbol{e}_{n+k} - \sum_{i=1}^n A_{ik}\boldsymbol{e}_i)$ and let

$$a = \sum_{i=1}^n a_i\boldsymbol{e}_i = \sum_{i=1}^n a_i'\left(\boldsymbol{e}_i + \sum_k A_{ik}\boldsymbol{e}_{n+u}\right) + \sum_{k=1}^{N-n} c_k\left(\boldsymbol{e}_{n+k} - \sum_i A_{ik}\boldsymbol{e}_i\right)$$

Then $a_i = a_i' - \sum_{k=1}^{N-n} c_kA_{ik}$ for $1 \leq i \leq n$ and

$$c_k = a_{n+k} - \sum_{i=1}^n A_{ik}a_i' = a_{n+k} - \sum_{i=1}^n A_{ik}\left(a_i + \sum_{j=1}^{N-n} c_jA_{ij}\right) \quad \text{for } 1 \leq k \leq N-n$$

Now $|c_k| \leq M\varepsilon\|a\|_n + \varepsilon n\|a\|_n + \varepsilon^2 n(N-n)\max_{1\leq j\leq N-n}|c_j|$ gives $\max_{1\leq k\leq N-n}\cdot$ $|c_k| \leq 2(M+n)\varepsilon\|a\|_n$ so that

$$\begin{aligned}
\|P_{L^\perp}(a)\| &= \left\|\sum_{k=1}^{N-n} c_k\left(\boldsymbol{e}_{n+k} - \sum_{i=1}^n A_{ik}\boldsymbol{e}_i\right)\right\| \\
&\leq 2(M+n)(N-n) \in \|a\|_n\sqrt{1+n\varepsilon^2} \\
&\leq \sin\psi\|a\|
\end{aligned}$$

and $A(a,L) \leq \psi$. //.

LEMMA 1.8. *Let* $p : X \times [0,1] \to Y$ *be a homotopy between the path-connected topological spaces* X *and* Y*. Fix* $x_0 \in X$ *and Let* $y_s = p(x_0, s)$ *then*

$$|\pi_1(Y, y_s) : p_{s\sharp}(\pi_1(X, x_0))| = |\pi_1(Y, y_0) : p_{0\sharp}(\pi_1(X, x_0))| \quad \text{for } 0 \le s \le 1.$$

PROOF. Let $\alpha(t) = p(x_0, t)$, $0 \le t \le s$, be the path joining y_0 and y_s so that

$$\theta_s : \pi_1(Y, y_s) \to \pi_1(Y, y_0)$$

defined by $\theta_s(\gamma) = \alpha \cdot \gamma \cdot \alpha^{-1}$ is a group isomorphism and we have a commutative diagram

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[rr, "p_{0\sharp}"] \arrow[dr, "p_{s\sharp}"'] & & \pi_1(Y, y_0) \\ & \pi_1(Y, y_s) \arrow[ur, "\theta_s"'] & \end{tikzcd}$$

Since for any $[\beta] \in \pi_1(X, x_0)$ we have a homotopy

$$\varphi : [0, s] \times [0,1] \to Y$$

$$\text{defined by } \varphi(t, m) = \begin{cases} p(x_0, 3tm) & 0 \le m \le 1/3 \\ p(\beta(3m-1), t) & 1/3 \le m \le 2/3 \\ p(x_0, 3t(1-m)) & 2/3 \le m \le 1 \end{cases}$$

and $\varphi(t, 0) = \varphi(t, 1) = y_0$, so that $[\alpha \cdot p_s(\beta) \cdot \alpha^{-1}] = [p_0(\beta)]$. Our result follows. //.

LEMMA 1.9 [6, p. 64]. *Suppose* A *convex* $\subseteq \boldsymbol{R}^n$ *and* $f : A \to \boldsymbol{R}^N$ *a continuous map such that* $\operatorname{Lim\,sup}_{z \to z_0} \|f(z) - f(z_0)\| / \|z - z_0\| \le \alpha$ *for all* $z_0 \in A$*, then*

$$\|f(z) - f(w)\| \le \alpha \|z - w\| \quad \text{for all } z, w \in A. \qquad //.$$

Now we can establish the smoothing theorem.

THEOREM 1.10 (Smoothing theorem). *Let* V *open* $\subset \overline{V} \subset V_2$ *open* $\subset \overline{V_2} \subset U$ *open* $\subset B^n$ *and let* Y *be a* C^∞ *n-manifold properly and smoothly embedded in* $\boldsymbol{R}^N$ *and let* $j : Y \to \boldsymbol{R}^N$ *be the canonical injection.*

Suppose $h : U \to Y \xrightarrow{j} \boldsymbol{R}^N$ *is a locally bilipschitz embedding such that* $L_c(h) \le \alpha \le \alpha_0(n) = \alpha_0$, $\alpha > 1$ *where* α_0 *satisfies*

$$0 < 2\alpha_0^2(\alpha_0^2 - 1)(n - 1 + (n + 5\sqrt{n})9n^2) < 1$$

then for all $0 < \mu < 1$*, there exists an isotopy* $\psi : U \times [0,1] \to h(U) \subseteq Y$ *satisfying*

1- $\psi_0 = h$
2- $\psi_s|_{V_2^c} \equiv h|_{V_2^c}$ *for all* $0 \le s \le 1$,
3- $\psi_1|_V$ *is a* C^∞ *diffeomorphism,*
4- *For all* $z \in U$, $0 \le s \le 1$, $\|(\psi_s)^{-1}h(z) - z\| \le \mu d(z, V_2^c)$,
5- $(\psi_s)^{-1}h$ *is locally bilipshitz and* $L_c((\psi_s)^{-1}h) \le \beta\alpha^4$ *for all* $0 \le s \le 1$ *where*:

$$\beta = \beta_n(\alpha) = \left(\sqrt{1 - 2\alpha^2(\alpha^2-1)(n-1)} - 3n\sqrt{2\alpha^2(\alpha^2-1)(n+5\sqrt{n})}\right)^{-1}$$

Proof. Let $\{w_j : j \ge 1\}$ and $\{w_j^0 : j \ge 1\}$ be the open covers of Y constructed in Lemma 1.3 with respect to the compact set $h(\overline{V}_2)$ of Y and with $\varepsilon < \min\{1/(3n(N-n)\alpha^2), \sin(\eta/2)/(4(N-n)(M+n)), \varepsilon_0\}$ where $M = (N-n)n\alpha/\sqrt{1/2\alpha^2 - 2(\alpha^2-1)(n-1)}$, $0 < \eta = \arcsin(1-c) < \pi/2$,

$$c = 3n\sqrt{\frac{2(\alpha^2-1)(n+5\sqrt{n})}{\frac{1}{\alpha^2} - 2(\alpha^2-1)(n-1)}}, \quad \frac{1}{2(1-10\varepsilon^2n^2(N-n)^2)} \cdot \Big[12(N-n)^2n^{3/2}\varepsilon$$

$$+ \sqrt{(12(N-n)^2n^{3/2}\varepsilon)^2 + 4(1-10\varepsilon^2n^2(N-n)^2)(1+3(N-n)^2\varepsilon)}\Big] < \alpha,$$

and

$$\frac{1}{2(1+15n^2(N-n)^2\varepsilon)} \cdot \Bigg[-4\alpha^2n(N-n)\varepsilon$$

$$+ \sqrt{(4\alpha^2n(N-n)\varepsilon)^2 + 4(1+15n^2(N-n)^2\varepsilon)\left(\frac{1}{(\beta\alpha^2)^2} - 8\alpha^2n(N-n)^2\varepsilon\right)}\Bigg]$$

$$> \frac{1}{\beta\alpha^3} \quad \text{for } 0 < \varepsilon \le \varepsilon_0$$

then $T = \bigcup_{j \ge 1} T_j$ is a tubular open neightbourhood of Y in $\boldsymbol{R}^N$ and there exists $\pi : T \to Y$ a C^∞ submersion such that for all $z \in T$

i) $\pi(z)$ is the unique point in Y such that $\|\pi(z) - z\| = d(z, Y)$

ii) For all $j \ge 1$, $\pi(T_j) = T_j \cap Y$ and

$$T_j \cap \pi^{-1}(\pi(z)) = (\pi(z) + N_{\pi(z)}(Y)) \cap T_j \quad \text{which defines a } C^\infty$$

chart q_j of $\pi^{-1}(\pi(z))$ at z and we get a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & T_z(\pi^{-1}(\pi(z))) & \longrightarrow & T_z(\boldsymbol{R}^N) & \xrightarrow{T(\sigma)} & T_{\pi(z)}(\boldsymbol{R}^N) \\
 & & \Big\uparrow{\scriptstyle \theta_{q_j}} & & \Big\uparrow{\scriptstyle \theta_{id}} & & \Big\uparrow{\scriptstyle \theta_{id}} \\
0 & \longrightarrow & \boldsymbol{R}^{N-n} & \longrightarrow & \boldsymbol{R}^N & = & \boldsymbol{R}^N \\
 & & (t_k) & \longmapsto & \sum_{k=1}^{N-n} t_k \boldsymbol{u}_k & &
\end{array}$$

where $\boldsymbol{u}_k = \boldsymbol{e}_{n+k} - \sum_{i=1}^{n} D_i f_j^{(i)}(p_1(\pi(z)))\boldsymbol{e}_i$ for $1 \le k \le N-n$ and

$$\sigma(x) = \pi(z) - z + x \quad \text{a translation of } \boldsymbol{R}^N \text{ so that}$$

$T_z(\pi^{-1}(\pi(z))) = \theta_{id}(N_{\pi(z)}(Y))$. Let $0 < \mu < 1$ and note that there exists

$$0 < \delta_0 < \min\{d(\overline{V}_2, U^c), d(h(\overline{V}_2), Y - h(U)), r\} \quad \text{where}$$

$r = \min\{1,$ Lebesgue number of the cover $\{T_j : 1 \le j \le s\}$ of $h(\overline{V}_2)\}$ such that for all $y \in \overline{V}_2$

$$\|x - y\| < \delta_0 \Rightarrow \frac{\|x - y\|}{\alpha} \le \|h(x) - h(y)\| \le \alpha\|x - y\|$$

and for all $h(y) \in h(\overline{V}_2)$

$$\|x - y\| < \delta_0 \Rightarrow \frac{\|x - y\|}{\alpha} \le \|h(x) - h(y)\| \le \alpha\|x - y\|.$$

Let t be the function defined in lemma 1.2 with respect to the open set U and define

$$G_k : U \to \boldsymbol{R}^N$$

by

$$G_k(y) = \begin{cases} \dfrac{1}{a_n(k\delta t(y))^n} \displaystyle\int \varphi\left(\frac{\|x - y\|}{k\delta t(y)}\right) h(x)\, d\lambda(x) & \text{if } t(y) > 0 \\ h(y) & \text{otherwise} \end{cases}$$

where $0 < k < \delta_0(\alpha - 1)\mu/(40 \cdot 2^n \cdot \alpha^4)$.

