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INVARIANT HOMOGENEOUS STRUCTURES ON HOMOGENEOUS REAL HYPERSURFACES IN A COMPLEX PROJECTIVE SPACE AND AN ODD-DIMENSIONAL SPHERE

By

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1. Introduction

In Riemannian geometry the theory of homogeneous spaces is a very interesting subject. Many geometers investigate homogeneous submanifolds in a complex projective space CP_n and get many fruitful results. CP_n has good geometric structures. One of them is a Kähler structure. These structures induce many geometric structures on submanifolds. For example, almost contact metric structures on real hypersurfaces are induced from the Kähler structure of CP_n. These structures are very useful to investigate geometries of real hypersurfaces. On the other hand, CP_n has the Hopf fibration whose total space is the odd-dimensional unit sphere S^{2n+1}. Its projection is a Riemannian submersion. The fundamental equations of Riemannian submersions are investigated by O'Neill [9]. The Hopf fibration is a useful tool when we study geometries of submanifolds in CP_n. Through the Hopf fibration informations of submanifolds in CP_n can be translated into informations of submanifolds in S^{2n+1} and vice versa. Using this method, R. Takagi [11] classified homogeneous real hypersurfaces in CP_n. By his theorem they are classified into 5 types of Riemannian submanifolds, say of type (A)–(E) (see §2 Theorem T).

The homogeneity of a Riemannian manifold can be studied by means of the existence of a so called homogeneous structure tensor (cf. [1] and [14]). So it is natural to expect that on each homogeneous manifold a homogeneous structure tensor will contain geometric informations about this space. Therefore it is an important problem to determine homogeneous structure tensors on homogeneous spaces. In the paper [6] the author gives a homogeneous structure on a homogeneous real hypersurface of type (A) (cf. §4). Using this tensor, we know that

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a real hypersurface of type (A) is naturally reductive. Further, in the paper [7] the author determines all naturally reductive homogeneous real hypersurfaces in CP_n.

Our aim in this paper is twofold. One of our purposes is to determine a homogeneous structure on a real hypersurface of type (B). This is expressed by using its almost contact metric structure and the shape operator. The result is as follows.

THEOREM 4.1. *The following tensor T^B defines an invariant homogeneous structure on a homogeneous real hypersurface M of type* (B)

$$T_X^B Y = \frac{\alpha}{2}\eta(X)\phi Y + \eta(Y)\phi AX - g(\phi AX, Y)\xi. \tag{4.1}$$

Here (ϕ, ξ, η, g) is an almost contact metric structure on M and α is the principal curvature in the direction of ξ (for details see §2).

Another purpose of this paper is to investigate relations between homogeneous structures of submanifolds in CP_n and S^{2n+1}. Here we prove

THEOREM 3.1. *Let T be an invariant homogeneous structure of a real hypersurface M on which ξ is principal in CP_n. Then the lift hypersurface M' in a unit sphere S^{2n+1} is locally homogeneous and the following tensor T' defines a homogeneous structure of M'.*

(3.1)
$$T'_X Y = (T_{\pi(X)}\pi(Y))^* - g'(X, V)(\phi\pi(Y))^* - g'(Y, V)(\phi\pi(X))^* + g(\phi\pi(X), \pi(Y))V.$$

Here π is the map from M' to M naturally induced by the Hopf fibration. V and $(\ )^*$ denote the vertical tangent vector of M' and the horizontal lift of a vector (for details see §2).

Further, using these observations, we obtain homogeneous structures of submanifolds in S^{2n+1} which are some of homogeneous hypersurfaces given by isotropy representations of compact Riemannian symmetric spaces of rank 2 (cf. [11]).

2. Preliminaries

In this section we explain preliminary results concerning Riemannian homogeneous structures, real hypersurfaces of a complex projective space and Hopf fibrations.

First, we recall a criterion for homogeneity of a Riemannian manifold obtained by Ambrose and Singer [1]. We start with

DEFINITION 2.1. A connected Riemannian manifold (M, g) is said to be homogeneous if the group $I(M)$ of isometries acts transitively on M.

