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ON GOLDBACH NUMBERS IN ARITHMETIC PROGRESSIONS

By

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1. Introduction

The Goldbach conjecture says that every large even integer is representable as the sum of two primes. If an even integer is written in this manner, we call it Goldbach number. In 1952 Ju. V. Linnik investigeted the distribution of Goldbach numbers. Let $G(q,k)$ denote the least Goldbach number g in an arithmetic progression $g \equiv k \pmod{q}$. Obviously, if q is even then k must be even. He [16] showed that

$$G(q,k) \ll q(\log q)^5$$

under the extended Riemann hypothesis.

In 1968 M. Jutila [13, 14] unconditionally proved that, for any $\varepsilon > 0$,

$$G(q,k) \ll q^{6/5+\varepsilon} \tag{1}$$

subject to prime q. The exponent 6/5 comes from the zero-density estimate for L-functions by M. N. Huxley [9]. The restriction to prime moduli is caused by I. M. Vinogradov's bound [24, p. 360] for character sums over shifted primes:

$$\sum_{p \le x} \chi(p-k) \tag{2}$$

where χ is a non-principal character to modulo q and $(k,q)=1$. Also see A. A. Karatsuba [15] and A. Hildebrand [8].

In 1986 Z. Kh. Rakhmonov [21] successfully estimated (2) for composite q, so as to remove the restriction on q from (1). Later he [22] gave, for prime moduli q, a simple proof of (2) with the exponent 3/2 in place of 6/5.

In the same spirit as H. Iwaniec and M. Jutila [12], we apply the sieve methods to this problem and present an improvement upon (1).

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THEOREM. *Let $J > 14/13$ be given. Then we have*

$$G(q,k) \ll q^J$$

for all prime $q \neq 2$ and all $1 \le k \le q$. Here the implied constant depends only on J.

Recently A. Perelli [19] studied the distribution of Goldbach numbers over polynomial sequences and showed some surprising results. However the estimates depend on the coefficients of polynomials. Our problem is then independent of [19].

We adapt the sieve identity method, which has been developed by several authors, see [2, 5, 6] for instance. Fatally we encounter the numerical integrations to compute. We control the arising patameters so that all numerical calculations can be done by hand. This leads us to the exponent 14/13. So there would be some room for a further refinement, by pushing our method more. However we stop at this place.

Inspired by K. Alladi [1, Lemma 3], we examine the incomplete sum

$$\sum_{\substack{d|n \\ d<\sqrt{n}}} \mu(d), \tag{3}$$

which seems to be easily handled, as the divisor function. And besides, the above sum vanishes whenever $\mu(n) = 1$, vide [23, Chap. 2, ex. 25]. We introduce (3) into the sieve of Eratosthenes. This produces a sort of sieve identity, which one may regard as a weighted sieve with Kuhn's type constant weight or a truncated iteration of Buchstab's identity, see [4, 17]. This observation makes a feature of this paper. Thus our aim is, in turn, to explore and to illustrate the effect of (3) on finding primes.

In addition, we employ the reversal rôles trick due to J.-r. Chen, vide [18], together with the Rosser-Iwaniec upper bound sieve [10, 17]. To deal with the remainder terms arising from the sieve methods, we use an idea of D. R. Heath-Brown and H. Iwaniec [7, 11].

We change a little the usual notation in the sieve theory. This will be explained in the next section. Except these, we use the standard notation in Number Theory. Especially, the letter p is reserved for primes. We write some absolute constant by using the letter K, which is not the same at each appearance. $a \equiv b(q)$ is short for $a \equiv b \pmod q$. $n \sim N$ means $N_1 < n \le N_2$ with some $N \le N_1$, $N_2 \le 2N$. For a set S, $|S|$ stands for its cardinality. We use the abbreviation $L = \log x$.

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2. Sieve of Eratosthenes

To begin with, for $z \geq 2$, we introduce the arithmetical functions:

$$\Phi_z(n) = \begin{cases} 1, & \text{if } p|n \text{ implies } p \geq z \text{ or } n = 1 \\ 0, & \text{otherwise,} \end{cases}$$

$$\Psi_z(n) = \begin{cases} 1, & \text{if } p|n \text{ implies } p < z \text{ or } n = 1 \\ 0, & \text{otherwise.} \end{cases}$$

We notice that, in the usual notation in the sieve theory, $S(\mathscr{A}, z) = \sum_{n \in \mathscr{A}} \Phi_z(n)$, and the sieve of Eratosthenes reads $\Phi_z(n) = \sum_{d|n} \mu(d)\Psi_z(d)$. We then observe that both Φ and Ψ are completely multiplicative. Let $p(n)$ denote, as usual, the least prime factor of an integer $n > 1$.

LEMMA 1.

$$\Phi_z(n) = 1 - \sum_{\substack{p|n \\ p<z}} \Phi_p\left(\frac{n}{p}\right).$$

LEMMA 2. *For $D > 2$, we have*

$$\Phi_z(n) = \sum_{\substack{d|n \\ d<D}} \mu(d)\Psi_z(d) + \sum_{\substack{d|n \\ d/p(d)<D\leq d}} \mu(d)\Psi_z(d)\Phi_{p(d)}\left(\frac{n}{d}\right).$$

LEMMA 3. *Suppose that, as $x \to \infty$, $z = z(x) \to \infty$ and $\log x/\log z > \log\log x$. Then we have*

$$\sum_{n\leq x} \Psi_z(n) \ll x \exp\left(-\frac{\log x}{\log z}\right).$$

Lemma 1 is Buchstab's identity. Lemma 2 may be produced by an iterative usage of Lemma 1. For an elegant simple proof, see [2]. Lemma 3 is [20, Kap. V, Lemma 5.2]. Lemmas 2 and 3 form a prototype of the fundamental lemma in the sieve theory, vide [20, Kap. VI, Satz 6.1]. Lemma 4 below is the core of our proof of Theorem and verified by a straightforward argument.

LEMMA 4.

$$\sum_{\substack{d|n \\ d<\sqrt{n}}} \mu(d) = \begin{cases} 0, & \text{if } \mu(n) = 1 \\ 1, & \text{if } n = p \\ 0, & \text{if } p|n \text{ with } \sqrt{n} < p < n \\ -2, & \text{if } n = p_1 p_2 p_3 \text{ with } p_3 < p_2 < p_1 < \sqrt{n}. \end{cases}$$

The above weight, however, takes various values for n having five prime factors. To state this, we define the set

$$\mathscr{E} = \{n \in N \mid n = p_1 p_2 p_3 p_4 p_5, p_5 < p_4 < p_3 < p_2 < p_1 < \sqrt{n}\}$$

and the subsets

$$\mathscr{E}_2 = \{n \in \mathscr{E} \mid p_1 p_2 < \sqrt{n}\}$$

$$\mathscr{E}_3 = \{n \in \mathscr{E} \mid p_1 p_3 < \sqrt{n}\}$$

$$\mathscr{E}_4 = \{n \in \mathscr{E} \mid p_1 p_4 < \sqrt{n}, p_2 p_3 < \sqrt{n}\}$$

$$\mathscr{E}_5 = \{n \in \mathscr{E} \mid p_1 p_5 > \sqrt{n}\}.$$

LEMMA 5. *In the above notation, we have that*

$$\mathscr{E}_2 \subset \mathscr{E}_3 \subset \mathscr{E}_4, \quad \mathscr{E}_4 \cap \mathscr{E}_5 = \varnothing,$$

and that, for any $n \in \mathscr{E}$,

$$\sum_{\substack{d|n \\ d<\sqrt{n}}} \mu(d) = \sum_{j=2,3,4} \sum_{n \in \mathscr{E}_j} 2 - \sum_{n \in \mathscr{E}_5} 2.$$