Note that $G_k|_{V_2^c} \equiv h|_{V_2^c}$ and by [3, p. 125] $G_k|_{t^{-1}((0,1])}$ is C^∞.

Claim 1. 1- $\|G_k(y) - h(y)\| \leq \alpha k\delta t(y)$ in particular G_k is continuous on U. 2- $\|G_k(y) - G_k(y')\| \leq \alpha^2\|y - y'\|$ for $\|y - y'\| < \delta_0/2$.

Proof. 1- We may assume $y \in \overline{V}_2$ with $t(y) > 0$, hence

$$\begin{aligned}\|G_k(y) - h(y)\| &= \left\|\frac{1}{a_n(k\delta t(y))^n}\int \varphi\left(\frac{\|x - y\|}{k\delta t(y)}\right)h(x)\,d\lambda(x) - h(y)\right\| \\ &= \frac{1}{a_n}\left\|\int \varphi(\|x\|)(h(y + k\delta t(y)x) - h(y))\,d\lambda(x)\right\| \quad \text{by [4, p. 155]} \\ &\leq \alpha k\delta t(y)\end{aligned}$$

by Cauchy-Schwarz inequality [3, p. 153].

2- Now suppose $y, y' \in U$, $\|y - y'\| < \frac{1}{2}\delta_0$, $t(y) > 0$ and $t(y') = 0$ then

$$\begin{aligned}&\|G_k(y) - G_k(y')\| \\ &\quad \leq \|G_k(y) - h(y)\| + \|h(y) - h(y')\| \\ &\quad \leq \alpha\|y - y'\|(k \cdot 10 \cdot 2^n + 1) \quad \text{by lemma 1.2} + [2, \text{p. } 160] \\ &\quad \leq \alpha^2\|y - y'\|.\end{aligned}$$

Also if $y, y' \in U$, $\|y - y'\| < \frac{1}{2}\delta_0$, $t(y)$ and $t(y')$ both >0 then

$$\begin{aligned}&\|G_k(y) - G_k(y')\| \\ &\quad \leq \sqrt{\frac{1}{a_n}\int \varphi\|h(y + k\delta t(y)x) - h(y' + k\delta t(y')x)\|^2\,d\lambda(x)} \\ &\quad \leq \alpha(1 + k \cdot 10 \cdot 2^n)\|y - y'\| \\ &\quad \leq \alpha^2\|y - y'\| \quad \text{as above.} \qquad //.\end{aligned}$$

Now define $\psi : U \times [0,1] \to Y$ by

$$\psi_s = \begin{cases} \pi \circ G_{sk} & 0 < s \leq 1 \\ h & s = 0 \end{cases}$$

and we follow [11] to show that ψ is the required map satisfying properties 1), 2), 3) of the theorem.

Claim 2. $\psi_s(t^{-1}((0,1])) \subseteq h(t^{-1}((0,1]))$.

PROOF. Let $0 < s \leq 1$ and $y \in \overline{V}_2$ with $t(y) > 0$ and suppose $\psi_s(y) \notin h(t^{-1}((0,1]))$. We have

$$\begin{aligned}\|\psi_s(y) - h(y)\| &\leq \|\pi \circ G_{sk}(y) - G_{sk}(y)\| + \|G_{sk}(y) - h_y)\|.\\ &\leq 2\|G_{sk}(y) - h(y)\|\\ &\leq 2\alpha sk\delta t(y) \quad \text{by claim 1}\\ &< \delta_0\end{aligned}$$

hence $\psi_s(y) = h(z)$ with $t(z) = 0$ and

$$\begin{aligned}\frac{d(y, t^{-1}(0))}{\alpha} \leq \frac{\|z - y\|}{\alpha} &\leq \|h(z) - h(y)\|\\ &= \|\psi_s(y) - h(y)\| \leq 2\alpha kd(y, t^{-1}(0))\end{aligned}$$

by lemma 1.2, which is absurd by choice of k. //.

By claim 2 we have $\psi : U \times [0,1] \to h(U)$.

CLAIM 3. ψ is continuous.

PROOF. Continuity on $U \times (0,1]$ follows from [3, p. 125].
Let $y \in h(\overline{V}_2)$, $\varepsilon > 0$ and Let $h(\overline{V}_2) \subset W$ open $\subset \overline{W} \subset \bigcup_{j=1}^{s} T_j$.
There exists $0 < \delta_1 < \min\{(1/2)\delta_0, \varepsilon\}$ such that $z, z' \in \overline{W}$, $\|z - z'\| < \delta_1 \Rightarrow \|\pi(z) - \pi(z')\| < \varepsilon/2$ then for $\|y - y'\| < \delta_1/\alpha^2$ and $s < 1/\alpha \min\{d(h(\overline{V}_2), W^c), \varepsilon/4\}$ we have $\|G_{sk}(y) - h(y)\| \leq \alpha sk\delta t(y) < d(h(\overline{V}_2), W^c)$ and $\|G_{sk}(y) - G_{sk}(y')\| \leq \alpha^2\|y - y'\| < \delta_1$ hence

$$\begin{aligned}\|\psi_s(y') - h(y)\| &\leq \|\psi_s(y') - \psi_s(y)\| + \|\psi_s(y) - h(y)\|\\ &< \varepsilon/2 + 2\alpha sk\delta t(y) < \varepsilon\end{aligned}$$

establishing continuity at $(y, 0)$. //.

Note that for all $0 < s \leq 1$

$$\psi_s|_{t^{-1}((0,1])} : t^{-1}((0,1]) \to h(t^{-1}((0,1])) \text{ is } C^\infty \text{ by [3, p. 125]}$$

and proper.

Claim 4. for all $0 < s \leq 1$, $\psi_s|_{t^{-1}((0,1])}$ is étale (= local diffeomorphism).

Proof. We shall prove this for $s = 1$, the proof for $0 < s < 1$ is obtained by replacing k by sk throughout.

Let $y \in U$, $t(y) > 0$, and set $d = k\delta t(y)$ and let h_σ be the d-simplicial approximation of h at y so that $h'_\sigma(y) \cdot z = h_\sigma(y+z) - h(y)$ and

$$\begin{aligned}
&\frac{1}{a_n(k\delta t(\zeta))^n}\int \varphi\left(\frac{\|x-\zeta\|}{k\delta t(\zeta)}\right) h_\sigma(x)\,d\lambda(x) \\
&\quad = h(y) + \sum_{i=1}^{n} \frac{1}{a_n(k\delta t(\zeta))^n}\int \varphi\frac{\langle x-y, v_i\rangle}{k\delta t(y)}\,d\lambda(x)(h(y+v_i)-h(y)) \\
&\quad = h_\sigma(\zeta),
\end{aligned}$$

so that

$$\begin{aligned}
&\|G'_k(y)\cdot z - h'_\sigma(y)\cdot z\| \\
&\quad = \frac{1}{a_n k^n \delta^n}\left\|\int \left(\frac{\varphi}{t(y)^n}\right)'(y)\cdot z(h(x)-h_\sigma(x))\,d\lambda(x)\right\| \\
&\quad = \Bigg\|\frac{-n}{t(y)}t'(y)\cdot z(G_k(y)-h(y)) + \frac{1}{a_n(k\delta t(y))^{n+2}} \\
&\qquad \int \varphi'(k\delta t(y)\frac{(y-x)\cdot z}{\|y-x\|} - \|x-y\|k\delta t'(y)\cdot z)(h(x)-h_\sigma(x))\,d\lambda(x)\Bigg\| \\
&\quad \leq \|z\|\left(10\cdot 2^n n\alpha k + \frac{2}{a_n(k\delta t(y))^{n+1}}\sqrt{\sum_{i=1}^{N}\left(\int |\varphi'||h(x)-h_\sigma(x)|_i\,d\lambda(x)\right)^2}\right)
\end{aligned}$$

by lemma 1.2 + claim 1

$$\leq \|z\|\left(\cdots + \frac{2}{a_n(k\delta t(y))^{n+1}}\sqrt{\int |\varphi'|\,d\lambda(x)\int |\varphi'|\,\|h(x)-h_\sigma(x)\|^2\,d\lambda(x)}\right)$$

by Cauchy-Schwarz inequality [3, p. 153]

$$\leq \|z\|\left(\cdots + \frac{2\sqrt{2(\alpha^2-1)(n+5\sqrt{n})}}{a_n(k\delta t(y))^{n+1}}\sqrt{\int |\phi'|\,d\lambda(x)\cdot\int |\varphi'|\|x-y\|^2\,d\lambda(x)}\right)$$

by Lemma 1.4

$$\leq \|z\| \left(\cdots + 2\sqrt{2(\alpha^2-1)(n+5\sqrt{n})} \cdot \sqrt{2}\,\frac{\left|\int_0^\infty \varphi'(r) r^n \, dr\right|}{\int_0^\infty \varphi(r) r^{n-1}\, dr} \right) \quad \text{by [4, p. 155]}$$

$$\leq 3n\sqrt{2(\alpha^2-1)(n+5\sqrt{n})}\|z\|$$

$$\leq 3n\sqrt{\frac{2(\alpha^2-1)(n+5\sqrt{n})}{\frac{1}{\alpha^2}-2(\alpha^2-1)(n-1)}} \cdot \|h'_\sigma(y)\cdot z\| \quad \text{by lemma 1.4}$$

$$= c\|h'_\sigma(y)\cdot z\|.$$

Since $0 < c < 1$ we have $G'_k(y)\cdot z \neq 0$ and lemma 1.5 gives $A(G'_k(y)\cdot z, h'_\sigma(y)\cdot z) \leq \frac{1}{2}\pi - \eta$.

