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ON LAGRANGIAN H-UMBILICAL SURFACES IN $CP^2(\tilde{c})$

By

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Abstract. A Lagrangian H-umbilical surface M is an isotropic surface in $CP^2(\tilde{c})$ if and only if M is a minimal surface in $CP^2(\tilde{c})$.

1. Introduction

Let M be an n-dimensional submanifold of a complex m-dimensional Kaehler manifold $\tilde{M}$ with complex structure J and Kaehler metric g. A submanifold M of a Kaehler manifold $\tilde{M}$ is said to be *totally real* if each tangent space of M is mapped into the normal space by the complex structure of $\tilde{M}$. The totally real submanifold M of $\tilde{M}$ is called *Lagrangian* if $n = m$. A Kaehler manifold of constant holomorphic sectional curvature $\tilde{c}$ is called a *complex space form* and will be denoted by $\tilde{M}(\tilde{c})$. Let $CP^m(\tilde{c})$ be a complex m-dimensional complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature $\tilde{c}$. Chen and Ogiue [1] classified totally umbilical submanifolds in $\tilde{M}(\tilde{c})$ $(\tilde{c} \neq 0)$ and proved that $\tilde{M}^m(\tilde{c})$ $(\tilde{c} \neq 0)$ $(m \geq 2)$ admits no totally umbilical, Lagrangian submanifolds except the totally geodesic ones. Recently, Chen [2] introduced the notion of Lagrangian H-umbilical submanifolds which is the simplest totally real submanifolds next to the totally geodesic ones in $\tilde{M}(\tilde{c})$ and classified Lagrangian H-umbilical submanifolds in $\tilde{M}(\tilde{c})$.

A *Lagrangian H-umbilical* submanifold of a Kaehler manifold $\tilde{M}^n$ is a non-totally geodesic Lagrangian submanifold whose second fundamental form takes the following simple form;

$$\begin{aligned} &\sigma(e_1, e_1) = \lambda J e_1, \quad \sigma(e_2, e_2) = \cdots = \sigma(e_n, e_n) = \mu J e_1 \\ &\sigma(e_1, e_j) = \mu J e_j, \quad \sigma(e_j, e_k) = 0, \quad j \neq k, j, k = 2, \ldots, n \end{aligned} \tag{1.1}$$

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for some suitable functions λ, μ with respect to some suitable orthonormal local frame field $\{e_i\}$.

From Theorem in Matsuyama [5], we see that any non-totally geodesic, minimal Lagrangian submanifold $M^n (n : \text{even})$ in $CP^n(\tilde{c})$ which has at most two principal curvatures in the direction of any normal is constant isotropic submanifold in $CP^n(\tilde{c})$ $(n \geq 4)$ or minimal Lagrangian H-umbilical surface in $CP^2(\tilde{c})$.

The aim of this paper is to study Lagrangian H-umbilical surfaces in terms of isotropic.

THEOREM 1.1. *Let M be a Lagrangian H-umbilical surface in $CP^2(\tilde{c})$. M is an isotropic surface in $CP^2(\tilde{c})$ if and only if M is a minimal surface in $CP^2(\tilde{c})$.*

COROLLARY 1.1. *A constant isotropic Lagrangian H-umbilical surface in $CP^2(\tilde{c})$ is locally congruent to a flat torus.*

COROLLARY 1.2. *An isotropic Lagrangian H-umbilical surface with constant scalar normal curvature in $CP^2(\tilde{c})$ is locally congruent to a flat torus.*

REMARK 1.1. More generally, Montiel and Urbano [6] completely classified a complete constant isotropic Lagrangian submanifold M^n in $CP^n(\tilde{c})$.

REMARK 1.2. Very recently, Chen [3] showed that non-totally geodesic minimal Lagrangian surfaces in any Kaehler surface are Lagrangian H-umbilical.

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2. Preliminaries

Let ∇ (resp. $\tilde{\nabla}$) be the covariant differentiation on M (resp. $\tilde{M}$). We denote by σ the second fundamental form of M in $\tilde{M}$. Then the Gauss formula and the Weingarten formula are given respectively by $\sigma(X, Y) = \tilde{\nabla}_X Y - \nabla_X Y$, $\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi$ for vector fields X, Y tangent to M and a normal vector field ξ normal to M, where $-A_\xi X$ (resp. $D_X \xi$) denotes the tangential (resp. normal) component of $\tilde{\nabla}_X \xi$. Let $\zeta = (1/n) \operatorname{trace} \sigma$ and $H = |\zeta|$ denote the mean curvature vector and the mean curvature of M in $\tilde{M}$, respectively. If the second fundamental form σ satisfies $\sigma(X, Y) = g(X, Y)\zeta$, then M is said to be *totally umbilical* submanifold in $\tilde{M}$. If the second fundamental form σ satisfies $g(\sigma(X, Y), \zeta) = g(X, Y)g(\zeta, \zeta)$, then M is said to be *pseudo-umbilical* submanifold in $\tilde{M}$. The submanifold M of $\tilde{M}$ is said to be a *λ-isotropic* submanifold if $|\sigma(X, X)| = \lambda$ for all unit tangent vectors X at each point.