On the other hand, local homogeneity is defined by

DEFINITION 2.2. A connected Riemannian manifold (M, g) is said to be locally homogeneous if, for each $p, q \in M$, there exists a neighborhood U of p, a neighborhood V of q and a local isometry $\phi : U \to V$ such that $\phi(p) = q$.

In the paper [1], Ambrose and Singer give a criterion for homogeneity of a Riemannian manifold:

THEOREM AS ([1]). *A connected, complete and simply connected Riemannian manifold M is homogeneous if and only if there exists a tensor field T of type $(1,2)$ on M such that*

(i) $g(T_X Y, Z) + g(Y, T_X Z) = 0$,

(ii) $(\nabla_X R)(Y, Z) = [T_X, R(Y, Z)] - R(T_X Y, Z) - R(Y, T_X Z)$,

(iii) $(\nabla_X T)_Y = [T_X, T_Y] - T_{T_X Y}$,

for $X, Y, Z \in \mathcal{X}(M)$. Here ∇ denotes the Levi Civita connection, R is the Riemannian curvature tensor of M and $\mathcal{X}(M)$ is the Lie algebra of all C^∞ vector fields over M.

Furthermore, without the topological conditions of completeness and simply connectedness, the three conditions (i)–(iii) give a criterion for local homogeneity of M.

REMARK 2.3. If we put $\tilde{\nabla} := \nabla - T$, then the conditions (i), (ii) and (iii) are equivalent to $\tilde{\nabla} g = 0$, $\tilde{\nabla} R = 0$ and $\tilde{\nabla} T = 0$, respectively.

Secondly, we turn to some preliminaries concerning real hypersurfaces of a complex projective space. Let $CP_n(4)$ be an n-dimensional complex projective space with constant holomorphic sectional curvature 4 and let J and $\bar{g}$ be its complex structure and metric, respectively. Further, let M be a connected submanifold of $CP_n(4)$ with real codimension 1, simply called a real hypersurface in the following. We denote by g the induced Riemannian metric of M and by v a local unit normal vector field of M in $CP_n(4)$.

The Gauss and Weingarten formulas are:

$$\overline{\nabla}_X Y = \nabla_X Y + g(AX, Y)v, \tag{2.1}$$

$$\overline{\nabla}_X v = -AX, \tag{2.2}$$

where $\overline{\nabla}$ and ∇ denote the Levi Civita connection on $CP_n(4)$ and M, respectively and A is the shape operator of M in $CP_n(4)$.

We define an almost contact metric structure (ϕ, ξ, η, g) of M as usual. That is,

$$\xi = -J\nu, \quad \eta(X) = g(X, \xi), \quad \phi X = (JX)^T, \quad \text{for } X \in TM,$$

where TM denotes the tangent bundle of M and $(\ )^T$ the tangential component of a vector. These structure tensors satisfy the following relations:

$$\begin{aligned} &\phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1, \\ &g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad X, Y \in TM, \end{aligned} \tag{2.3}$$

where I denotes the identity mapping of TM.

From (2.1) we easily have

$$(\nabla_X \phi) Y = \eta(Y) AX - g(AX, Y)\xi, \tag{2.4}$$

$$\nabla_X \xi = \phi AX \tag{2.5}$$

for tangent vectors $X, Y \in TM$.

For a homogeneous structure we define

DEFINITION 2.4. A homogeneous structure tensor T on a real hypersurface in CP_n is said to be invariant if all structure tensors (ϕ, ξ, η, g) are parallel with respect to the connection $\tilde{\nabla} = \nabla - T$.

In our case the Gauss and Codazzi equations of M become

$$\begin{aligned} R(X, Y)Z = {} & g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ & - 2g(\phi X, Y)\phi Z + g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \tag{2.6}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi. \tag{2.7}$$

Homogeneous real hypersurfaces of $CP_n(4)$ are completely classified. In [11] R. Takagi obtained the following:

THEOREM T ([11]). *Let M be a homogeneous real hypersurface of $CP_n(4)$. Then M is locally congruent to one of the following spaces*:

(A) *a tube of radius r over a totally geodesic $CP_k(4)$ $(0 \le k \le n-1)$, $0 < r < \pi/2$;*