Note that $h(y + d\bar{B}^n) \subseteq T_j$ for some $1 \leq j \leq s$ so that by Lemma 1.3

$$\|h(y+\boldsymbol{v}_i) - h(y)\|_{N-n}^2 \leq \varepsilon^2 (N-n) n \|h(y+v_i) - h(y)\|_n^2$$

and

$$\|h_\sigma(y+z) - h(y)\|_{N-n} \leq \varepsilon(N-n)\sqrt{n}\sum_{i=1}^n |z_i| \|h(y+\boldsymbol{v}_i) - h(y)\|_n$$

$$\leq \varepsilon(N-n)n\alpha\|z\|$$

$$\leq \frac{\varepsilon(N-n)n\alpha}{\sqrt{\frac{1}{\alpha^2}-2(\alpha^2-1)(n-1)}}\|h_\sigma(y+z)-h(y)\| \quad \text{by lemma 1.4}$$

$$\leq \frac{\varepsilon(N-n)n\alpha}{\sqrt{\frac{1}{\alpha^2}-2(\alpha^2-1)(n-1)-\varepsilon^2(N-n)^2n^2\alpha^2}}\|h_\sigma(y+z)-h(y)\|_n$$

$$< \frac{(N-n)n\alpha}{\sqrt{\frac{1}{2\alpha^2}-2(\alpha^2-1)(n-1)}}\varepsilon\|h_\sigma(y+z)-h(y)\|_n$$

$$= M\varepsilon\|h_\sigma(y+z)-h(y)\|_n.$$

Recall that from Lemma 1.3 we have a commutative diagram

$$\begin{array}{ccccc}
0 \longrightarrow & T_{\psi_1(y)}(Y) = T_{\psi_1(y)}(w_j \cap Y) & \xrightarrow{T(j)} & T_{\psi_1(y)}(w_j) = T_{\psi_1(y)}(\boldsymbol{R}^N) \\
& \Big\uparrow \theta_m & & \Big\uparrow \theta_{id} \\
0 \longrightarrow & \boldsymbol{R}^n & \longrightarrow & \boldsymbol{R}^N \\
& \boldsymbol{x} & \longmapsto & \sum_{i=1}^{n} x_i \left(\boldsymbol{e}_i + \sum_{k=1}^{N-n} D_i f_k^{(i)}(p_1 \psi_1(y)) \boldsymbol{e}_{n+k} \right)
\end{array}$$

where m is the C^∞ chart corresponding to the C^∞ diffeomorphism

$$r_j B^n \to r_j B^N \cap Y$$

$$x \mapsto (x, f^{(j)}(x)).$$

Now $M_{\psi_1(y)} = \sum_{i=1}^{n} \boldsymbol{R}\left(\boldsymbol{e}_i + \sum_{k=1}^{N-n} D_i f_k^{(j)}(p_1 \psi_1(y)) \boldsymbol{e}_{n+k} \right)$ and Lemma 1.7 gives $A(h'_\sigma(y) \cdot z, M_{\psi_1(y)}) \leq \frac{1}{2}\eta$ therefore by Lemma 1.6 we get $A(G'_k(y) \cdot z, M_{\psi_1(y)}) \leq \frac{1}{2}(\pi - \eta)$, hence $G'_k(y) \cdot z \notin N_{\psi_1(y)}(Y) = \theta_{id}^{-1}(ker(T_{G_k(y)}(\pi)))$ and for all $z \in \boldsymbol{R}^n$ $0 \neq T_{G_k(y)}(\pi) \circ \theta_{id}(G'_k(y) \cdot z) = T_y(\psi_1) \circ \theta_{id}(z)$ and $T_y(\psi_1)$ is a bijection as desired. //.

Now by Claim 4, for all $0 < s \leq 1$ $\psi_s : t^{-1}((0,1]) \to h(t^{-1}((0,1]))$ is proper, C^∞ and étale hence a finitely sheeted covering, and claim 3+ Lemma 1.8 show that ψ_s is a bijection hence a C^∞ diffeomorphism.

To prove property 4), Let $z \in V_2$ and $t(z) > 0$ then $h(z) = \psi_s(z_s)$ for some $z_s \in V_2$, $t(z_s) > 0$ and

$$\begin{aligned}
\|h(z_s) - h(z)\| &= \|h(z_s) - \psi_s(z_s)\| \\
&\leq 2\alpha s k \delta t(z_s) \quad \text{by claim 2} \\
&< \delta_0
\end{aligned}$$

hence

$$\begin{aligned}
\|z_s - z\| &\leq \alpha \|h(z_s) - h(z)\| \\
&\leq 2\alpha^2 sk(\delta t(z) + 10 \cdot 2^n \|z - z_s\|) \quad \text{by lemma 1.2}
\end{aligned}$$

or

$$\|z_s - z\| \leq 4\alpha^2 skd(z, t^{-1}(0))$$

and

$$\|(\psi_s)^{-1} h(z) - z\| < \mu d(z, V_2^c)$$

as desired.

We prove property 5) for $s = 1$, the proof for $0 < s < 1$ is obtained by replacing k by sk throughout. We write ψ for ψ_1 in what follows.

By lemma 1.9 it suffices to show that for all $z_0 \in U$

$$\frac{\|z - z_0\|}{\beta\alpha^4} \leq \|\psi^{-1}h(z) - \psi^{-1}h(z_0)\| \leq \beta\alpha^4\|z - z_0\|$$

for all z in some neighbourhood of z_0.

CASE I. $t(z_0) = 0$

We may assume $z_0 \in \overline{V}_2$. Let $z \in U$, $\|z - z_0\| < \delta_0/2\alpha$, $\psi^{-1}h(z) = z_1$, then

$$\begin{aligned}\|h(z_1) - h(z_0)\| &\leq \|h(z_1) - h(z)\| + \|h(z) - h(z_0)\| \\ &\leq 2\alpha k\delta t(z_1) + \alpha\|z - z_0\| < \delta_0\end{aligned}$$

hence

$$\begin{aligned}\|z_1 - z_0\| &\leq \alpha\|h(z_1) - h(z_0)\| \\ &\leq 2\alpha^2 k 10.2^n\|z_1 - z_0\| + \alpha^2\|z - z_0\| \quad \text{by lemma 1.2}\end{aligned}$$

and $\|z_1 - z_0\| \leq \alpha^3\|z - z_0\|$. Similarly,

$$\|z_1 - z_0\| \geq \frac{\|z - z_0\|}{\alpha^3}$$

hence,

$$\frac{\|z - z_0\|}{\alpha^3} \leq \|\psi^{-1}h(z) - \psi^{-1}h(z_0)\| \leq \alpha^3\|z_1 - z_0\|.$$

CASE II. $t(z_0) > 0$

Let h_σ be the d-simplicial approximation of h at z_0, $d = k\delta t(z_0)$, then

$$\begin{aligned}\|G_k'(z_0) \cdot z\| &\geq \|h_\sigma'(z_0) \cdot z\| - \|(G_k'(z_0) - h_\sigma'(z_0)) \cdot z\| \\ &\geq \left(\sqrt{\frac{1}{\alpha^2} - 2(\alpha^2 - 1)(n - 1)} - 3n\sqrt{2(\alpha^2 - 1)(n + 5\sqrt{n})}\right)\|z\| \\ &\qquad \text{by lemma 1.4} + \text{claim 4} \\ &= \frac{\|z\|}{\beta\alpha}.\end{aligned}$$

Let $B_p(z_0) \subset t^{-1}((0, 1])$ such that $p < \delta_0/2\alpha$, $\psi^{-1}h(B_p(z_0)) \subset B_{p_1}(\psi^{-1}h(z_0)) \subset t^{-1}((0, 1])$, $p_1 < \delta_0/2$ and the oscillation of G_k' on $B_{p_1}(\psi^{-1}h(z_0)) < (\alpha - 1)/\beta\alpha^2$.

Now for $z \in B_p(z_0)$, set $\psi^{-1}h(z) = z^1$ and $\psi^{-1}h(z_0) = z_0^1$, then

$$\begin{aligned}
&\|G_k(z^1) - G_k(z_0^1)\| \\
&\quad \geq \|G_k'(z_0^1) \cdot (z^1 - z_0^1)\| - \|G_k(z^1) - G_k(z_0^1) - G_k'(z_0^1) \cdot (z^1 - z_0^1)\| \\
&\quad \geq \frac{\|z^1 - z_0^1\|}{\beta\alpha} - \sup_{\zeta \in [z_0^1, z^1]} \|G_k'(\zeta) - G_k'(z_0^1)\| \cdot \|z^1 - z_0^1\| \quad \text{by [2, p. 162]} \\
&\quad \geq \frac{\|z^1 - z_0^1\|}{\beta\alpha^2}
\end{aligned}$$

and

$$\begin{aligned}
&\|G_k(z^1) - h(z_0)\| \\
&\quad \leq \|G_k(z^1) - \psi(z^1)\| + \|h(z) - h(z_0)\| \\
&\quad \leq \alpha k \delta t(z^1) + \alpha\|z - z_0\| < r
\end{aligned}$$

so that

$$\begin{aligned}
&h(B_p(z_0)) \cup h(\psi^{-1}h(B_p(z_0))) \cup G_k(\psi^{-1}h(B_p(z_0))) \\
&\quad \subseteq B_r(h(z_0)) \subseteq T_j \quad \text{for some } 1 \leq j \leq s.
\end{aligned}$$