PROOF. The first part immediately follows from the definition of $\mathscr{E}_j$'s. To see the second part, write $V(n)$ for the left hand side. For a set S of integers, let $|S|(n)$ denote the characteristic function of S, so that $|S| = \sum_{n \in S} 1 = \sum |S|(n)$. Put

$$\mathscr{E}^{ij} = \{n \in \mathscr{E} \mid p_i p_j > \sqrt{n}\},$$

for $1 \leq i < j \leq 5$. Then, for any $n \in \mathscr{E}$, the expression

$$\sum\sum_{1 \leq i < j \leq 5} |\mathscr{E}^{ij}|(n)$$

is the number of divisors $d|n$ with $\nu(d) = 2$ and $d > \sqrt{n}$, which are not counted by $V(n)$. In view of the correspondence $d \leftrightarrow n/d$, this is also equal to the number of divisors $d|n$ with $\nu(d) = 3$ and $d < \sqrt{n}$, which are counted by $V(n)$. And, $V(n)$ counts all divisors $d|n$ with $\nu(d) = 1$, since $p_1 < \sqrt{n}$, and no divisor $d|n$ with $\nu(d) = 4$. Hence we find that

$$V(n) = 1 - \binom{5}{1} + \sum\sum_{1 \le i < j \le 5} (1 - |\mathscr{E}^{ij}|(n)) - \sum\sum_{1 \le i < j \le 5} |\mathscr{E}^{ij}|(n) + 0$$

$$= 6 - 2 \sum\sum_{1 \le i < j \le 5} |\mathscr{E}^{ij}|(n).$$

Now, since $n = p_1 p_2 p_3 p_4 p_5 > (p_2 p_4)^2$, $\mathscr{E}^{24}$ is empty, whence $\mathscr{E}^{ij} = \varnothing$ if $i + j \ge 7$. Since $p_1 p_4 > \sqrt{n}$ implies $p_2 p_3 < n/p_1 p_4 < \sqrt{n}$, we have $\mathscr{E}^{14} \cap \mathscr{E}^{23} = \varnothing$ or $|\mathscr{E}^{14}|(n) + |\mathscr{E}^{23}|(n) = |\mathscr{E}^{14} \cup \mathscr{E}^{23}|(n)$. Thus it turns out that

$$V(n) = 2\{(1 - |\mathscr{E}^{12}|(n)) + (1 - |\mathscr{E}^{13}|(n)) + (1 - (|\mathscr{E}^{14}|(n) + |\mathscr{E}^{23}|(n))) + |\mathscr{E}^{15}|(n)\}$$

$$= 2(|\mathscr{E}_2|(n) + |\mathscr{E}_3|(n) + |\mathscr{E}_4|(n) - |\mathscr{E}_5|(n)),$$

as claimed.

3. Proof of Theorem

First of all, we notice that $G(q, 0) = 2q$ for all prime q.

Let $6/5 \ge J > 14/13$ be given. Let q be a large prime, and $1 \le k \le q - 1$. Put

$$x = x(q) = \tfrac{1}{4} q^J; \quad \mathscr{I} = (x, x + y]; \quad y = xL^{-4},$$

so that $x^{5/6} \ll q \ll x^{13/14 - \eta}$ with some $\eta = \eta(J) > 0$. Our goal is to show that the set $\{(p, p_1) \in \mathscr{I}^2 \mid p + p_1 \equiv k \pmod{q}\}$ is not empty. Then we would find a Goldbach number $p + p_1 \le 2(x + y) \le 4x = q^J$ satisfying $p + p_1 \equiv k \pmod{q}$.

To this end we define

$$Q = \sum\sum_{\substack{p, r \in \mathscr{I} \\ p + r \equiv k(q)}} \left(\sum_{\substack{d|r \\ d < X}} \mu(d) \right) \Phi_Y(r); \quad X = x^{1/2}; \quad Y = x^\gamma, \quad \gamma = \frac{12}{77}. \tag{4}$$

Since $1/7 < \gamma < 1/6$ or $Y = x^\gamma > (2x)^{1/7}$, Q counts $r \in \mathscr{I}$ having at most six prime factors. The design of our proof is to estimate Q in two different ways.

In the first stage we shall give an asymptotic formula for Q. We postpone our proof of the following evaluation until section 5.

PROPOSITION 1.

$$Q = \frac{y}{(q-1)L} \sum_{r \in \mathscr{I}} \Bigg(\sum_{\substack{d \mid r \\ d < X}} \mu(d) \Bigg) \Phi_Y(r) + O(yq^{-1}L^{-4}).$$

Write B for the above sum over $r \in \mathscr{I}$. We replace the condition $d < X = x^{1/2}$ by $d < \sqrt{r}$. Since $\sqrt{r} - x^{1/2} \ll x^{1/2}L^{-4}$, the resulting error is

$$\sum_{|d - x^{1/2}| \ll x^{1/2}L^{-4}} |\{r \in \mathscr{I} \mid d|r\}| \ll x^{1/2}L^{-4} \cdot yx^{-1/2} \ll yL^{-4}.$$

Next we may restrict $r \in \mathscr{I}$ to squarefree integers, at the cost of $O(yY^{-1} + x^{1/2})$. Then Lemmas 4 and 5 show that

$$B = \sum_{r \in \mathscr{I}} \mu^2(r) \Bigg(\sum_{\substack{d \mid r \\ d < \sqrt{r}}} \mu(d) \Bigg) \Phi_Y(r) + O(yL^{-4})$$

$$= \sum_{p \in \mathscr{I}} (+1) + \sum\sum\sum_{\substack{p_1p_2p_3 \in \mathscr{I} \\ Y \le p_3 < p_2 < p_1 < p_2p_3}} (-2) + \Bigg(\sum_{j=2,3,4} \sum_{r \in \mathscr{A}_j} (+2) + \sum_{r \in \mathscr{A}_5} (-2) \Bigg) + O(yL^{-4})$$

where $\mathscr{A}_j = \mathscr{E}_j \cap \mathscr{I}$, $j = 2, 3, 4, 5$. By the prime number theorem and partial summation, we then obtain that

$$B = (1 - 2C_3 + O(L^{-1}))\frac{y}{L} + 2\Bigg(\sum_{j=2,3,4} |\mathscr{A}_j| - |\mathscr{A}_5| \Bigg) \tag{5}$$

where

$$C_3 = \iint_{\substack{\gamma < t_3 < t_2 \\ 2t_2 + t_3 < 1 \\ t_2 + t_3 > 1/2}} \frac{dt_2 dt_3}{(1 - t_2 - t_3)t_2t_3}.$$

We here note that, by Lemma 5,

$$\sum_{j=2,3,4,5} |\mathscr{A}_j| \le 3(|\mathscr{A}_4| + |\mathscr{A}_5|) = 3|\mathscr{A}_4 \cup \mathscr{A}_5| \le 3|\mathscr{E} \cap \mathscr{I}|. \tag{6}$$

As above, summing by parts and the prime number theorem yield that

$$|\mathscr{E} \cap \mathscr{I}| = \sum\sum\sum\sum\sum_{\substack{p_1p_2p_3p_4p_5 \in \mathscr{I} \\ Y \le p_5 < p_4 < p_3 < p_2 < p_1}} 1 = (C_5 + O(L^{-1}))\frac{y}{L} \tag{7}$$

where

$$C_5 = \iiiint\limits_{\substack{\gamma<t_5<t_4<t_3<t_2 \\ 2t_2+t_3+t_4+t_5<1}} \frac{dt_2dt_3dt_4dt_5}{(1-t_2-t_3-t_4-t_5)t_2t_3t_4t_5}.$$

We proceed to bound C_3 and C_5 crudely. First we consider C_3:

$$\int_\gamma^{1/3}\left(\int_{\max(1/2-t_3,t_3)}^{(1-t_3)/2} \frac{dt_2}{(1-t_2-t_3)t_2}\right)\frac{dt_3}{t_3}.$$