(B) *a tube of radius r over a complex quadric Q_{n-1}, $0 < r < \pi/4$;*

(C) *a tube of radius r over $CP_1 \times CP_{(n-1)/2}$, $n \ge 5$ is odd, $0 < r < \pi/4$;*

(D) *a tube of radius r over a complex Grassmann* $G_{2,5}(\boldsymbol{C})$, $n=9$, $0<r<\pi/4$;
(E) *a tube of radius r over a Hermitian symmetric space* $SO(10)/U(5)$, $n=15$, $0<r<\pi/4$.
Here $\boldsymbol{C}P_0$ *means a single point.*

Homogeneous real hypersurfaces have other representations obtained by using the Hopf fibrations. For later use, we only write such representations in the case of real hypersurfaces of type (A) and type (B).

For real hypersurfaces of type (A) we have the following commutative diagram:

$$\begin{array}{ccc} S^{2p+1}(r_1)\times S^{2q+1}(r_2) & \longrightarrow & S^{2n+1}(1) \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi'} \\ M & \longrightarrow & \boldsymbol{C}P_n(4), \end{array} \tag{2.8}$$

where $r_1^2+r_2^2=1$ and $p+q=n-1$.

For real hypersurfaces of type (B), we have

$$\begin{array}{ccc} SO(2)\times SO(n+1)/\boldsymbol{Z}_2\times SO(n-1) & \longrightarrow & S^{2n+1} \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi'} \\ M & \longrightarrow & \boldsymbol{C}P_n(4), \end{array} \tag{2.9}$$

where $\boldsymbol{Z}_2$ denotes the finite group of order 2. In both diagrams the map π' is the Hopf fibration (cf. [8], [11], [13]).

About the decomposition of the tangent space into the eigenspaces of the shape operator of a homogeneous real hypersurface, we know the following:

THEOREM 2.5([12]). *The tangent space of the homogeneous real hypersurfaces can be decomposed as follows*:
for type (A): $TM=\boldsymbol{R}\xi\oplus T_x\oplus T_{-1/x}$, $A\xi=(x-1/x)\xi$, $x>0$;
for type (B): $TM=\boldsymbol{R}\xi\oplus T_x\oplus T_{-1/x}$, $A\xi=(-4x/(x^2-1))\xi$, $0<x<1$;
for type (C), (D) *and* (E): $\begin{cases} TM=\boldsymbol{R}\xi\oplus T_x\oplus T_{-1/x}\oplus T_{(x+1)/(1-x)}\oplus T_{(x-1)/(x+1)}, \\ A\xi=(-4x/(x^2-1))\xi, 0<x<1, \end{cases}$
where T_λ *denotes the eigenspace of the shape operator with the principal curvature* λ. *Further, for type* (B)–(E) *we have* $\phi T_x=T_{-1/x}$ (cf. [5]).

In what follows we denote the principal curvature in the direction of the vector ξ by α, that is, $A\xi=\alpha\xi$. From Theorem 2.5 we have

PROPOSITION 2.6. *The shape operator of a homogeneous real hypersurface of type* (B) *satisfies the following relations*:

$$\phi A + A\phi = -\frac{4}{\alpha}\phi, \tag{2.10}$$

$$\phi\left(A^2 + \frac{4}{\alpha}A - I\right) = 0. \tag{2.11}$$

For the covariant derivative of the shape operator A, we have:

PROPOSITION 2.7([3]). *Let M be a homogeneous real hypersurface of type* (B). *Then the shape operator A of M satisfies*

(2.12)
$$(\nabla_X A)Y = -\frac{\alpha}{4}\{2\eta(X)(A\phi - \phi A)Y + \eta(Y)(A\phi - 3\phi A)X + g((A\phi - 3\phi A)X, Y)\xi\}.$$