Now Lemma 1.3 gives

$$(G_k(z^1))_i = (\psi(z^1))_i - \sum_{k=1}^{N-n} t_k D_i f_k^{(j)}(p_1\psi(z^1)) \quad \text{and}$$

$$(G_k(z^1))_{n+k} = f_k^{(j)}(p_1\psi(z^1)) + t_k$$

for $1 \leq i \leq n$ and $1 \leq k \leq N - n$, with similar expressions for $G_k(z_0^1)$. We get

$$\begin{aligned}
\|\psi(z^1) - \psi(z_0^1)\|^2 &= \sum_{k=1}^{N-n} (f_k^{(j)}(p_1\psi(z^1)) - f_k^{(j)}(p_1\psi(z_0^1)))^2 \\
&\quad + \sum_{i=1}^{n} \Bigg[(G_k(z^1) - G_k(z_0^1))_i + \sum_{k=1}^{N-n} \{(G_k(z^1) - G_k(z_0^1))_{n+k} \\
&\qquad + (f_k^{(j)}(p_1\psi(z_0^1)) - f_k^{(j)}(p_1\psi(z^1)))\} D_i f_k^{(j)}(p_1\psi(z^1)) \\
&\qquad + ((G_k(z_0^1))_{n+k} - f_k^{(j)}(p_1\psi(z_0^1)))(D_i f_k^{(j)}(p_1\psi(z^1)) \\
&\qquad - D_i f_k^{(j)}(p_1\psi(z_0^1))) \Bigg]^2
\end{aligned}$$

and

$$\begin{aligned}&\|\psi(z^1)-\psi(z_0^1)\|^2(1-10\varepsilon^2n^2(N-n)^2)\\&\quad\le\|G_k(z^1)-G_k(z_0^1)\|^2(1+3(N-n)^2\varepsilon)\\&\qquad+\|G_k(z^1)-G_k(z_0^1)\|\|\psi(z^1)-\psi(z_0^1)\|12(N-n)^2n^{3/2}\varepsilon\end{aligned}$$

hence

$$\begin{aligned}&\|\psi(z^1)-\psi(z_0^1)\|\\&\quad\le\frac{\|G_k(z^1)-G_k(z_0^1)\|}{2(1-10\varepsilon^2n^2(N-n)^2)}\cdot\Big[12(N-n)^2n^{3/2}\varepsilon\\&\qquad+\sqrt{(12(N-n)^2n^{3/2}\varepsilon)^2+4(1-10\varepsilon^2n^2(N-n)^2)(1+3(N-n)^2\varepsilon)}\Big]\\&\quad\le\alpha\|G_k(z^1)-G_k(z_0^1)\|\\&\quad\le\alpha^3\|z^1-z_0^1\|\quad\text{by claim 1}\end{aligned}$$

and

$$\|\psi^{-1}h(z)-\psi^{-1}h(z_0)\|\ge\frac{\|z-z_0\|}{\alpha^4}\ldots\ldots\ldots\ldots\ldots(*).$$

Also

$$\begin{aligned}&\|G_k(z^1)-G_k(z_0^1)\|_n^2\\&\quad=\sum_{i=1}^{n}\Big[(\psi(z^1)-\psi(z_0^1))_i-\sum_{k=1}^{N-n}\{(G_k(z^1)-G_k(z_0^1))_{n+k}\\&\qquad+(f_k^{(j)}(p_1\psi(z_0^1))-f_k^{(j)}(p_1\psi(z^1)))\}D_if_k^{(j)}(p_1\psi(z^1))\\&\qquad-\sum_{k=1}^{N-n}((G_k(z_0^1))_{n+k}-f_k^{(j)}(p_1\psi(z_0^1)))(D_if_k^{(j)}(p_1\psi(z^1))\\&\qquad-D_if_k^{(j)}(p_1\psi(z_0^1)))\Big]^2\\&\quad\le\|\psi(z^1)-\psi(z_0^1)\|^2(1+15n^2(N-n)^2\varepsilon)\\&\qquad+\|G_k(z^1)-G_k(z_0^1)\|^2n(N-n)^2\varepsilon\\&\qquad+\|G_k(z^1)-G_k(z_0^1)\|\,\|\psi(z^1)-\psi(z_0^1)\|4n(N-n)\varepsilon.\end{aligned}$$

Note that

$$
\begin{aligned}
\left(\frac{\|z^1 - z_0^1\|}{\beta\alpha^2}\right)^2 &\le \|G_k(z^1) - G_k(z_0^1)\|^2 \\
&= \|G_k(z^1) - G_k(z_0^1)\|_n^2 \\
&\quad + \sum_{k=1}^{N-n}\left(\frac{1}{a_n}\int \varphi(\|x\|)(h(z^1 + k\delta t(z^1)x) - h(z_0^1 + k\delta t(z_0^1)x))_{n+k}\,d\lambda(x)\right)^2 \\
&\le \|G_k(z^1) - G_k(z_0^1)\|_n^2 + (N-n)(\varepsilon\sqrt{n}\cdot\alpha(1 + k10.2^n)\|z^1 - z_0^1\|)^2 \\
&\le \|\psi(z^1) - \psi(z_0^1)\|^2(1 + 15n^2(N-n)^2\varepsilon) + 8\alpha^2 n(N-n)^2\varepsilon\|z^1 - z_0^1\|^2 \\
&\quad + 4\alpha^2 n(N-n)\varepsilon\|z^1 - z_0^1\|\,\|\psi(z^1) - \psi(z_0^1)\|
\end{aligned}
$$

hence

$$
\begin{aligned}
&\|\psi(z^1) - \psi(z_0^1)\| \\
&\quad \ge \frac{\|z^1 - z_0^1\|}{2(1 + 15n^2(N-n)^2\varepsilon)}\cdot\Big[-4\alpha^2 n(N-n)\varepsilon \\
&\qquad + \sqrt{(4\alpha^2 n(N-n)\varepsilon)^2 + 4(1 + 15n^2(N-n)^2\varepsilon)(1/(\beta\alpha^2)^2 - 8\alpha^2 n(N-n)^2\varepsilon)}\Big] \\
&\quad \ge \frac{\|z^1 - z_0^1\|}{\beta\alpha^3}
\end{aligned}
$$

and

$$\|z - z_0\| \ge \frac{\|z^1 - z_0^1\|}{\beta\alpha^4}$$

or

$$\|\psi^{-1}h(z) - \psi^{-1}h(z_0)\| \le \beta\alpha^4\|z - z_0\| \,\dots\dots\dots\dots\dots\dots\,(^{**}).$$

Now (*) and (**) give

$$\frac{\|z - z_0\|}{\alpha^4} \le \|\psi^{-1}h(z) - \psi^{-1}h(z_0)\| \le \beta\alpha^4\|z - z_0\|. \qquad //.$$

2- Here we also need several lemmas to establish the smoothing criterion.

LEMMA 2.1. *Let M be a connected C^∞ Riemannian n-manifold with Riemannian distance d, then for all $x \in M$ and all $\varepsilon > 0$ there exists some $\delta > 0$ such that $B_\delta(o_x) \subset \Omega \cap T_x(M)$ and*

$$\left|\frac{d(\exp_x(\boldsymbol{h})), \exp_x(\boldsymbol{k})}{\|\boldsymbol{h}-\boldsymbol{k}\|} - 1\right| \leq \varepsilon$$

for all $\boldsymbol{h}$, $\boldsymbol{k} \in B_\delta(\boldsymbol{o}_x)$, $\boldsymbol{h} \neq \boldsymbol{k}$, *and there exists* $c_0 = (X_0, \varphi_0)$, *a* C^∞ *chart of* M, $x \in X_0$, *such that*

$$\left|\frac{d(z,w)}{\|\varphi_0(z) - \varphi_0(w)\|} - 1\right| \leq \varepsilon$$

for all z, $w \in X_0$, $z \neq w$.

PROOF. Note that there exists $0 < r < 1$, $\overline{B_r(\boldsymbol{o}_x)} \subseteq \Omega \cap T_x(M)$ such that for all $0 < \rho \leq r$, $\exp_x| : B_\rho(\boldsymbol{o}_x) \to B_\rho(x)$ is a C^∞ diffeomorphism [5, p. 351] and $B_\rho(x)$ is strictly geodesically convex [5, p. 356] (i.e. $B_\rho(x)$ is convex with respect to the geodesic field of the Levi-Civita connection of M and the induced Riemannian structure on $B_\rho(x)$ has a Riemannian distance $= d|_{B_\rho(x)}$); hence for all z, $w \in B_\rho(x)$ there exists a unique geodesic arc γ_{zw} in $B_\rho(x)$ joining z and w and $d(z,w) = \mathrm{Lt}(\gamma_{zw})$ [5, p. 25, 355] where Lt denotes the length of γ_{zw}.

Let $c = (U, \varphi)$ be a C^∞ chart of M such that $B_r(x) \subset U$ and $\theta_c : \boldsymbol{R}^n \to T_x(M)$ is an isometry which defines a C^∞ structure for $T_x(M)$. Recall that if E is an n-dimentional vector space and $a \in E$ then there exists a canonical bijection $\tau_a : T_a(E) \to E$ [4, p. 23].

Now $c_1 = (B_r(x), \theta_c^{-1} \circ \exp_x^{-1})$ is a C^∞ chart of M and if

$$B_r(x) \times \boldsymbol{R}^n \to o_M^{-1}(B_r(x))$$

$$(z, \boldsymbol{h}) \mapsto \theta_{c_1,z}(\boldsymbol{h})$$

is the fibered chart corresponding to c_1, then the local expression of the Levi-Civita connection with respect to this fibered chart is

$$\boldsymbol{C}_z((z,\boldsymbol{v}),(z,\boldsymbol{w})) = ((z,\boldsymbol{w}),(\boldsymbol{v}, -\Gamma_z(\boldsymbol{v},\boldsymbol{w})))$$

where $\Gamma_z(\boldsymbol{v},\boldsymbol{w}) = \sum_{i,p,q} \Gamma_{pq}^i(z) v_p w_q \boldsymbol{e}_i$ and Γ_{pq}^i are C^∞ functions on $B_r(x)$.