We simply estimate the inner integral by the maximum of the integrand times the length of the interval. We see that the integrand monotonically decreases throughout the interval under consideration. Hence,

$$\begin{aligned} C_3 &\le \int_\gamma^{1/4} \frac{(1-t)/2-(1/2-t)}{(1-(1/2-t)-t)(1/2-t)t}\,dt + \int_{1/4}^{1/3}\frac{(1-t)/2-t}{(1-2t)t^2}\,dt \\ &= \int_\gamma^{1/4}\frac{dt}{1/2-t} + \int_{1/4}^{1/3}\frac{1}{2}\left(\frac{1}{t^2}-\frac{2}{1-2t}-\frac{1}{t}\right)dt \\ &= \Big/_{\gamma}^{1/4} -\log\left(\frac{1}{2}-t\right) + \Big/_{1/4}^{1/3}\frac{1}{2}\left(-\frac{1}{t}+\log\left(\frac{1-2t}{t}\right)\right) \\ &= \log(2-4\gamma)+\frac{1}{2}(1-\log 2). \end{aligned}$$

Since $\gamma = 12/77$, we have

$$C_3 \le \log\frac{106}{77}+\frac{1}{2}(1-\log 2) < 0.475. \tag{8}$$

We turn to C_5:

$$\int_\gamma^{1/5}\int_{t_5}^{(1-t_5)/4}\int_{t_4}^{(1-t_4-t_5)/3}\int_{t_3}^{(1-t_3-t_4-t_5)/2} \frac{dt_2dt_3dt_4dt_5}{(1-t_2-t_3-t_4-t_5)t_2t_3t_4t_5}.$$

As for the integrand, we see that

$$\begin{aligned} (1-t_2-t_3-t_4-t_5)t_2t_3t_4t_5 &\ge (1-2t_3-t_4-t_5)t_3^2t_4t_5 \\ &\ge (1-3t_4-t_5)t_4^3t_5 \\ &\ge (1-4t_5)t_5^4 \\ &\ge (1-4\gamma)\gamma^4 \end{aligned}$$

in the region under consideration. The volume of integral domain is equal to

$$\int_{\gamma}^{1/5}\int_{t_5}^{(1-t_5)/4}\int_{t_4}^{(1-t_4-t_5)/3}\frac{1}{2}(1-3t_3-t_4-t_5)\,dt_3\,dt_4dt_5$$
$$=\int_{\gamma}^{1/5}\int_{t_5}^{(1-t_5)/4}\Big/_{t_4}^{(1-t_4-t_5)/3}\frac{-1}{2\cdot 2\cdot 3}(1-3t_3-t_4-t_5)^2\,dt_4dt_5$$
$$=\int_{\gamma}^{1/5}\int_{t_5}^{(1-t_5)/4}\frac{1}{2\cdot 2\cdot 3}(1-4t_4-t_5)^2\,dt_4dt_5$$
$$=\int_{\gamma}^{1/5}\Big/_{t_5}^{(1-t_5)/4}\frac{-1}{2\cdot 2\cdot 3\cdot 3\cdot 4}(1-4t_4-t_5)^3\,dt_5$$
$$=\int_{\gamma}^{1/5}\frac{1}{2\cdot 2\cdot 3\cdot 3\cdot 4}(1-5t_5)^3\,dt_5$$
$$=\Big/_{\gamma}^{1/5}\frac{-1}{2\cdot 2\cdot 3\cdot 3\cdot 4\cdot 4\cdot 5}(1-5t_5)^4$$
$$=\frac{(1-5\gamma)^4}{4!5!}.$$

Hence, $\gamma = 12/77$ gives that

$$C_5 \le \frac{1}{(1-4\gamma)\gamma^4}\frac{(1-5\gamma)^4}{4!5!} = \frac{77}{29}\left(\frac{17}{12}\right)^4\frac{1}{2880} < 0.004. \tag{9}$$

This finishes all our task of bounding the numerical integrations.

Next stage of our proof is to separate Q according to the number of prime factors of $r \in \mathscr{I}$. Turning back to the definition (4) of Q, we use Lemmas 4 and 5 to obtain that

$$Q = \sum_{\substack{p,r\in\mathscr{I}\\ p+r\equiv k(q)}}\sum \mu^2(r)\Bigg(\sum_{\substack{d|r\\ d<\sqrt{r}}}\mu(d)\Bigg)\Phi_Y(r) + O(y^2q^{-1}L^{-4})$$
$$=\sum_{\substack{p,p_1\in\mathscr{I}\\ p+p_1\equiv k(q)}}\sum(+1)+\sum_{\substack{p\in\mathscr{I}\ p_1p_2p_3\in\mathscr{I}\\ Y\le p_3<p_2<p_1<p_2p_3\\ p+p_1p_2p_3\equiv k(q)}}\sum\sum\sum(-2)$$
$$+\Bigg(\sum_{j=2,3,4}\sum_{\substack{p\in\mathscr{I}\ r\in\mathscr{A}_j\\ p+r\equiv k(q)}}\sum(+2)+\sum_{\substack{p\in\mathscr{I}\ r\in\mathscr{A}_5\\ p+r\equiv k(q)}}\sum(-2)\Bigg)+O(y^2q^{-1}L^{-4})$$
$$=P-2P_1+2\Bigg(\sum_{j=2,3,4}P_j-P_5\Bigg)+O(y^2q^{-1}L^{-4}),\quad\text{say.}$$

P is our target. Trivially, $P_1 \geq 0$ and $P_5 \geq 0$. In section 6, we shall show the following upper bound, which is 8/3 times as large as the expected main term.

PROPOSITION 2. *For* $j = 2, 3, 4$, *we have*

$$P_j \leq \frac{8}{3}\frac{y}{(q-1)L}|\mathscr{A}_j| + O(y^2 q^{-1} L^{-4}).$$

We therefore have that

$$Q \leq P + 2\cdot\frac{8}{3}\left(\sum_{j=2,3,4}|\mathscr{A}_j|\right)\frac{y}{(q-1)L} + O(y^2 q^{-1} L^{-4}). \tag{10}$$

We are now in the final step. It follows from Proposition 1 and (5) that

$$Q = (1 - 2C_3 + O(L^{-1}))\frac{y^2}{(q-1)L^2} + 2\left(\sum_{j=2,3,4}|\mathscr{A}_j| - |\mathscr{A}_5|\right)\frac{y}{(q-1)L}.$$

Combining this with (10), we deduce that

$$P \geq (1 - 2C_3 + O(L^{-1}))\frac{y^2}{(q-1)L^2} - 2\left(\left(\frac{8}{3} - 1\right)\sum_{j=2,3,4}|\mathscr{A}_j| + |\mathscr{A}_5|\right)\frac{y}{(q-1)L}.$$

Moreover (6) and (7) show that

$$2\left(\left(\frac{8}{3} - 1\right)\sum_{j=2,3,4}|\mathscr{A}_j| + |\mathscr{A}_5|\right) \leq 2\cdot\frac{5}{3}\cdot 3|\mathscr{E}\cap\mathscr{I}| = 10(C_5 + O(L^{-1}))\frac{y}{L}.$$

We thus reach the inequality

$$P \geq (1 - 2C_3 - 10C_5 + O(L^{-1}))\frac{y^2}{(q-1)L^2}.$$

Our crude bounds (8) and (9) yield that $1 - 2C_3 - 10C_5 > 1 - 2\times 0.475 - 10\times 0.004 = 0.01$. Consequently, $P = |\{(p, p_1) \in \mathscr{I}^2 \mid p + p_1 \equiv k \pmod{q}\}| > 0$. This completes our proof of Theorem, apart from the verification of Propositions 1 and 2.

4. Character Sums

In this section we consider some sums involving characters to modulo q, so as to provide for our proof of Propositions 1 and 2. We hold the notation and the conditions introduced in the previous sections. Especially, $J, q, k, x, \mathscr{I}$, and $\mathscr{E}$. Let χ be a non-principal character to modulo q.