Finally, we explain some fundamental equations of the Hopf fibration and their submanifolds. For details see [9] and [10]. Let $\pi' : S^{2n+1} \to \boldsymbol{C}P_n$ be the Hopf fibration. Further, let $(J, \bar{g}, \overline{\nabla})$ be the triple determined by the complex structure, the Riemannian metric of constant holomorphic sectional curvature 4 and the Levi Civita connection of $\boldsymbol{C}P_n$. Moreover, let $(\overline{g'}, \overline{\nabla'})$ be the pair formed by the metric of constant sectional curvature 1 and the Levi Civita connection of S^{2n+1}. For a real hypersurface M of $\boldsymbol{C}P_n$ we have the following commutative diagram:

$$\begin{array}{ccccc} (g', \nabla') & M' & \xrightarrow{\ \iota'\ } & S^{2n+1}(1) & (\overline{g'}, \overline{\nabla'}) \\ & \pi\big\downarrow & & \big\downarrow\pi' & \\ (\phi, \xi, \eta, g, \nabla) & M & \xrightarrow{\ \iota\ } & \boldsymbol{C}P_n(4) & (J, \bar{g}, \overline{\nabla}). \end{array} \tag{2.13}$$

Here $(\phi, \xi, \eta, g, \nabla)$ denotes the almost contact metric structure and the Levi Civita connection of M, M' is the inverse image of M by π' and (g', ∇') denotes the pair of the Riemannian metric and the Levi Civita connection of M'.

In the following, for a vector $X \in T\boldsymbol{C}P_n$ (resp. $\in TM$) X^* denotes the horizontal lift of X in TS^{2n+1} (resp. in TM'). Further, for a point $z \in S^{2n+1}$ (resp. $\in M'$) $V_z = -iz$ denotes a vertical tangent vector at $z \in S^{2n+1}$ (resp. $\in M'$), where i is the complex structure of $\boldsymbol{C}^{n+1}$ acting canonically on the unit sphere $S^{2n+1} \subset \boldsymbol{C}^{n+1}$. For $X \in TS^{2n+1}$ we have $\overline{\nabla'}_X V = (-J\pi'(X))^*$, where J and $(\ )^*$ denote the complex structure of $\boldsymbol{C}P_n$ and the horizontal lift of a vector, respectively and π' also denotes the differential of π'. Then the fundamental equations of the submersions π' and π are

$$\overline{\nabla}'_{X^*} Y^* = (\overline{\nabla}_X Y)^* + \bar{g}(JX, Y)V, \quad X, Y \in TCP_n, \tag{2.14}$$

$$\nabla'_{X^*} Y^* = (\nabla_X Y)^* + g(\phi X, Y)V, \quad X, Y \in TM. \tag{2.15}$$

For a unit normal vector field v of M in CP_n the horizontal lift v^* defines a unit normal vector field of M' in S^{2n+1}. From (2.14) we can easily get the following relations between the shape operator A' of M' and the shape operator A of M:

$$A'X^* = (AX)^* - \eta(X)V. \tag{2.16}$$

Further, since $\overline{\nabla}'_V v^* = \overline{\nabla}'_{v^*} V = (-Jv)^* = \xi^*$, we have

$$A'V = -\xi^*. \tag{2.17}$$

So we obtain

$$A'Z = (A\pi(Z))^* - g(\pi(Z), \xi)V - g'(Z, V)\xi^*, \quad Z \in TM', \tag{2.18}$$

where the differential of π is denoted by the same letter π.

Using (2.15), we have

$$\pi(\nabla'_X Z) = \nabla_{\pi(X)}\pi(Z) - g'(V, Z)\phi\pi(X) - g'(V, X)\phi\pi(Z). \tag{2.19}$$

The covariant derivative of the shape operator A' of M' has the following formula.

LEMMA 2.8([2]). *Let A' be the shape operator of M' in S^{2n+1}. Then we have*

$$\begin{aligned}(\nabla'_X A')Y = {} & \{(\nabla_{\pi(X)}A)\pi(Y) + \eta(\pi(Y))\phi\pi(X) + g(\phi\pi(X), \pi(Y))\xi\}^* \\ & - g'(X, V)\{(\phi A - A\phi)\pi(Y)\}^* - g'(Y, V)\{(\phi A - A\phi)\pi(X)\}^* \\ & - g(\pi(Y), (\phi A - A\phi)\pi(X))V.\end{aligned} \tag{2.20}$$

PROOF. This follows from a straightforward calculation by using (2.18), (2.19) and the definition of $\nabla'A'$ (for details see [2]). ■

3. Relations between Homogeneous Structures in the Hopf Fibration

In this section we obtain the relations between homogeneous structures of real hypersurfaces in $CP_n(4)$ and the corresponding lifts hypersurfaces in S^{2n+1}.