We denote by $\tilde{R}$ and R the Riemannian curvature for $\tilde{\nabla}$ and ∇ respectively. Then the Gauss equation is given by

(2.1)
$$g(\tilde{R}(X, Y)Z, W) = g(R(X, Y)Z, W) + g(\sigma(X, Z), \sigma(Y, W)) - g(\sigma(Y, Z), \sigma(X, W))$$

for all vector fields X, Y, Z and W tangent to M. We denote by $\tilde{M}(\tilde{c})$ a complex m-dimensional complex-space-form of constant holomorphic sectional curvature $\tilde{c}$. We have

$$\begin{aligned}\tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z} = (\tilde{c}/4)\{&g(\tilde{Y}, \tilde{Z})\tilde{X} - g(\tilde{X}, \tilde{Z})\tilde{Y} + g(J\tilde{Y}, \tilde{Z})J\tilde{X} \\ &- g(J\tilde{X}, \tilde{Z})J\tilde{Y} + 2g(\tilde{X}, J\tilde{Y})J\tilde{Z}\}\end{aligned} \tag{2.2}$$

for all vector fields $\tilde{X}, \tilde{Y}$ and $\tilde{Z}$ on $\tilde{M}(\tilde{c})$.

We prepare the following result.

THEOREM 2.1 [4]. *Let M be an n-dimensional real space form of constant curvature c. If M is an isotropic Lagrangian submanifold of $CP^n(\tilde{c})$, then M is parallel. Thus M is totally geodesic or $n = 2$ and M is locally congruent to a flat torus $T^2(c = 0)$.*

3. Proof of Theorem 1.1

Let M be a Lagrangian H-umbilical surface in $CP^2(\tilde{c})$. We choose a local orthonormal frame field

$$e_1, e_2, e_3 = Je_1, e_4 = Je_2$$

of $CP^2(\tilde{c})$ such that e_1, e_2 are tangent to M. By (1.1), the surface in $CP^2(\tilde{c})$ satisfies

$$\begin{cases} \sigma(e_1, e_1) = \lambda e_3 \\ \sigma(e_1, e_2) = \qquad \mu e_4 \\ \sigma(e_2, e_2) = \mu e_3 \end{cases} \tag{3.1}$$

for some suitable functions λ and μ with respect to some suitable orthonormal local frame field $\{e_i\}$. Now, the Gauss curvature K is given by

$$K = g(R(e_1, e_2)e_2, e_1) \tag{3.2}$$

By (2.1), (2.2) and (3.2) we get the Gauss curvature

$$K = \tilde{c}/4 + \sum_{\alpha=3}^{4}\{h_{11}^{\alpha}h_{22}^{\alpha} - (h_{12}^{\alpha})^2\} \tag{3.3}$$

where $h_{ij}^{\alpha} = g(\sigma(e_i, e_j), e_\alpha)$.

By (3.1) and (3.3) we have

$$K = \tilde{c}/4 + \mu(\lambda - \mu) \tag{3.4}$$

By (3.1), for any unit tangent vector $e = (ke_1 + le_2)/\sqrt{k^2 + l^2}$, where k, l are some real numbers, we get (see [7])

$$|\sigma(e, e)|^2 = (k^4\lambda^2 + 2k^2l^2\lambda\mu + l^4\mu^2 + 4k^2l^2\mu^2)/(k^2 + l^2)^2 \tag{3.5}$$

On the other hand, we get

$$|\sigma(e_1, e_1)|^2 = \lambda^2 \tag{3.6}$$

$$|\sigma(e_2, e_2)|^2 = \mu^2 \tag{3.7}$$

If the surface is isotropic, by (3.6) and (3.7) we have

$$\mu = \pm\lambda$$

The case (i): $\mu = \lambda$

By (3.4), we get nonzero constant Gauss curvature $K = \tilde{c}/4$. By Theorem 2.1, we see that the Lagrangian H-umbilical surface is a totally geodesic surface in $\boldsymbol{C}P^2(\tilde{c})$. This is a contradiction for definition (1.1).

The case (ii): $\mu = -\lambda$

We see that the surface is minimal in $\boldsymbol{C}P^2(\tilde{c})$.

Conversely, if the surface is a minimal surface, then $\mu = -\lambda$ and by (3.5), we get

$$|\sigma(e, e)|^2 = \lambda^2$$

This completes the proof of Theorem 1.1.

Now, we shall show Corollary 1.1. Since the surface M is constant λ-isotropic, by Theorem 1.1 we see that M is minimal and $\mu = -\lambda$. So, by (3.4) we have constant Gauss curvature $K = \tilde{c}/4 - 2\lambda^2$. Thus, the assertion of Corollary 1.1 follows immediately from Theorem 2.1.

Now, we shall show Corollary 1.2. The scalar normal curvature is given by

$$K_N = \sum_{\alpha,\beta=3}^{4} \left\{ \sum_{i=1}^{2} (h_{1i}^{\alpha} h_{2i}^{\beta} - h_{1i}^{\beta} h_{2i}^{\alpha}) \right\}^2 \tag{3.8}$$

Since the surface M is isotropic, by Theorem 1.1 we see that M is minimal and $\mu = -\lambda$. So, by (3.1) and (3.8) we have $K_N = 4\lambda^4$. Thus the assertion of Corollary 1.2 follows from Corollary 1.1.

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