LEMMA 6. *Let (a_m) and (b_n) be arbitrary sequences. Suppose that $1 \le M$, $N \le q/2$. Then we have that*

$$\sum_{\substack{mn \in \mathscr{I} \\ m \sim M}} \sum_{n \sim N} \chi(mn - k) a_m b_n \ll q^{1/2} L \|a\| \|b\| + x^{1/2} L \max_{\substack{m \sim M \\ n \sim N}} |a_m| \, |b_n|$$

where $\|\cdot\|$ denotes the ℓ^2-norm.

PROOF. We treat the condition $mn \in \mathscr{I}$ in the same way as [3, p. 165]. The sum to estimate is then bounded by

$$\int_{|t| \le x^{1/2}} |C(t)| \min(L, |t|^{-1}) \, dt + x^{1/2} L \max_{m \sim M \, n \sim N} |a_m| \, |b_n|$$

where

$$C = C(t) = \sum_{m \sim M} \sum_{n \sim N} \chi(mn - k) a_m b_n (mn)^{it}.$$

Thus it is sufficient to show that $|C|^2 \ll q \|a\|^2 \|b\|^2$ uniformly for t.

Cauchy's inequality shows that

$$|C|^2 \le \|a\|^2 \sum_{m \sim M} \left| \sum_{n \sim N} \chi(mn - k) b_n n^{it} \right|^2.$$

Since $1 \le M < m \le 2M \le q$, the summation range for m may be widen up to $1 \le m \le q$. Expanding the square and bringing the sum over m inside, we have that

$$|C|^2 \ll \|a\|^2 \sum_{n, n' \sim N} \sum |b_n| |b_{n'}| \left| \sum_{1 \le m \le q} \chi(mn - k) \bar{\chi}(mn' - k) \right|.$$

If $n \not\equiv n' \pmod q$ then the inner sum becomes the Jacobstahl sum, which is bounded. In the alternative case, $n \equiv n' \pmod q$ implies $n = n'$, because of $|n - n'| \le N < q$. Hence we conclude that

$$\begin{aligned} |C|^2 &\ll \|a\|^2 \left\{ \left(\sum_{n \sim N} |b_n| \right)^2 + q \sum_{n \sim N} |b_n|^2 \right\} \\ &\ll \|a\|^2 (N + q) \|b\|^2 \\ &\ll q \|a\|^2 \|b\|^2. \end{aligned}$$

LEMMA 7.
$$\sum_{p\in\mathcal{I}} \chi(p-k)\log p \ll (qx)^{1/2}L^5$$

This is [24, p. 364, Theorem 8] or [22]. We shall sketch the proof. We may replace $\log p$ by the von Mangoldt function Λ, with the error $O(x^{1/2})$. We next decompose it by using R. C. Vaughan's identity. In the notation of [3, §24], we take $U = V = x^{1/3}$. Then the character sum in question becomes

$$\ll L \sum_{m\le x^{1/3}} \max_t \left| \sum_{\substack{mn\in\mathcal{I}\\ n>t}} \chi(mn-k)\right| + \left| \sum\sum_{\substack{mn\in\mathcal{I}\\ x^{1/3}<m,n\le 2x^{2/3}}} \chi(mn-k)a_m b_n \right| + x^{1/2}$$

where $a_m \ll \Lambda(m)+1$, $b_n \ll \tau(n)+\log n$. The Pólya-Vinogradov inequality shows that the above first term is $O(Lx^{1/3}\cdot q^{1/2}L)$, which is satisfactory. To bound the second term, we may appeal to Lemma 6, since $4x^{2/3} < q$. Thus, Lemma 7 follows.

LEMMA 8. *For $j = 2,3,4$, we have*
$$\sum_{n\in\mathcal{A}_j} \chi(n-k) \ll (qx)^{1/2}L^4.$$

PROOF. We begin by recalling the definition of $\mathcal{A}_j = \mathcal{E}_j \cup \mathcal{I}$ in sections 2 and 3. We give the detail for $j=4$ only. The others are similar. The sum to estimate is
$$F = \sum\sum\sum\sum\sum_{p_1p_2p_3p_4p_5\in\mathcal{I}}{}^{*} \chi(p_1p_2p_3p_4p_5-k)$$
where $*$ indicates the conditions:
$$Y\le p_5<p_4<p_3<p_2<p_1<p_2p_3p_4p_5;\quad p_1p_4<p_2p_3p_5;\quad p_2p_3<p_1p_4p_5.$$
We note that $p_1 > x^{1/5}$ and $p_2p_3p_4p_5 > Y^4$. On putting $f_n(p_1) = \sum\sum\sum\sum^{*}_{p_2p_3p_4p_5=n} 1$, we see that

(11) $$F = \sum\sum_{\substack{p_1n\in\mathcal{I}\\ p_1>x^{1/5}n>Y^4}} \chi(p_1n-k)f_n(p_1)$$
$$= \sum\sum_{\substack{pn\in\mathcal{I}\\ p>x^{1/5}n>Y^4}} \chi(pn-k)\sum_{\ell} f_n(\ell)\int_0^1 e^{2\pi i(\ell-p)t}\,dt$$
$$= \int_0^1 \left(\sum\sum_{\substack{pn\in\mathcal{I}\\ p>x^{1/5}n>Y^4}} \chi(pn-k)a_p(t)b_n(t)\right) dt = \int_0^1 F_0(t)\,dt, \quad \text{say},$$

where

$$a_p(t) = e^{-2\pi ipt} \ll 1, \tag{12}$$

$$b_n(t) = \sum_{\ell} f_n(\ell) e^{2\pi i\ell t} = \sum\sum\sum\sum_{\substack{p_2p_3p_4p_5=n \\ Y \le p_5 < p_4 < p_3 < p_2}} \sum_{\substack{p_2 < \ell < n \\ p_2p_3/(p_4p_5) < \ell < p_2p_3p_5/p_4}} e^{2\pi i\ell t} \tag{13}$$

$$\ll \min(x, \|t\|^{-1}).$$

Here $\|t\|$ denotes the distance from t to the nearest integer. Then, by a dyadic decomposition of summation ranges, we find that

$$F_0(t) \ll L^2 \sup_{\substack{x^{1/5} \le M, N \le x^{4/5} \\ x \ll MN \ll x}} \left| \sum\sum_{\substack{mn \in \mathscr{I} \\ m \sim M\, n \sim N}} \chi(mn-k) a_m(t) b_n(t) \right|.$$

Since $2x^{4/5} < q$, Lemma 6 yields that

$$F_0(t) \ll L^2 \sup_{M,N} (qMN)^{1/2} L \max_{m \sim M} |a_m(t)| \max_{n \sim N} |b_n(t)|$$

$$\ll (qx)^{1/2} L^3 \min(x, \|t\|^{-1})$$

because of (12) and (13). Hence, combining this with (11), we get the required bound.

Next we consider mean values of character sums. Let χ_0 denote the principal character. We employ the Dirichlet polynomial method as in [10, §2].

LEMMA 9. *Let (δ_l) and (α_m) be arbitrary sequences with $\delta_l, \alpha_m \ll 1$. Suppose that $1 \ll M \ll x^{15/28}$ and $x \ll MN \ll x$. Then we have that, for any $A > 0$,*

$$G = \sum_{\substack{\chi (\mathrm{mod}\, q) \\ \chi \ne \chi_0}} \left| \sum_{l \in \mathscr{I}} \chi(l-k)\delta_l \right| \left| \sum_{m \sim M} \chi(m)\alpha_m \right| \left| \sum_{n \sim N} \chi(n) \right| \ll x^2 L^{-A}.$$

PROOF. We first note that $x^{15/28} \ll x^{1-\eta/2} q^{-1/2}$. The mean value estimate shows that

$$\sum_{\chi (\mathrm{mod}\, q)} \left| \sum_{l \in \mathscr{I}} \chi(l-k)\delta_l \right|^2 \ll x^2; \quad \sum_{\chi (\mathrm{mod}\, q)} \left| \sum_{m \sim M} \chi(m)\alpha_m \right|^4 \ll (q + M^2) M^2 L^3.$$

It follows from the fourth power means for the Dirichlet L-functions that

$$\sum_{\chi(\mathrm{mod}\, q)} \left| \sum_{n \sim N} \chi(n) \right|^4 \ll qN^2L^8.$$

By Hölder's inequality and the above estimates, we then have that

$$\begin{aligned} G &\leq (x^2)^{1/2}((q+M^2)M^2L^3)^{1/4}(qN^2L^8)^{1/4} \\ &\ll x(q^2+qM^2)^{1/4}(MN)^{1/2}L^3 \\ &\ll x(q^2+x^{2-\eta})^{1/4}x^{1/2}L^3 \\ &\ll x^{2-\eta/4}L^3, \end{aligned}$$

as claimed.