We have the following:

THEOREM 3.1. *Let T be an invariant homogeneous structure on a real hypersurface M on which ξ is principal in CP_n. Then the lift hypersurface M' in a unit sphere S^{2n+1} is locally homogeneous and the following tensor T' defines a homogeneous structure on M'*:

(3.1)
$$T'_X Y = (T_{\pi(X)}\pi(Y))^* - g'(X,V)(\phi\pi(Y))^* - g'(Y,V)(\phi\pi(X))^* + g(\phi\pi(X),\pi(Y))V.$$

PROOF. We have to prove the conditions (i)–(iii) of the Theorem AS. First, we prove (i). By straightforward calculation we have

$$g'(T'_X Y, Z) + g'(Y, T'_X Z) = g(T_{\pi(X)}\pi(Y), \pi(Z)) + g(\pi(Y), T_{\pi(X)}\pi(Z)).$$

By our hypothesis the right-hand side of this equation vanishes. This prove (i).
Secondly, we prove (ii). For this purpose it suffices to prove $\nabla'_X A' = T'_X A'$. Using (2.5), (2.18) and (3.1), we obtain

$$\begin{aligned}(T'_X A')Y = {} & \{(T_{\pi(X)}A)\pi(Y) + \eta(\pi(Y))\phi\pi(X) + g(\phi\pi(X),\pi(Y))\xi\}^* \\ & - g'(X,V)\{(\phi A - A\phi)\pi(Y)\}^* - g'(Y,V)\{(\phi A - A\phi)\pi(X)\}^* \\ & - g(\pi(Y),(\phi A - A\phi)\pi(X))V.\end{aligned}$$

Combining this with (2.20), we have

$$\nabla'_X A' - T'_X A' = (\tilde{\nabla}_{\pi(X)}A)^*. \tag{3.2}$$

By our assumption and (2.5) we have

$$T_W\xi = \phi A W, \quad W \in TM. \tag{3.3}$$

Taking the covariant differentiation of (3.3), we obtain

$$(\tilde{\nabla}_{\pi(X)}T)_W\xi + T_W(\tilde{\nabla}_{\pi(X)}\xi) = (\tilde{\nabla}_{\pi(X)}\phi)AW + \phi(\tilde{\nabla}_{\pi(X)}A)W. \tag{3.4}$$

Since T, ϕ and ξ are parallel with respect to the connection $\tilde{\nabla}$, (3.4) reduces to

$$\phi(\tilde{\nabla}_{\pi(X)}A)W = 0. \tag{3.5}$$

From (2.3) and (3.5) we have

$$(\tilde{\nabla}_{\pi(X)}A)W = g((\tilde{\nabla}_{\pi(X)}A)\xi, W)\xi. \tag{3.6}$$

On the other hand we have the following:

$$\begin{aligned}(\tilde{\nabla}_{\pi(X)}A)\xi &= \tilde{\nabla}_{\pi(X)}(\alpha\xi) - A(\tilde{\nabla}_{\pi(X)}\xi)\\ &= \alpha\tilde{\nabla}_{\pi(X)}\xi\\ &= 0.\end{aligned}$$

Here we use the fact that the principal curvature α in the direction ξ is constant (cf. [5] p. 533 Lemma 2.4).

So we have $\tilde{\nabla}_{\pi(X)}A = 0$. Combining this with (3.2) we have (ii).

Finally, we prove (iii). For this purpose we define the following two tensors ϕ^* and T^*:

$$\phi^* X = (\phi\pi(X))^*, \quad T_X^* Y = (T_{\pi(X)}\pi(Y))^*,$$

where $X, Y \in TM'$. To prove (iii) it suffices to verify that these tensors and V are parallel with respect to the connection $\tilde{\nabla}' = \nabla' - T'$.