Also $X_i(z) = \theta_{c_1,z}\ (\boldsymbol{e}_i)$, $z \in B_r(x)$, $1 \leq i \leq n$, is a frame of $T(M)$ over $B_r(x)$ and the curvature tensor of the Levi-Civita connection has the expression $(\boldsymbol{r} \cdot (X_j \wedge X_k)) \cdot X_i = \sum_p r_{ijk}^p X_p$ where r_{ijk}^p are C^∞ functions on $B_r(x)$.

Now $\{\theta_c(\boldsymbol{e}_i) : 1 \leq i \leq n\}$ is an orthonormal basis for $T_x(M)$ and if $\boldsymbol{h}_x \in B_r(\boldsymbol{o}_x)$ and if

$$\upsilon : (-r, r) \to B_r(x)$$

$$t \mapsto \exp_x(t\boldsymbol{h}_x)$$

is the corresponding geodesic, then with respect to the fibered chart above $v'(t)$ has the expression $\eta \mapsto (v(t), \eta\theta_c^{-1}(\boldsymbol{h}_x))$.

Also if $\boldsymbol{u}_i$ is the parallel transport of $\theta_c(\boldsymbol{e}_i)$ along the geodesic v for $1 \le i \le n$ then $\{\boldsymbol{u}_i : 1 \le i \le n\}$ form a basis of $T_{v(t)}(M)$ [5, p. 30] and the local expression of the $\boldsymbol{u}_i$'s with respect to the fibered chart above are $t \mapsto (v(t), \boldsymbol{v}_i(t))$ where

$$\boldsymbol{v}_i'(t) + \sum_{j,p,q} \Gamma_{pq}^j(v(t))(\theta_c^{-1}(\boldsymbol{h}_x))_p(\boldsymbol{v}_i(t))_q \boldsymbol{e}_j = 0$$

hence $\boldsymbol{v}_i = \boldsymbol{v}_i(t, \boldsymbol{h}_x)$ is continuous on $(-r, r) \times (B_r(\boldsymbol{o}_x) - \{\boldsymbol{o}_x\})$ by [2, p. 296] so that $\boldsymbol{u}_i(t, \boldsymbol{h}_x) = \sum_{j=1}^n u_{ij}(t, \boldsymbol{h}_x) X_j(v(t))$ where $u_{ij}(t, \boldsymbol{h}_x)$ are continuous on $(-r, r) \times (B_r(\boldsymbol{o}_x) - \{\boldsymbol{o}_x\})$ and by Cramer's rule we get $(\boldsymbol{r} \cdot (\boldsymbol{u}_j \wedge \boldsymbol{u}_k)) \cdot \boldsymbol{u}_p = \sum_{i=1}^n s_{pjk}^i \boldsymbol{u}_i(t)$ where s_{pjk}^i are continuous on $(-r, r) \times (B_r(\boldsymbol{o}_x) - \{\boldsymbol{o}_x\})$ and for each $\boldsymbol{h}_x \in B_r(\boldsymbol{o}_x) - \{\boldsymbol{o}_x\}$, $s_{pjk}^i(\cdot, \boldsymbol{h}_x)$ are C^∞ functions. With these notations and terminology we have the following claim.

Claim. There exists $0 < \delta < r$ such that for all $\boldsymbol{h} \in B_\delta(\boldsymbol{o}_x)$ and all $\boldsymbol{k} \in T_x(M) - \{\boldsymbol{o}_x\}$ we have

$$1 - \varepsilon \le \frac{\|T_{\boldsymbol{h}}(\exp_x)\tau_{\boldsymbol{h}}^{-1}(\boldsymbol{k})\|}{\|\boldsymbol{k}\|} \le 1 + \varepsilon$$

Proof. Since $T_{\boldsymbol{o}_x}(\exp_x) \circ \tau_{\boldsymbol{o}_x}^{-1} = 1$ by [5, p. 22] it suffices to show that there exists $0 < \delta < \rho < r$ such that $1 - \varepsilon < \|T_{t\boldsymbol{h}}(\exp_x) \cdot \tau_{t\boldsymbol{h}}^{-1}(\boldsymbol{k})\| < 1 + \varepsilon$ for $0 < |t| < \delta$ and all $\boldsymbol{h}, \boldsymbol{k} \in T_x(M)$, $\|\boldsymbol{h}\| = \rho$, $\|\boldsymbol{k}\| = 1$. There exists J an open interval containing 0 in $\boldsymbol{R}$ such that $f : (-r, r) \times J \to M$ defined by $f(t, \zeta) = \exp_x t(\boldsymbol{h} + \zeta\boldsymbol{k})$ is a one-parameter family of geodesics and

$$\begin{aligned} \boldsymbol{w} : (-r, r) &\to T(M) \\ t &\mapsto f_\zeta'(t, 0) = tT_{t\boldsymbol{h}}(\exp_x) \cdot \tau_{t\boldsymbol{h}}^{-1}(\boldsymbol{k}) \end{aligned}$$

is the unique Jacobi field along the geodesic $v = f(\cdot, 0)$ such that $\boldsymbol{w}(0) = \boldsymbol{o}_x$ and $(\nabla_E \boldsymbol{w})(0) = \boldsymbol{k}$ [5, p. 36], where $(E(t))$ is the unique vector field on $\boldsymbol{R}$ such that $\tau_t(E(t)) = \boldsymbol{e}_1$ and ∇_E is the covariant derivative, for the Levi-Civita connection on M, of $\boldsymbol{w}$ at t along the tangent vector $E(t)$ [4, p. 321].

Let

$$\boldsymbol{h} = \sum_{i=1}^n h_i \theta_c(\boldsymbol{e}_i), \quad \boldsymbol{k} = \sum_{i=1}^n k_i \theta_c(\boldsymbol{e}_i).$$

We have $w(t) = \sum_{i=1}^{n} w_i(t)u_i(t)$ where w_i's are C^∞ functions, hence

$$\nabla_E w = \sum_{i=1}^{n} w_i'(t)u_i(t)$$

and

$$\nabla_E(\nabla_E w) = \sum_{i=1}^{n} w_i''(t)u_i(t) \quad \text{since } \nabla_E u_i = 0.$$

But $\nabla_E(\nabla_E w) = (r \cdot (v' \wedge w)) \cdot v'$ and $v'(t) = \sum_{i=1}^{n} h_i u_i(t)$ since v is a geodesic, therefore $w_i''(t) = \sum_{k=1}^{n}(\sum_{p,j}^{n} s_{pjk}^{i}(t)h_j h_p)w_k(t)$ with $w_i(0) = 0 = w_i''(0)$, $w_i'(0) = k_i$ and $w_i = w_i(t, h, k)$ is continuous on $(-r, r) \times (B_r(o_x) - \{o_x\}) \times T_x(M)$ by [2, p. 296] for $1 \leq i \leq n$.

Now Taylor formula [2, p. 190] shows that for $1 \leq i \leq n$ $w_i(t) = tk_i + t^2 \cdot \int_0^1 (1 - \zeta)w_i''(\zeta t)\, d\zeta$ hence $\mathrm{Lim}_{t \to 0}\, 1/t^2(w_i(t) - tk_i) = 0$ uniformly for all $h, k \in T_x(M)$, $\|h\| = \rho$, $\|k\| = 1$.

Let $\alpha_i(t) = w_i(t) - tk_i$ for $1 \leq i \leq n$ and $\alpha(t) = \sum_{i=1}^{n} \alpha_i(t)\theta_c(e_i)$ so that $s_t = tk + \alpha(t) \in T_x(M)$. Let $\overline{w}$ be the unique parallel transport along v such that $\overline{w}(0) = s_t$ [5, p. 30], then $\overline{w}(t) = w(t) = \sum_{i=1}^{n} w_i(t)u_i(t)$ and since $\nabla_E \overline{w} = 0$ we get $\|w(t)\| = \|s_t\| = \|tk + \alpha(t)\|$ so that $\mathrm{Lim}_{t \to 0}\|w(t)\|/|t| = 1$ uniformly for all $h, k \in T_x(M)$, $\|h\| = \rho$, $\|k\| = 1$, hence there exists some $0 < \delta < \rho < r$ such that $1 - \varepsilon < \|w(t)\|/|t| < 1 + \varepsilon$ for $0 < |t| < \delta$ and all $h, k \in T_x(M)$, $\|h\| = \rho$, $\|k\| = 1$ as desired. //.

Observe that if $\gamma : [0, 1] \to B_\delta(o_x)$ is any piecewise smooth curve, then

$$\mathrm{Lt}(\gamma) = \sqrt{\int_0^1 \|\theta_c^{-1}\tau_{\gamma(t)}(\exp_x) \cdot (\gamma'(t))\|^2\, dt} \quad \text{and}$$

$$\mathrm{Lt}(\exp_x \circ \gamma) = \sqrt{\int_0^1 \|T_{\gamma(t)}(\exp_x) \cdot (\gamma'(t))\|^2\, dt}$$

so that $\mathrm{Lt}(\gamma) \geq \|\gamma(0) - \gamma(1)\|$ and if further $\tau_{\gamma(t)}(\gamma'(t)) \neq o_x$ for all $0 \leq t \leq 1$ then

$$1 - \varepsilon \leq \frac{\mathrm{Lt}(\exp_x \circ \gamma)}{\mathrm{Lt}(\gamma)} \leq 1 + \varepsilon$$

by the above claim.