LEMMA 10. *Let* (δ_l), (α_m) *and* (β_n) *be arbitrary sequences satisfying* δ_l, $\alpha_m \ll 1$ *and* $\beta_n \ll \tau(n)$. *Suppose that*

$$\sum_{l \in \mathscr{I}} \chi(l-k)\delta_l \ll (qx)^{1/2}L^4$$

for any non-principal character χ $(\mathrm{mod}\, q)$. *Let*

$$x^{1/14} \ll M \ll x^{12/77} (= Y), \quad x \ll MN \ll x.$$

Then we have that, for any $A > 0$,

$$H = \sum_{\substack{\chi(\mathrm{mod}\, q) \\ \chi \neq \chi_0}} \left| \sum_{l \in \mathscr{I}} \chi(l-k)\delta_l \right| \left| \sum_{m \sim M} \chi(m)\alpha_m \right| \left| \sum_{n \sim N} \chi(n)\beta_n \right| \ll x^2L^{-A}.$$

PROOF. Let $I(\chi)$, $M(\chi)$ and $N(\chi)$ denote the above three character sums, in the order. Let S_0 be the set of $\chi(\mathrm{mod}\, q)$, $\chi \neq \chi_0$, such that $I(\chi) \ll L$ or $M(\chi) \ll L$ or $N(\chi) \ll L$. It is easy to see that the contribution of $\chi \in S_0$ to H is

$$\ll Lx(qx)^{1/2} \ll x^2L^{-A}$$

for any $A > 0$, since $q \ll x^{13/14-\eta}$. For parameters $U, V, W \gg L$, let $S = S(U, V, W)$ denote the set of $\chi(\mathrm{mod}\, q)$, $\chi \neq \chi_0$, such that

$$U < |I(\chi)| \leq 2U \quad \text{and} \quad V < |M(\chi)| \leq 2V \quad \text{and} \quad W < |N(\chi)| \leq 2W.$$

It follows from our assmption and trivial bounds that

$$U \ll (qx)^{1/2}L^4, \quad V \ll M, \quad W \ll NL. \tag{14}$$

Thus, we have that

$$H \ll x^2 L^{-A} + L^3 \sup_{U,V,W} |S(U,V,W)| UVW \tag{15}$$

where the supremum is taken over U, V, W satisfying $U, V, W \gg L$ and (14).

As before in Lemma 9, we see

$$|S| \ll \frac{x^2}{U^2}.$$

By the mean value estimate and the large value estimate, we find that

$$|S| \ll \frac{(q+M^h)M^h}{V^{2h}} L^{35}; \quad |S| \ll \frac{M^{2h}}{V^{2h}} L^{35} + \frac{qM^{4h}}{V^{6h}} L^{105},$$

with $h = 6$. Similarly,

$$|S| \ll \frac{(q+N)N}{W^2} L^3; \quad |S| \ll \frac{N^2}{W^2} L^3 + \frac{qN^4}{W^6} L^9.$$

Hence, (15) becomes

$$H \ll x^2 L^{-A} + L^{108} \sup_{U,V,W} T(U,V,W) UVW \tag{16}$$

where

$$T = T(U,V,W) = \min\left(\frac{x^2}{U^2}, \frac{(q+M^h)M^h}{V^{2h}}, \frac{M^{2h}}{V^{2h}} + \frac{qM^{4h}}{V^{6h}}, \frac{(q+N)N}{W^2}, \frac{N^2}{W^2} + \frac{qN^4}{W^6}\right).$$

We have four cases to consider, and use the inequality: For $a_1, \dots, a_n > 0$,

$$\min(a_1, \dots, a_n) \le a_1^{s_1} \cdots a_n^{s_n}; \quad s_1 + \cdots + s_n = 1,\ 0 \le s_1, \dots, s_n \le 1.$$

CASE 1. $T \le 2M^{2h}/V^{2h}$ and $T \le 2N^2/W^2$. In this case, we have that

$$\begin{aligned} T &\le 2\min\left(\frac{x^2}{U^2}, \frac{M^{2h}}{V^{2h}}, \frac{N^2}{W^2}\right) \\ &\le 2\left(\frac{x^2}{U^2}\right)^{1/2-1/2h} \left(\frac{M^{2h}}{V^{2h}}\right)^{1/2h} \left(\frac{N^2}{W^2}\right)^{1/2} \\ &= 2(UVW)^{-1} x \left(\frac{U}{x}\right)^{1/h} (MN). \end{aligned}$$

Hence, by (14),

$$TUVW \ll x^2 (qx^{-1}L^8)^{1/12} \ll x^2 (x^{-1/14})^{1/12}.$$

CASE 2. $T > 2M^{2h}/V^{2h}$ and $T > 2N^2/W^2$.

$$\begin{aligned} T &\le 2\min\left(\frac{x^2}{U^2}, \frac{qM^h}{V^{2h}}, \frac{qM^{4h}}{V^{6h}}, \frac{qN}{W^2}, \frac{qN^4}{W^6}\right) \\ &\le 2\left(\frac{x^2}{U^2}\right)^{1/2}\left(\frac{qM^h}{V^{2h}}\right)^{(2h-1)/4h(h+1)}\left(\frac{qM^{4h}}{V^{6h}}\right)^{1/4h(h+1)} \\ &\quad\times\left(\frac{qN}{W^2}\right)^{(2h-1)/4(h+1)}\left(\frac{qN^4}{W^6}\right)^{1/4(h+1)} \\ &= 2(UVW)^{-1}xq^{1/2}(MN)^{(2h+3)/4(h+1)}. \end{aligned}$$

Thus, we have that

$$\begin{aligned} TUVW &\ll xq^{1/2}x^{15/28} \\ &\ll x^2(qx^{-13/14})^{1/2} \\ &\ll x^{2-\eta/2}. \end{aligned}$$

CASE 3. $T \le 2M^{2h}/V^{2h}$ and $T > 2N^2/W^2$.

$$\begin{aligned} T &\le 2\min\left(\frac{x^2}{U^2}, \frac{M^{2h}}{V^{2h}}, \frac{qN}{W^2}, \frac{qN^4}{W^6}\right) \\ &\le 2\left(\frac{x^2}{U^2}\right)^{1/2}\left(\frac{M^{2h}}{V^{2h}}\right)^{1/2h}\left(\frac{qN}{W^2}\right)^{(2h-3)/4h}\left(\frac{qN^4}{W^6}\right)^{1/4h} \\ &= 2(UVW)^{-1}xMq^{(2h-2)/4h}N^{(2h+1)/4h}. \end{aligned}$$

Since $N \gg xM^{-1} \gg xY^{-1} = x^{65/77}$, we see that

$$\begin{aligned} TUVW &\ll xMq^{5/12}N^{13/24} \\ &\ll x(MN)(qN^{-11/10})^{5/12} \\ &\ll x^2(qx^{-13/14})^{5/12} \\ &\ll x^{2-5\eta/12}. \end{aligned}$$

CASE 4. $T > 2M^{2h}/V^{2h}$ and $T \le 2N^2/W^2$.

$$T \le 2\min\left(\frac{x^2}{U^2}, \frac{qM^h}{V^{2h}}, \frac{qM^{4h}}{V^{6h}}, \frac{N^2}{W^2}\right)$$

$$\le 2\left(\frac{x^2}{U^2}\right)^{1/2-1/6h}\left(\frac{qM^{4h}}{V^{6h}}\right)^{1/6h}\left(\frac{N^2}{W^2}\right)^{1/2}$$

$$= 2(UVW)^{-1}x\left(\frac{U}{x}\right)^{1/3h}q^{1/6h}M^{2/3}N.$$

By (14), we have that

$$TUVW \ll x(qU^2x^{-2})^{1/36}M^{-1/3}(MN)$$

$$\ll x^2(q^2x^{-1}L^8M^{-12})^{1/36}$$

$$\ll x^2L(q^2x^{-13/7})^{1/36}$$

$$\ll x^{2-\eta/18}L,$$

because of $M^{12} \gg x^{6/7}$.