First, we prove $\tilde{\nabla}'_X V = 0$. By the definition of T' we have $T'_X V = -(\phi\pi(X))^*$. Since the right-hand side of this equals to $\nabla'_X V$ (see [9]), we get the assertion.

Next, we prove the parallelism of the tensor ϕ^*. Using (2.15) and (2.18), we have

$$\begin{aligned}(\nabla'_X\phi^*)Y &= \nabla'_X(\phi^* Y) - \phi^*(\nabla'_X Y)\\ &= \{(\nabla_{\pi(X)}\phi)\pi(Y)\}^* - g'(Y, V)\pi(X)^* + g(\pi(X), \pi(Y))V.\end{aligned}$$

On the other hand, we obtain

$$\begin{aligned}(T'_X\phi^*)Y &= T'_X((\phi\pi(Y))^*) - \phi^*(T'_X Y)\\ &= \{(T_{\pi(X)}\phi)\pi(Y)\}^* - g'(Y, V)\pi(X)^* + g(\pi(X), \pi(Y))V.\end{aligned}$$

So we get

$$(\tilde{\nabla}'_X\phi^*)Y = (\nabla'_X\phi^*)Y - (T'_X\phi^*)Y = 0.$$

Now, we prove $(\tilde{\nabla}'_X T^*)_Y Z = 0$. According to (2.15) and (2.18), we obtain the following expression after a long and straightforward calculation.

$$\begin{aligned}(\nabla'_X T^*)_Y Z &= \nabla'_X(T_Y^* Z) - T^*_{\nabla'_X Y} Z - T_Y^*(\nabla'_X Z)\\ &= \{(\nabla_{\pi(X)}T)_{\pi(Y)}\pi(Z)\}^*\\ &\quad + g'(X, V)\{T_{\pi(Y)}(\phi\pi(Z)) + T_{\phi\pi(Y)}\pi(Z) - \phi T_{\pi(Y)}\pi(Z)\}^*\\ &\quad + g'(Y, V)\{T_{\phi\pi(X)}\pi(Z)\}^* + g'(Z, V)\{T_{\pi(Y)}(\phi\pi(Z))\}^*\\ &\quad + g(\phi\pi(X), T_{\pi(Y)}\pi(Z))V.\end{aligned}$$

On the other hand, we obtain

$$
\begin{aligned}
(T'_X T^*)_Y Z &= T'_X T^*_Y Z - T^*_{T'_X Y} Z - T^*_Y T'_X Z \\
&= \{(T_{\pi(X)} T)_{\pi(Y)} \pi(Z)\}^* \\
&\quad + g'(X, V)\{T_{\pi(Y)}(\phi\pi(Z)) + T_{\phi\pi(Y)}\pi(Z) - \phi T_{\pi(Y)}\pi(Z)\}^* \\
&\quad + g'(Y, V)\{T_{\phi\pi(X)}\pi(Z)\}^* + g'(Z, V)\{T_{\pi(Y)}(\phi\pi(Z))\}^* \\
&\quad + g(\phi\pi(X), T_{\pi(Y)}\pi(Z)) V.
\end{aligned}
$$

So we get

$$
\begin{aligned}
(\tilde{\nabla}'_X T^*)_Y Z &= (\nabla'_X T^*)_Y Z - (T'_X T^*)_Y Z \\
&= \{(\tilde{\nabla}_{\pi(X)} T)_{\pi(Y)} \pi(Z)\}^*.
\end{aligned}
$$

By the hypothesis the right-hand side of this vanishes. The theorem is now proved by all the above arguments. ■

4. Homogeneous Structures on Real Hypersurfaces

In this section we obtain an invariant homogeneous structure on a homogeneous real hypersurface of type (B). After that we give homogeneous structures of some type of homogeneous hypersurfaces in S^{2n+1}.