Now Let $\boldsymbol{h}, \boldsymbol{k} \in B_\delta(\boldsymbol{o}_x)$, $\boldsymbol{h} \neq \boldsymbol{k}$ and Let $h' = \exp_x(\boldsymbol{h})$, $k' = \exp_x(\boldsymbol{k})$,

$$\gamma_1 : t \mapsto (1-t)\boldsymbol{h} + t\boldsymbol{k},$$

$$\gamma_2 : t \mapsto (\exp_x)^{-1}(\gamma_{h',k'}(t))$$

and $c_0 = (X_0, \varphi_0) = (B_\delta(x), \theta_c^{-1} \circ (\exp_x)^{-1})$. We have

$$\frac{d(h',k')}{\|\varphi_0(h') - \varphi_0(k')\|} = \frac{d(h',k')}{\|\boldsymbol{h} - \boldsymbol{k}\|} \geq \frac{\mathrm{Lt}(\exp_x \circ \gamma_2)}{\mathrm{Lt}(\gamma_2)}$$

and $\mathrm{Lt}(\exp_x \circ \gamma_1)/\mathrm{Lt}(\gamma_1) \geq d(h',k')/\|\boldsymbol{h}-\boldsymbol{k}\|$. Note that $\tau_{\gamma_i(t)}(\gamma_i'(t)) \neq \boldsymbol{o}_x$ for $i = 1,2$ so that $1 - \varepsilon \leq d(h',k')/\|\boldsymbol{h}-\boldsymbol{k}\| \leq 1 + \varepsilon$ as desired. //.

LEMMA 2.2. *Let W open $\subset \overline{W} \subset U$ open connected $\subset B^n$ and Let Y be a C^∞ n-manifold and $h : U \to Y$ an embedding such that $h(U) = \bigcup_{i\geq 1} Y_i$ where for all $i \geq 1$, (Y_i, φ_i) is a C^∞ chart of Y and $\varphi_i \circ h|_{h^{-1}(Y_i)}$ is bilipschitz with $L(\varphi_i \circ h|_{h^{-1}(Y_i)}) \leq \alpha$, $\alpha > 1$, then there exists a Riemannian distance d on $h(U)$ such that $h|_W$ is locally bilipschitz and $L_c(h|_W, \|\cdot\|, d) \leq \alpha^4$.*

PROOF. By [3, p. 20] there exists $\mathscr{A} = \{c_i = (Y_i', \varphi_i') : i \geq 1\}$ a C^∞ atlas of $h(U)$ such that

1- $\{Y_i' : i \geq 1\}$ is a locally finite open cover of $h(U)$.

2- $\overline{Y_i'} \subseteq Y_{j(i)}$, $\varphi_i' = \varphi_{j(i)}|_{Y_i'}$ where $j(i) = \min\{j \geq 1 : \overline{Y_i'} \subset Y_j\}$.

Let $\{f_i : i \geq 1\}$ be a C^∞ partition of unit subordinate to the cover $\{Y_i' : i \geq 1\}$ of $h(U)$ [4, p. 16].

Note that the tangent bundle of $h(U)$ is defined by the family of C^∞ diffeomorphisms [4, p. 104]

$$\sigma_i : Y_i' \times \boldsymbol{R}^n \to o_{h(U)}^{-1}(Y_i')$$

$$(x, h) \mapsto \theta_{c_i, x}(\boldsymbol{h}).$$

Define a Riemannian metric an $h(U)$ by [5, p. 264]

$$\boldsymbol{g}(x) = \sum_{i\geq 1} f_i(x)\boldsymbol{g}_i(x)$$

where $\langle \boldsymbol{g}_i(x), \sigma_i(x, \boldsymbol{h}) \otimes \sigma_i(x, \boldsymbol{k})\rangle = \langle \boldsymbol{h}, \boldsymbol{k}\rangle$, so that if $\gamma : [a, b] \to h(U)$ is any piecewise C^∞ curve then its length is given by

$$\begin{aligned} \mathrm{Lt}(\gamma) &= \int_a^b \sqrt{\langle \boldsymbol{g}(\gamma(t)), \gamma'(t) \otimes \gamma'(t)\rangle}\, dt \\ &= \int_a^b \sqrt{\sum_{i \geq 1} f_i(\gamma(t)) \|D(\varphi_i \circ \gamma)(t)\|^2}\, dt \end{aligned} \tag{2.1}$$

Note that for all $i, j \geq 1$ such that $Y_i' \cap Y_j' \neq \varnothing$ the transition homeomorphism $\varphi_j' \circ \varphi_i'^{-1} : \varphi_i'(Y_i' \cap Y_j') \to \varphi_j'(Y_j' \cap Y_j')$ is bilipschitz and $L(\varphi_j' \circ \varphi_i'^{-1}) \leq \alpha^2$ so that

$$(1/\alpha^3)\|t\| \leq \|D(\varphi_j' \circ \varphi_i'^{-1})(x) \cdot t\| \leq \alpha^3 \|t\| \tag{2.2}$$

for all $x \in \varphi_j'(Y_i' \cap Y_j')$ and all $t \in \boldsymbol{R}^n$.

Let $r =$ Lebesgue number of the cover $\{Y_i' : i \geq 1\}$ of $h(\overline{W})$ with respect to the Riemannian distance d, then there exists a finite family of C^∞ charts $\{(Z_j, \psi_j) : 1 \leq j \leq s\}$ such that:

1- $h(\overline{W}) \subseteq \bigcup_{J=1}^{s} Z_j$

2- For all $1 \leq j \leq s$, $Z_j \subseteq B_{r/4}(z_j)$, $z_j \in h(\overline{W})$, $\psi_j(Z_j)$ is an open ball in $\boldsymbol{R}^n$ where $\psi_j = \varphi_{i(j)}'|_{Z_j}$ and $i(j) = \min\{1 \leq i : B_r(z_j) \subset Y_i'\}$.

Now it suffices to show that for $1 \leq j \leq s$, ψ_j is bilipschitz and $L(\psi_j, d, \|\cdot\|) \leq \alpha^3$. Let $x, y \in Z_j$ and let

$$\begin{aligned} \gamma : [0, 1] &\to \psi_j(Z_j) \\ t &\mapsto (1 - t)\psi_j(x) + t\psi_j(y) \end{aligned}$$

hence

$$\begin{aligned} d(x, y) &\leq \mathrm{Lt}((\psi_j)^{-1} \circ \gamma) \\ &\leq \alpha^3 \|\psi_j(x) - \psi_j(y)\| \quad \text{by equations } 2.1 + 2.2, \end{aligned}$$

also since $d(x, y) < r/2$, there exists a piecewise C^∞ curve $p : [a, b] \to h(U)$ such that $p(a) = x$, $p(b) = y$ and $\mathrm{Lt}(p) < r/2$, hence $d(x)$, $p(t)) \leq \mathrm{Lt}(p) < r/2$ for all $a \leq t \leq b$ and $p([a, b]) \subseteq Y_{i(j)}'$ so that $\mathrm{Lt}(p) \geq 1/\alpha^3 \int_a^b \|D(\varphi_{i(j)} \circ p)(t)\|\, dt \geq 1/\alpha^3 \|\psi_j(x) - \psi_j(y)\|$ and $d(x, y) \geq 1/\alpha^3 \|\psi_j(x) - \psi_j(y)\|$. //.

LEMMA 2.3. *Let Y be a connected C^∞ Riemannian n-manifold with Riemannian distance d and Let $\alpha > 1$ then there exists a locally bilipschitz isometric embedding $\psi : Y \to \boldsymbol{R}^N$ such that $L_c(\psi, d, \|\cdot\|) \leq \alpha$.*

PROOF. By virtue of Nash embedding theorem [9] we may assume that Y is a C^∞ Riemannian submanifold of $\boldsymbol{R}^N$. Let $\varepsilon > 0$ such that

$$\sqrt{\frac{1-\varepsilon^2 n(N-n)}{1+\varepsilon^2 n(N-n)}} \geq \frac{1}{\alpha}.$$

The argument of lemma 1.3 shows that there exists $\mathscr{A} = \{B_j : j \geq 1\}$ a covering of Y such that: for all $j \geq 1$, $B_j = B^N_{s_j}(z_j) \cap Y$, $z_j \in Y$ and modulo an affine transformation of $\boldsymbol{R}^N$ there exists

$$\begin{aligned} \psi_j : B^n_{s_j}(0) &\to B_j \\ x &\mapsto (x, f^{(j)}(x)) \end{aligned}$$

C^∞ diffeomorphism and $f^{(j)}(0) = 0$, $f^{(j)'}(0) = 0$, $\sup\{|D_i f^{(j)}_k(\zeta)| : 1 \leq i \leq n, 1 \leq k \leq N-n, \|\zeta\| < s_j\} < \varepsilon$. Let $\mathscr{B} = \{B_{r_i}(y_i) : i \geq 1\}$ be a refinement of $\mathscr{A}$ by strictly geodesically convex balls [5, p. 356]. Now Let $z, w \in B_{r_i}(y_i) \subset B_j = B^N_{s_j}(z_j) \cap Y \subset \boldsymbol{R}^N$ for some $j \geq 1$. We have $w = \exp_z \boldsymbol{h}_z$ for some $\boldsymbol{h}_z \in T_z(Y)$ and

$$\begin{aligned} \gamma : [0,1] &\to B_{r_i}(y_i) \\ t &\mapsto \exp_z(t\boldsymbol{h}_z) \end{aligned}$$

is the unique geodesic arc from z to w [5, p. 25] and

$$\begin{aligned} d(z,w) &= d(z, \exp_z(\boldsymbol{h}_z)) = \|\boldsymbol{h}_z\| \quad [5,\ \text{p. } 355] \\ &= \mathrm{Lt}(\gamma) \geq \|z-w\|. \end{aligned}$$

Note that

$$\|z-w\|^2 \geq (1-\varepsilon^2(N-n)n)\|z-w\|_n^2$$

and if

$$\begin{aligned} \varphi : I &\to \boldsymbol{R}^n \\ t &\mapsto (1-t)p_1(z) + tp_1(w) \end{aligned}$$

then

$$\begin{aligned} \|z-w\|_n &\geq \frac{1}{\sqrt{1+\varepsilon^2 n(N-n)}} \int_0^1 \|(\psi_j \circ \varphi)'(t)\|\, dt \\ &= \frac{\mathrm{Lt}(\psi_j \circ \varphi)}{\sqrt{1+\varepsilon^2 n(N-n)}} \geq \frac{d(z,w)}{\sqrt{1+\varepsilon^2 n(N-n)}} \end{aligned}$$

and $\sqrt{\dfrac{1-\varepsilon^2 n(N-n)}{1+\varepsilon^2 n(N-n)}}\, d(z,w) \leq \|z-w\| \leq d(z,w)$. Our result follows. //.