Combining all the above four cases with (16), we get the required bound.

5. Proof of Proposition 1

First of all, we recall the definition (4) of Q.

(17) $$Q = \sum_{p\in\mathscr{I}} \sum_{\substack{r\in\mathscr{I} \\ r\equiv k-p(q)}} W(r); \quad W(r) := \left(\sum_{\substack{d\mid r \\ d<X}} \mu(d)\right)\Phi_Y(r).$$

On $p \in \mathscr{I}$, we attach the weight $\xi_p := \log p/\log x = 1 + O(L^{-5})$, and impose the restriction $(k-p, q) = 1$. The resulting cost is

(18) $$\ll y^2q^{-1}L^{-5} + y^2q^{-2}.$$

We may then treat the congruence $r \equiv k - p \pmod q$ by means of character sums. The leading term would come from the principal character:

(19) $$\frac{1}{q-1}\sum_{\substack{p\in\mathscr{I} \\ (k-p,q)=1}} \frac{\log p}{\log x} \sum_{\substack{r\in\mathscr{I} \\ (r,q)=1}} W(r) = \frac{y}{(q-1)L}\sum_{r\in\mathscr{I}} W(r) + O(y^2q^{-1}L^{-5}),$$

by the prime number theorem. On putting

$$R = \sum_{\substack{p \in \mathscr{I} \\ (k-p,q)=1}} \xi_p \left(\sum_{\substack{r \in \mathscr{I} \\ r \equiv k-p(q)}} W(r) - \frac{1}{q-1} \sum_{\substack{r \in \mathscr{I} \\ (r,q)=1}} W(r) \right), \tag{20}$$

we have, by (17), (18) and (19), that

$$Q = \frac{y}{(q-1)L} \sum_{r \in \mathscr{I}} W(r) + R + O(y^2 q^{-1} L^{-5}).$$

Our task is thus to show

$$R \ll y^2 q^{-1} L^{-4}. \tag{21}$$

Before expressing R by characters, we need some elemantary preparation. We shall decompose W given by (17) into some sums of the convoluton of two arithmetical functions. We are demanded to remove the dependence between these functions, as well as the variables.

Since Φ_Y is completely multiplicative, we see

$$W(r) = \sum_{\substack{dl=r \\ d<X}}\sum \mu(d)\Phi_Y(d)\Phi_Y(l). \tag{22}$$

Let $\mathscr{M}$ be the sequence $(Y(1+L^{-10})^{-g})_{1 \le g \le G}$ and $Z = Y(1+L^{-10})^{-G}$, where G is the integer defined by the inequality

$$Y(1+L^{-10})^{-(G+1)} < \exp\left(\frac{L}{(\log L)^2}\right) \le Y(1+L^{-10})^{-G} (= Z). \tag{23}$$

Then, we see $G \ll L^{11}$. The interval $[Z, Y)$ is thus divided into the $O(L^{11})$ subintervals $[M, M(1+L^{-10}))$, $M \in \mathscr{M}$.

Now, by Lemma 1, we have that

$$\Phi_Y(l) = \Phi_Z(l) - \sum_{\substack{p \mid l \\ Z \le p < Y}} \Phi_p\left(\frac{l}{p}\right). \tag{24}$$

We approximate $\Phi_Z(l)$ by

$$\sum_{\substack{e \mid l \\ e<V}} \mu(e)\Psi_Z(e); \quad V = x^{1/28}. \tag{25}$$

It follows from Lemma 2 that the resulting error is

$$\sum_{\substack{e \mid l \\ e/p(e)<V \le e}} \mu(e)\Psi_Z(e)\Phi_{p(e)}\left(\frac{l}{e}\right) \ll \sum_{\substack{e \mid l \\ V \le e < VZ}} \Psi_Z(e),$$

since $p(e) \geq Z$ implies $\Psi_Z(e) = 0$. The second term of (24) is written as

$$\sum_{M \in \mathscr{M}} \sum_{\substack{pf=l \\ M \leq p < M(1+L^{-10})}} \sum \Phi_p(f).$$

When $M \leq p < M(1 + L^{-10})$, we replace $\Phi_p(f)$ by $\Phi_M(f)$ at the cost of

$$\sum_{M \in \mathscr{M}} \sum_{\substack{pf=l \\ M \leq p < M(1+L^{-10})}} \sum \left(\sum_{\substack{p'|f \\ M \leq p' < p}} \Phi_{p'}\left(\frac{f}{p'}\right)\right) \ll \sum_{M \in \mathscr{M}} \sum_{\substack{p'p|l \\ M \leq p' < p < M(1+L^{-10})}} \sum 1,$$

because of Lemma 1 again. On summing up the above rearrangement, we deduce that

$$\begin{aligned}
(26) \qquad W(r) &= \sum_{\substack{m|r \\ m<XV}} \alpha(m) + \sum_{M \in \mathscr{M}} \sum_{\substack{p|r \\ M \leq p < M(1+L^{-10})}} \beta_M\left(\frac{r}{p}\right) \\
&\quad + O\left(\sum_{\substack{m|r \\ V \leq m < VZ}} \Psi_Z(m)\tau\left(\frac{r}{m}\right)\right) \\
&\quad + O\left(\sum_{M \in \mathscr{M}} \sum_{\substack{p'p|r \\ M \leq p' < p < M(1+L^{-10})}} \sum \tau\left(\frac{r}{p'p}\right)\right) \\
&= W_1(r) + W_2(r) + W_3(r) + W_4(r), \quad \text{say,}
\end{aligned}$$

where

$$(27) \qquad \alpha(m) = \sum_{\substack{de=m \\ d<Xe<V}} \sum \mu(d)\Phi_Y(d)\mu(e)\Psi_Z(e) \ll \sum_{de=m} \sum \Phi_Y(d)\Psi_Y(e) = 1,$$

$$(28) \qquad \beta_M(n) = \sum_{\substack{df=n \\ d<X}} \sum \mu(d)\Phi_Y(d)\Phi_M(f) \ll \tau(n).$$

Let R_j be the similar expression to R given by (20) with W_j in place of W. Then R is the sum of R_j's, $j = 1, 2, 3, 4$.

We first consider R_4. Recalling (26), we have that

$$\begin{aligned}
\sum_{r \in \mathscr{I}} |W_4(r)| &\ll \sum_{M \in \mathscr{M}} \sum_{M \leq p' < p < M(1+L^{-10})} \sum \sum_{p'pn \in \mathscr{I}} \tau(n) \\
&\ll yL \sum_{M \in \mathscr{M}} \sum_{M \leq p' < p < M(1+L^{-10})} \sum \frac{1}{p'p}\left(\frac{\log p}{\log p'}\right)
\end{aligned}$$

$$\ll yL \max_{M \in \mathcal{M}} \left(\sum_{M \le p < M(1+L^{-10})} \frac{\log p}{p} \right) \sum_{Z \le p' < Y} \frac{1}{p' \log p'}$$
$$\ll yL^{-9},$$

by the prime number theorem. Hence,

$$R_4 \ll \sum_{p \in \mathcal{I}} \left(\sum_{\substack{r \in \mathcal{I} \\ r \equiv k-p(q)}} |W_4(r)| + \frac{1}{q} \sum_{r \in \mathcal{I}} |W_4(r)| \right)$$
$$\ll \frac{y}{q} \sum_{r \in \mathcal{I}} |W_4(r)|$$
$$\ll y^2 q^{-1} L^{-9},$$

which is satisfactory for (21).