LEMMA 2.4. *Let W open $\subset \overline{W} \subset U$ open $\subset B^n$ and Let Y be a C^∞ n-manifold and let $h : U \to Y$ be an embedding. Suppose that there exists C^∞ charts of Y, $c_i = (Y_i, \varphi_i)$, $i \geq 1$, such that $h(U) = \bigcup_{i\geq 1} Y_i$ and for each $i \geq 1$ $\varphi_i \circ h|_{h^{-1}(Y_i)}$ is Locally bilipschitz with $L_c(\varphi_i \circ h|_{h^{-1}(Y_i)}) \leq \alpha$, $\alpha > 1$, then there exists a proper C^∞ embedding $\psi : Y \to \boldsymbol{R}^N$ such that $\psi \circ h|_W$ is locally bilipschitz with $L_c(\psi \circ h|_W) \leq \alpha^5$.*

PROOF. By Whitney embedding theorem [4, p. 185] it suffices to show that there exists a C^∞ embedding $\psi : h(U) \to \boldsymbol{R}^N$ such that $\psi \circ h|_W$ is locally bilipschitz with $L_c(\psi \circ h|_W) \leq \alpha^5$. We may assume that U is connected and that $\varphi_i \circ h|_{h^{-1}(Y_i)}$ is bilipschitz with $L(\varphi_i \circ h|_{h^{-1}(Y_i)}) \leq \alpha$. Now it suffices to invoke lemma 2.2 + lemma 2.3 to establish our claim. //.

Now we can establish our smoothing criterion.

THEOREM 2.5. *Let X be a topological n-manifold and let α_0 satisfies $0 < 2\alpha_0^2(\alpha_0^2 - 1)(n - 1 + 9n^2(n + 5\sqrt{n})) < 1$ then the following statements are equivalent*:

1- *X has a smooth structure.*

2- $\inf\{L(\mathscr{A}) : \mathscr{A}$ *Lipschitz atlas of $X\} = 1$,*

3- *X has a Lipschitz atlas $\mathscr{A}$ with $L(\mathscr{A}) \leq \alpha$,*

4- *X has a locally Lipschitz atlas $\mathscr{A}$ with $L_c(\mathscr{A}) \leq \alpha$, where*

$$\alpha^5(\beta_n(\alpha)\alpha^{20})^{5/19((20)^{n-1}-1)} \leq \alpha_0$$

and

$$\beta_n(\alpha) = \left(\sqrt{1 - 2\alpha^2(\alpha^2-1)(n-1)} - 3n\sqrt{2\alpha^2(\alpha^2-1)(n+5\sqrt{n})}\right)^{-1}$$

PROOF. Clearly we may assume that X is connected. 1) $\Rightarrow$ 2) Give X a Riemannian structure [5, p. 264] so that X is a connected C^∞ Riemannian n-manifold with Riemannian distance d.

Let $0 < \varepsilon < 1$, then by lemma 2.1 for all $x \in X$, there exists a C^∞ chart of X, $c_x = (X_x, \varphi_x)$, $x \in X_x$, such that φ_x is bilipschitz and

$$L(\varphi_x, d, \|\cdot\|) \le 1 + 1/3\varepsilon.$$

The collection of these C^∞ charts form a Lipschitz atlas of X with Lipschitz size $\le 1 + \varepsilon$.

Now it suffices to show that 4) $\Rightarrow$ 1). Let $\mathscr{B} = \{(V_i, \varphi_i) : i \ge 1\}$ be a locally Lipschitz atlas of X with $L_c(\mathscr{B}) \le \alpha$ and let $\varphi_i(V_i) = W_i \subset B^n$ for all $i \ge 1$. By abuse of notation we also let $\mathscr{B} = \{V_i : i \ge 1\}$. Since X has a Lebesgue covering dimension $= n$ by [8, p. 17, 27, 97], we may assume by Milnor lemma [10, lemma 2.4] that $\mathscr{B}$ is locally finite and that $\mathscr{B} = \bigcup_{k=0}^{n} \mathscr{A}_k$ where each $\mathscr{A}_k = \{V_i : i \equiv k \operatorname{mod}(n+1)\}$, a pairwise dijoint subfamily of $\mathscr{B}$.

Let $\mathscr{B}^i$, $0 \le i \le n$, be open covers of X such that $\mathscr{B}^0 = \mathscr{B}$, $\mathscr{B}^i = \{V_j^i : j \ge 1\}$, $V_j^0 = V_j$ for $j \ge 1$ and $\mathscr{B}^{i+1}$ is a shrinking of $\mathscr{B}^i$ for $0 \le i < n$ (i.e $\overline{V_j^{i+1}} \subseteq V_j^i$ for all $j \ge 1$) by [3, p. 21]. For j, $k \ge 1$, $0 \le i < n$ let $\overline{V_j^{i+1}} \subseteq V_j^{i'}$ open $\subset \overline{V_j^{i'}} \subset V_j^i$, $W_j^{i'} = \varphi_j(V_j^{i'})$ and $W_{jk}^{i'} = \varphi_k(V_j^{i'} \cap V_k^{i'})$. Also for j, $k \ge 1$, $0 \le i \le n$ Let $W_j^i = \varphi_j(V_j^i)$, $\varphi_j^i = \varphi_j|_{V_{jk}^i}$, $W_{jk}^i = \varphi_k(V_j^i \cap V_k^i)$ and $h_{jk}^i : W_{jk}^i \to W_{kj}^i$ be defined by $h_{jk}^i = \varphi_j \circ \varphi_k^{-1}|_{W_{jk}^i}$ so that for all $0 \le i \le n$ $X = (\sum_0 W_j^i \oplus \sum_1 W_j^i \oplus \cdots \oplus \sum_n W_j^i) \operatorname{mod}(\{W_{kj}^i\}, \{h_{kj}^i\})$ where $\sum_k$ denotes the topological sum over all $j \equiv k \operatorname{mod}(n+1)$.

To construct a C^∞ structure on X it suffices to establish the following assertion: For all $0 \le i \le n$, Let $X^i = (\sum_0 W_j^i \oplus \sum_1 W_j^i \oplus \cdots \oplus \sum_i W_j^i) \cdot \operatorname{mod}(\{W_{kj}^i\}, \{h_{kj}^i\})$ and Let $p_j^i : W_j^i \to X^i$ be the corresponding open embedding for all $j \equiv k \le i \operatorname{mod}(n+1)$, then there exists a homeomorphism

$$g^i : X^i \to B^i = \left(\sum_0 B_j \oplus \cdots \oplus \sum_i B_j\right) \operatorname{mod}(\{B_{kj}\}, \{f_{kj}\})$$

where B_j open $\subset B^n$ for all $j \ge 1$, B_{kj} open $\subset B_j$ for all $j, k \ge 1$ and $f_{kj} : B_{kj} \to B_{jk}$ is a C^∞ diffeomorphism such that

1- $B_{jj} = B_j$, $f_{jj} = \mathrm{id}$

2- For all $j, k, m \ge 1$ we have a commutative diagram of C^∞ diffeomorphisms

$$\begin{tikzcd} B_{kj} \cap B_{mj} \arrow[rr, "f_{mj}^{k}"] \arrow[dr, "f_{kj}^{m}"'] & & B_{jm} \cap B_{km} \\ & B_{mk} \cap B_{jk} \arrow[ur, "f_{km}^{j}"'] & \end{tikzcd}$$

where $f_{mj}^k = f_{mj}|_{B_{kj} \cap B_{mj}}$ and for all $s \equiv k < i \bmod(n+1)$ we have a commutative diagram of homeomorphisms defining g_s^i,

$$\begin{array}{ccc} p_s^i(W_s^i) & \xrightarrow{g^i|} & q_s(B_s) \\ {\scriptstyle p_s^i}\uparrow & & \uparrow{\scriptstyle q_s} \\ W_s^i & \xrightarrow{g_s^i} & B_s \end{array}$$

where $q_s : B_s \to B^i$ is the corresponding open embedding and g_s^i is locally bilipschitz with

$$L_c(g_s^i) \leq (\beta_n(\alpha)\alpha^{20})^{1/19((20)^i - 1)} = I_i.$$

We proceed by induction on i, $0 \leq i \leq n$. If $i = 0$, there is nothing to prove since $\mathscr{A}_0$ is a disjoint family of open subsets of X which are chart domains.

Assume our assertion holds for some $0 \leq i < n$. For all $j \equiv i + 1 \bmod(n+1)$ define the embedding

$$h_j : \bigcup_i W_{sj}^i \to B_i$$

by $h_j|_{W_{sj}^i} = g^i \circ p_s^i \circ h_{sj}^i = q_s \circ g_s^i \circ h_{sj}^i$ where $\bigcup_i$ denotes the union of all sets indexed by s where $s \equiv k \leq i \bmod(n+1)$.

By definition of g^i, h_j is clearly well-defined and, since $\mathscr{B}$ is locally finite, $\overline{\bigcup_i W_{sj}^{i+1}} = \bigcup_i \overline{W_{sj}^{i+1}}$ compact $\subseteq \bigcup_i W_{sj}^{i'}$ for all $j \equiv i + 1 \bmod(n+1)$. Also $\{(q_s(B_s), q_s^{-1}) : s \equiv k \leq i \bmod(n+1)\}$ is a C^∞ atlas of B^i and by the induction hypothesis $q_s^{-1} \circ h_j|_{W_{sj}^i}$ is locally bilipschitz with $L_c(q_s^{-1} \circ h_j|_{W_{sj}^i}) \leq I_i.\ \alpha$ for all $s \equiv k \leq i \bmod(n+1)$.