We proceed to R_3. We remember (26), (23) and (25). Then, Lemma 3 shows that

$$\sum_{r \in \mathcal{I}} |W_3(r)| \ll \sum_{V \le m < VZ} \Psi_Z(m) \sum_{mn \in \mathcal{I}} \tau(n)$$
$$\ll yL \sum_{V \le m < VZ} \frac{\Psi_Z(m)}{m}$$
$$\ll yL \exp\left(-\frac{\log V}{\log Z} \right)$$
$$\ll yL \exp(-K(\log L)^2),$$

Hence, as above, W_3 makes a neglisible contribution to R.

We turn to R_1. We further divide W_1 given by (26) according as $m \le U := x^{1/14}$ or $U < m < XV$. Here note that $U = x^{1/14} \ll x^{1-\eta} q^{-1}$ and $XV = x^{15/28} \ll x^{1-\eta/2} q^{-1/2}$. We then split up the summation ranges by powers of $1 + L^{-10}$, if m is large. For simplicity, we write $h \approx H$ for $H < h \le H(1 + L^{-10})$. Thus,

$$(29) \quad R_1 \ll \sum_{\substack{p \in \mathcal{I} \\ (k-p,q)=1}} \left| \sum\sum_{\substack{mn \in \mathcal{I} \\ mn \equiv k-p(q) \\ m \le U}} \alpha(m) - \frac{1}{q-1} \sum\sum_{\substack{mn \in \mathcal{I} \\ (mn,q)=1 \\ m \le U}} \alpha(m) \right|$$
$$+ \sum_M \sum_N \left| \sum_{\substack{p \in \mathcal{I} \\ (k-p,q)=1}} \xi_p \left(\sum\sum_{\substack{mn \in \mathcal{I} \\ mn \equiv k-p(q) \\ m \approx M\, n \approx N}} \alpha(m) - \frac{1}{q-1} \sum\sum_{\substack{mn \in \mathcal{I} \\ (mn,q)=1 \\ m \approx M\, n \approx N}} \alpha(m) \right) \right|$$
$$= S_1 + S_2, \quad \text{say},$$

where M and N run through the powers of $1 + L^{-10}$ subject to $U \ll M \ll XV$ and $x \ll MN \ll x$.

We bound S_1 first. Since $(k - p, q) = 1$, if $(m, q) > 1$ then the congruence $mn \equiv k - p \pmod{q}$ of n has no solution. If $(m, q) = 1$ then

$$\sum_{\substack{mn \in \mathscr{I} \\ mn \equiv k-p(q)}} 1 = \frac{y}{qm} + O(1) = \frac{1}{q-1} \sum_{\substack{mn \in \mathscr{I} \\ (n,q)=1}} 1. \tag{30}$$

We therefore see that $S_1 \ll yU \ll yx^{1-\eta}q^{-1}$, which is also neglisible.

Next we consider S_2. We need to drop the condition $mn \in \mathscr{I}$, in order to make the variables independent. We obserb that $mn \in \mathscr{I}$ is absorbed by the condition $x \le MN$ and $MN(1 + L^{-10})^2 \le x + y$. On the other hand, the condition $MN < x$ or $MN(1 + L^{-10})^2 > x + y$ means $x < mn \le MN(1 + L^{-10})^2 < x(1 + L^{-10})^2$ or $(x + y)(1 + L^{-10})^{-2} < MN < mn \le x + y$. Hence, such terms contribute to S_2 at most

$$yq^{-1} \sum\sum_{\substack{MN<x \text{ or} \\ MN(1+L^{-10})^2>x+y}} \sum\sum_{\substack{mn \in \mathscr{I} \\ m \approx M \\ n \approx N}} 1 \ll yq^{-1} \sum_{\substack{x<r<x(1+L^{-10})^2 \text{ or} \\ (x+y)(1+L^{-10})^{-2}<r\le x+y}} \tau(r)$$

$$\ll yq^{-1}xL^{-10}L.$$

Consequently, (29) becomes

$$R_1 \ll y^2q^{-1}L^{-5} + L^{22} \sup_{M,N,\alpha} \left| \sum_{\substack{p \in \mathscr{I} \\ (k-p,q)=1}} \xi_p \left(\sum\sum_{\substack{mn \equiv k-p(q) \\ m \approx M\, n \approx N}} \alpha_m - \frac{1}{q-1} \sum\sum_{\substack{(mn,q)=1 \\ m \approx M\, n \approx N}} \alpha_m \right) \right|$$

where the supremum is taken over all parameters M and N satisfying $x^{1/14} \ll M \ll x^{15/28}$ and $x \ll MN \ll x$, and all sequences (α_m) with $\alpha_m \ll 1$.

At last we were ready for expressing R_1 by characters. It turns out that

$$R_1 \ll y^2q^{-1}L^{-5} + L^{22} \sup_{M,N,\alpha} q^{-1} \sum_{\substack{\chi (\mathrm{mod}\, q) \\ \chi \ne \chi_0}} \left| \sum_{p \in \mathscr{I}} \chi(p-k)\xi_p \right| \left| \sum_{m \approx M} \chi(m)\alpha_m \right| \left| \sum_{n \approx N} \chi(n) \right|. \tag{31}$$

Then Lemma 9 yields the desired bound for R_1.

Finally we proceed to consider R_2. The initial step is the same as above. Thus, in place of (31), we have that

(32)

$$R_2 \ll y^2 q^{-1} L^{-4} + L^{22} \sup_{M,N,\beta} q^{-1} \sum_{\substack{\chi (\mathrm{mod}\, q) \\ \chi \neq \chi_0}} \left| \sum_{p \in \mathscr{S}} \chi(p-k)\xi_p \right| \left| \sum_{p \approx M} \chi(p) \right| \left| \sum_{n \approx N} \chi(n)\beta_n \right|$$

where the supremum is taken over all parameters M and N satisfying $Z \ll M \ll Y$ and $x \ll MN \ll x$, and all sequences (β_n) with $\beta_n \ll \tau(n)$.

We first deal with the case $Z \ll M \leq U = x^{1/14}$. Since $N \gg x/M \gg x^{13/14} \gg q$, the mean square estimate shows that

$$\sum_{\chi (\mathrm{mod}\, q)} \left| \sum_{n \approx N} \chi(n)\beta_n \right|^2 \ll (q+N)NL^3 \ll N^2 L^3. \tag{33}$$

Choose the integer H with $M^{H-1} < q \leq M^H$, so that $10 \leq H \ll (\log x)/(\log Z) \ll (\log L)^2$. Let ϖ denote the characteristic function of primes in $[M, M(1+L^{-10}))$, and ϖ_H the H-fold convolution of ϖ. Then, $\varpi_H(n) \leq H!$ for all n. The mean value estimate yields that

$$\sum_{\chi (\mathrm{mod}\, q)} \left| \sum_{M \leq p < M(1+L^{-10})} \chi(p) \right|^{2H} \ll (q + M^H) \sum_{M^H \leq n < M^H(1+L^{-10})^H} \varpi_H(n)^2 \tag{34}$$

$$\ll (M^H H!)^2.$$

Now, by Hölder's inequality, we have that the averaged character sum in (32) is at most

$$\left(\sum_{\chi \neq \chi_0} \left| \sum_{p \in \mathscr{S}} \chi(p-k)\xi_p \right|^{2+2/(H-1)} \right)^{(H-1)/2H}$$

$$\times \left(\sum_{\chi} \left| \sum_{p \approx M} \chi(p) \right|^{2H} \right)^{1/2H} \left(\sum_{\chi} \left| \sum_{n \approx N} \chi(n)\beta_n \right|^2 \right)^{1/2}.$$

In view of Lemma 7, (33) and (34), the above becomes

$$\ll ((qxL^8)^{1/(H-1)} x^2)^{(H-1)/2H} ((M^H H!)^2)^{1/2H} (N^2 L^3)^{1/2}$$

$$\ll (q/x)^{1/2H} x(MN)(H!)^{1/H} L^2$$

$$\ll x^{-1/28H} x^2 H L^2$$

$$\ll \exp(-KL(\log L)^{-2}) x^2 L^3,$$

which gives the acceptable estimation of R_2 in the case $Z \ll M \leq U$.