Now apply lemma 2.4 with the substitution $W \mapsto \overline{\bigcup_i W_{sj}^{i'}}$, $U \mapsto \bigcup_i W_{sj}^i$, $h \mapsto h_j$ and $Y \mapsto B^i$, then there exists a proper C^∞ embedding $\psi : B^i \to \boldsymbol{R}^N$ such that $\psi \circ h_j|_{\bigcup_i W_{sj}^{i'}}$ is locally bilipschitz with $L_c(\psi \circ h_j|_{\bigcup_i W_{sj}^{i'}}) \leq (I_i\alpha)^5$. Since $(I_i\alpha)^5 \leq \alpha_0$ by hypothesis, the smoothing theorem, theorem 1.10, provides a homeomorphism $\psi_1 : \bigcup_i W_{sj}^{i'} \to h_j(\bigcup_i W_{sj}^{i'}) \subset B^i$ such that $\psi_1|_{\bigcup_i W_{sj}^{i+1}}$ is a C^∞ diffeomorphism and $h_j' : W_j^i \to W_j^i$ defined by

$$h_j'(x) = \begin{cases} \psi_1^{-1} h_j(x) & \text{if } x \in \bigcup_i W_{sj}^{i'} \\ x & \text{otherwise} \end{cases}$$

is a locally bilipschitz homeomorphism and $L_c(h_j') \leq \beta_n(\alpha)(I_i\alpha)^{20} = I_{i+1}$.

Now we have the following commutative diagram where the horizontal arrows are homeomorphisms.

$$X^i \oplus \sum_{i+1} W_j^i \xrightarrow{g^i \oplus \sum_{i+1} h_j} B^i \oplus \sum_{i+1} W_j^i$$

$$\text{quotient map} \downarrow \qquad\qquad \downarrow \text{quotient map}$$

$$\Big(X^i \oplus \sum_{i+1} W_j^i\Big) \operatorname{mod}(\{W_{sj}^i\},\{p_s^i h_{sj}^i\}) \to \Big(B^i \oplus \sum_{i+1} W_j^i\Big) \operatorname{mod}(\{h_j'(W_{sj}^i)\},\{g^i p_s^i h_{sj}^i h_j'^{-1}\})$$

$$\text{Remark B} \Big\| \qquad\qquad \Big\| \text{Remark B}$$

$$\Big(\sum_0 W_j^i \oplus \cdots \oplus \sum_{i+1} W_j^i\Big) \operatorname{mod}(\{W_{ts}^i\},\{h_{ts}^i\}) \to \Big(\sum_0 B_j \oplus \cdots \oplus \sum_{i+1} B_j\Big) \operatorname{mod}$$
$$(\{B_{ts}, h_j'(W_{sj}^i) : t,s \equiv k \le i \operatorname{mod}(n+1), j \equiv i+1 \operatorname{mod}(n+1)\},$$
$$\{f_{ts}, g_s^i h_{sj}^i h_j'^{-1} : t,s \equiv k \le i \operatorname{mod}(n+1), j \equiv i+1 \operatorname{mod}(n+1)\})$$

$$\begin{matrix}\text{Open} \\ \text{embedding} \\ \text{Remark A}\end{matrix} \uparrow \qquad\qquad \uparrow \begin{matrix}\text{Open} \\ \text{embedding} \\ \text{Remark A}\end{matrix}$$

$$X^{i+1} \xrightarrow{g^{i+1}} B^{i+1} = \Big(\sum_0 B_j' \oplus \cdots \oplus \sum_{i+1} B_j'\Big) \operatorname{mod}$$
$$(\{B_{ts}', h_j'(W_{sj}^{i+1}) : t,s \equiv k \le i \operatorname{mod}(n+1), j \equiv i+1 \operatorname{mod}(n+1)\})$$
$$\{f_{ts}', g_s^{i+1} h_{sj}^{i+1} h_j'^{-1} : t,s \equiv k \le i \operatorname{mod}(n+1), j \equiv i+1 \operatorname{mod}(n+1)\})$$

where $B_s' = g_s^i(W_s^{i+1})$ for $s \equiv k \le i \operatorname{mod}(n+1)$, $B_j = W_j^i$, $B_j' = h_j'(W_j^{i+1})$, for $j \equiv i+1 \operatorname{mod}(n+1)$, $B_{ts}' = g_s^i(W_{ts}^{i+1})$, $f_{ts}' = f_{ts}|_{B_{ts}'}$ and $g_s^{i+1} = g_s^i|_{W_s^{i+1}}$.

Note that $g_s^{i+1} \circ h_{sj}^{i+1} \circ h_j'^{-1} : h_j'(W_{sj}^{i+1}) \to g_s^{i+1}(W_{js}^{i+1})$ and $g_s^{i+1} \circ h_{sj}^{i+1} \circ h_j'^{-1} = q_s^{-1} \circ \psi_1$ a C^∞ diffeomorphism for $j \equiv i+1 \operatorname{mod}(n+1)$ and we have a commutative diagram of homeomorphisms

$$\begin{array}{ccc} p_j^{i+1}(W_j^{i+1}) & \xrightarrow{g^{i+1}|} & q_j(B_j') \\ {\scriptstyle p_j^{i+1}} \uparrow & & \uparrow {\scriptstyle q_j} \\ W_j^{i+1} & \xrightarrow{g_j^{i+1} = h_j'} & B_j' \end{array}$$

and g_j^{i+1} is locally bilipschitz with $L_c(g_j^{i+1}) \le I_{i+1}$ as desired. //.

REMARK 2.6. We can give an alternative proof to the implication 1) $\Rightarrow$ 2) in theorem 2.5 that is independent of lemma 2.1 as follows.

Let $\sigma > 0$. We have X a C^∞ n-manifold and we let $e : X \to \boldsymbol{R}^N$ be a proper C^∞ embedding and $\pi/2 > \psi \geq \arcsin 1/(1+1/3\sigma)$, then for $\varepsilon < \cos\psi/(2(N-n)\cdot(n+\sqrt{n(N-n)}))$ we let T be the tubular neighbourhood of X in $\boldsymbol{R}^N$ constructed in lemma 1.3. We keep the notations of that Lemma. Define the function $\varphi : e(X) \to \boldsymbol{G}_{N,N-n}(\boldsymbol{R})$, where $\boldsymbol{G}_{N,N-n}(\boldsymbol{R})$ is the real Grassmannian of indicies $N, N-n$, as follows. For all $y \in e(X)$, $y \in T_j$ for some $j \geq 1$, set $\varphi(y) =$ the vector subspace of $\boldsymbol{R}^N$ generated by the set $(\pi^{-1}(y) \cap T_j) - y$. Clearly φ is well-defined and $\varphi(y) \in \boldsymbol{G}_{N,N-n}(\boldsymbol{R})$ for all $y \in e(X)$. Note that

$$\varphi(y) = \sum_{k=1}^{N-n} \boldsymbol{R}\left(\boldsymbol{e}_{n+k} - \sum_{i=1}^{n} D_i f_k^{(j)}(p_1(y))\boldsymbol{e}_i\right)$$

and since $\boldsymbol{G}_{N,N-n}(\boldsymbol{R}) = \boldsymbol{GL}_N(\boldsymbol{R})/(\boldsymbol{GL}_{N-n}(\boldsymbol{R}) \times \boldsymbol{GL}_n(\boldsymbol{R}) \times \boldsymbol{R}^{n(N-n)})$ [4, p. 70], the function $\varphi|_{T_j \cap e(X)}$ factors as the composition

$$T_j \cap e(X) \xrightarrow{\varphi_1} \boldsymbol{GL}_N(\boldsymbol{R}) \xrightarrow{\pi_1} \boldsymbol{G}_{N,N-n}(\boldsymbol{R})$$

where π_1 is the canonical C^∞ submersion and

$$\varphi_1(y) = \left[\begin{array}{c|c} -(D_i f_k^{(j)}(p_1(y)))^T & I_n \\ \hdashline I_{N-n} & (D_i f_k^{(j)}(p_1(y)))^T \end{array}\right]$$

so that the function φ is C^∞.

Now it suffices to show that for all $y \in e(X)$ there exists a chart (W_y, φ_y) such that $y \in W_y$ and φ_y is bilipschitz with $L(\varphi_y) \leq 1 + 1/3\sigma$. Let $y \in e(X)$ and let W_y be an open neighbourhood of y in $e(X)$ such that $W_y \subseteq (a_j + 1/2\varepsilon_j B^N) \cap e(X)$ for some $j \geq 1$, hence for all z, $w \in W_y$, $z \neq w$, we have $\|z-w\|_{N-n} < \sqrt{n(N-n)}\varepsilon\|z-w\|_n$ and since $\varphi(y) = \sum_{k=1}^{N-n} \boldsymbol{R}(\boldsymbol{e}_{n+k} - \sum_{i=1}^{n} D_i f_k^{(j)}(p_1(y))\boldsymbol{e}_i)$ where $\max\{|D_i f_k^{(j)}(p_1(y))| : 1 \leq i \leq n, 1 \leq k \leq N-n\} < \varepsilon$ and $\varepsilon < \cos\psi/4(N-n)(n+\sqrt{n(N-n)})$ we get, by the proof of lemma 1.7, $A(z-w, \varphi(y)) \geq \psi$.

This shows that the map φ constructed above is a C^∞ transverse field of X in $\boldsymbol{R}^N$ with respect to the embedding $e : X \to \boldsymbol{R}^N$ [see 13].

Let $\varphi_y = \theta \circ P_{(\varphi(y))^\perp}|_{W_y}$ where θ is an arbitrary isometry of $P_{(\varphi(y))^\perp}$ onto $\boldsymbol{R}^n$, then $\|\varphi_y(z) - \varphi_y(w)\| = \|P_{(\varphi(y))^\perp}(z-w)\| = \|z-w\| \sin A(z-w, \varphi(y))$ and $\|z-w\| \geq \|\varphi_y(z) - \varphi_y(w)\| \geq \|z-w\| \sin\psi \geq \|z-w\|/(1+1/3\sigma)$ so that (W_y, φ_y) is indeed a chart of $e(X)$ and φ_y is bilipschitz with $L(\varphi_y) \leq 1 + 1/3\sigma$ as desired.

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