It remains to consider the case $U < M \ll Y$. In this case, all assumptions of Lemma 10 are fulfilled, because of Lemma 7. Thus we have that $R_2 \ll y^2 q^{-1} L^{-4}$, as claimed. Consequently, Proposition 1 follows.

6. Proof of Proposition 2

To handle P_j, we need the reversal rôles trick. It follows from the definition that

$$P_j = \sum_{\substack{p \in \mathscr{I} \\ p+r \equiv k(q)}} \sum_{r \in \mathscr{A}_j} 1 = \sum_{\substack{r \in \mathscr{A}_j \\ (k-r,q)=1}} \sum_{\substack{p \in \mathscr{I} \\ p \equiv k-r(q)}} 1 + O(y^2 q^{-2}). \tag{35}$$

We shall estimate the inner sum by means of the Rosser-Iwaniec sieve [11, 17]. Let $\varepsilon > 0$ be a fixed small number. Put

$$z = x^{1/3}; \quad D = x^{(60/77)(1-\varepsilon)} = Y^{5(1-\varepsilon)}.$$

Then, since $2 < s = \log D / \log z < 3$, the upper bound sieve yields that

$$\sum_{\substack{l \in \mathscr{I} \\ l \equiv k-r(q)}} \Phi_z(l) \le \frac{2 + K\varepsilon}{\log D} \frac{y}{q-1} + \sum_{(d,q)=1} \lambda_d \left(\sum_{\substack{l \in \mathscr{I} \\ l \equiv k-r(q) \\ d \mid l}} 1 - \frac{y}{qd} \right)$$

where $(\lambda_d) = (\lambda_d(z, D))$ satisfies that

$$\lambda_d = 0 \quad \text{if} \quad d > D; \quad |\lambda_d| \le \mu^2(d) \Psi_z(d).$$

We then rewrite the above remainder term by using (30). Thus, providing that ε is small enough, we obtain that

$$\sum_{\substack{p \in \mathscr{I} \\ p \equiv k-r(q)}} 1 \le \frac{8}{3} \frac{y}{(q-1)L} + \left(\sum_{\substack{l \in \mathscr{I} \\ l \equiv k-r(q)}} \left(\sum_{d \mid l} \lambda_d \right) - \frac{1}{q-1} \sum_{\substack{l \in \mathscr{I} \\ (l,q)=1}} \left(\sum_{d \mid l} \lambda_d \right) \right) + O(Dq^{-1}).$$

Substituting this into (35), we have that

$$P_j \le \frac{8}{3} \frac{y}{(q-1)L} |\mathscr{A}_j| + T + O(y^2 q^{-2}) + O(Dyq^{-1}) \tag{36}$$

where

$$T = \sum_{\substack{r \in \mathscr{A}_j \\ (k-r,q)=1}} \left(\sum_{\substack{l \in \mathscr{I} \\ l \equiv k-r(q)}} U(l) - \frac{1}{q-1} \sum_{\substack{l \in \mathscr{I} \\ (l,q)=1}} U(l) \right)$$

with $U(l) = \sum_{d \mid l} \lambda_d$.

According to [11], (λ_d) has the following form.

$$\lambda_d = \sum_{\Delta=(D_1,\dots,D_r)} \sum_{v<D^\varepsilon} \sum_{\substack{D_i \le p_i < D_i^{1+\varepsilon^9} \\ 1\le i\le r \\ vp_1\cdots p_r = d}} \cdots \sum \rho_v(\varepsilon, \Delta)$$

where $\Delta = (D_1, \dots, D_r)$ runs through the subsequences of $(D^{\varepsilon^2(1+\varepsilon^9)^n})_{n\ge 0}$, including the empty one, such that

$$D_1 \ge D_2 \ge \cdots \ge D_r,$$

$$D_1 D_2 \cdots D_{2m-1}^3 \le D \quad \text{for all } 1 \le m \le (r+1)/2.$$

Here $\rho_v(\varepsilon, \Delta)$ depends only on v, ε and Δ, and satisfies $|\rho_v(\varepsilon, \Delta)| \le \mu^2(v)\Psi_{D^{\varepsilon^2}}(v)$. And the number of $\Delta = (D_1, \dots, D_r)$'s is less than $\exp(8\varepsilon^{-3})$.

Now, as before in section 5, we shall split up U and appeal to Lemmas 9 and 10. We write $U(l) = U_1(l) + U_2(l)$, where U_1 is the form arising from $\Delta = \varnothing$, $(D_1), (D_1, D_2)$, and U_2 from $\Delta = (D_1, \dots, D_r)$ with $r \ge 3$.

When $\Delta = \varnothing$, $(D_1), (D_1, D_2)$, d takes v, vp_1, vp_1p_2. Since $D_1 \ge D_2$ and $D_1^3 \le D$, we see that $d \le D^\varepsilon (D_1 D_2)^{1+\varepsilon^9} \le D^\varepsilon (D^{2/3})^{(1+\varepsilon^9)} = D'$, say. Hence U_1 may be written as

$$U_1(l) = \sum_{\Delta=\varnothing, (D_1), (D_1, D_2)} \sum_{\substack{dn=l \\ d \le D'}} \sum \alpha_d(\varepsilon, \Delta)$$

with $\alpha_d \ll 1$. Here we notice that

$$D' \le x^\varepsilon (x^{40/77})^{(1-\varepsilon)(1+\varepsilon^9)} \le x^{40/77+\varepsilon} \le x^{15/28}$$

providing that ε is small enough. Therefore U_1 is the same type as W_1 defined by (26) in the previous section.

We proceed to consider U_2. From the definition, we find that

$$U_2(l) = \sum_{\substack{\Delta=(D_1,\dots,D_r) \\ r\ge 3}} \sum_{D_r \le p_r < D_r^{1+\varepsilon^9}} \sum_{\substack{n \\ p_r n = l}} \beta_n(\varepsilon, \Delta)$$

where

$$\beta_n(\varepsilon, \Delta) = \sum_{v<D^\varepsilon} \sum_{\substack{D_i \le p_i < D_i^{1+\varepsilon^9} \\ 1 \le i \le r-1 \\ vp_1\cdots p_{r-1}h = n}} \cdots \sum \sum_h \rho_v(\varepsilon, \Delta) \ll \tau(n).$$

Since $D_3 \ge D_r$ and $D_3^5 \le D_1 D_2 D_3^3 \le D$, we see $D_r \le D^{1/5}$. Then,

$$x^{K\varepsilon^2} \le D^{\varepsilon^2} \le D_r \le p_r < D_r^{1+\varepsilon^9} < D^{(1+\varepsilon^9)/5} = Y^{(1-\varepsilon)(1+\varepsilon^9)} \le Y = x^{12/77}.$$

Hence U_2 is also the same type as W_2 given by (26). It turns out that U is divided into the suitable form for the argument of proving Proposition 1.

Furthermore Lemma 8 shows that the character sum $\sum_{r \in \mathscr{A}_j} \chi(r-k)$ fills the rôle of one over shifted primes in our proof of Proposition 1. Therefore the same argument as that in the previous section leads to

$$T \ll y^2 q^{-1} L^{-4}.$$

Combining this with (36), we have the required upper bound for P_j's.

This completes our proof of Theorem.

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