First, we have the following:

THEOREM 4.1. *The following tensor T^B defines an invariant homogeneous structure on a homogeneous real hypersurface M of type* (B)

$$
T^B_X Y = \frac{\alpha}{2}\eta(X)\phi Y + \eta(Y)\phi AX - g(\phi AX, Y)\xi. \tag{4.1}
$$

Its explicit components are given by

$$
\begin{cases}
T^B_{e_i} e_j = T^B_{\phi e_i}\phi e_j = T^B_\xi \xi = 0, \\
T^B_{e_i}\xi = x\phi e_i, T^B_{\phi e_i}\xi = \dfrac{1}{x} e_i, \\
T^B_\xi e_i = \dfrac{\alpha}{2}\phi e_i, T^B_\xi \phi e_i = -\dfrac{\alpha}{2} e_i \\
T^B_{e_i}\phi e_j = -x\delta_{ij}\xi, T^B_{\phi e_i} e_j = -\dfrac{1}{x}\delta_{ij}\xi,
\end{cases} \tag{4.2}
$$

where $e_1, \ldots, e_{n-1}, \phi e_1, \ldots, \phi e_{n-1}, \xi$ *is a local field of orthonormal frames such that* $e_1, \ldots, e_{n-1}$ *(resp.* $\phi e_1, \ldots, \phi e_{n-1}$*) is an orthonormal basis of* T_x *(resp.* $T_{-1/x}$*).*

PROOF. In order to prove our theorem it suffices to prove the following four equations:

$$\tilde{\nabla}_X g = 0, \quad \tilde{\nabla}_X \xi = 0, \quad \tilde{\nabla}_X \phi = 0, \quad \tilde{\nabla}_X A = 0, \tag{4.3}$$

where $X \in TM$ and $\tilde{\nabla} = \nabla - T^B$ (see Remark 2.3).

First, we shall prove $\tilde{\nabla}_X g = 0$. By the definition of T^B we have

$$g(T_X^B Y, Z) + g(Y, T_X^B Z) = \frac{\alpha}{2}\eta(X)\{g(\phi Y, Z) + g(Y, \phi Z)\},$$

and the right-hand side of this equation vanishes, since ϕ is a skew-symmetric transformation.

Secondly, we prove $\tilde{\nabla}_X \xi = 0$. By straightforward calculation we get

$$T_X^B \xi = \phi A X = \nabla_X \xi.$$

Here we use (2.5). So we have our assertion.

Thirdly, we prove $\tilde{\nabla}_X \phi = 0$. By a straightforward calculation we have

$$\begin{aligned}(T_X^B \phi) Y &= T_X^B(\phi Y) - \phi(T_X^B Y)\\ &= \eta(Y) A X - g(AX, Y)\xi.\end{aligned}$$

Compairing this with (2.4), we obtain

$$\tilde{\nabla}_X \phi = 0.$$

Finally, we prove $\tilde{\nabla}_X A = 0$. By a straightforward calculation we obtain

$$\begin{aligned}(T_X^B A) Y &= T_X^B(AY) - A(T_X^B Y)\\ &= \frac{\alpha}{2}\eta(X)(\phi A - A\phi) Y + \eta(Y)(\alpha\phi A - A\phi A) X\\ &\quad + g((\alpha\phi A - A\phi A)X, Y)\xi.\end{aligned} \tag{4.4}$$

On the other hand, using (2.10) and (2.11), we have

$$\alpha\phi A - A\phi A = \left(-\frac{\alpha}{4}\right)(A\phi - 3\phi A). \tag{4.5}$$

Substituting (4.5) into the right-hand side of (4.4), and using (2.12), we get

$$(T_X^B A)Y = (\nabla_X A)Y.$$

So we have the assertion.

According to (2.6) and (4.3), the metric g, the curvature R and the tensor T^B on M are all parallel with respect to the connection $\tilde{\nabla} = \nabla - T^B$. These facts prove our theorem. ∎

Combining Theorem 3.1 and Theorem 4.1, we have

COROLLARY 4.2. *The following tensor T defines a homogeneous structure on* $SO(2) \times SO(n+1)/\boldsymbol{Z}_2 \times SO(n-1)$.

$$T_X Y = (T^B_{\pi(X)}\pi(Y))^* - g'(X, V)(\phi\pi(Y))^* - g'(Y, V)(\phi\pi(X))^* + g(\phi\pi(X), \pi(Y))V.$$

In the paper [6], the author proves the following.

THEOREM 4.3([6]). *Let M be a homogeneous real hypersurface of type* (A). *Then*

$$T_X^A Y = \eta(Y)\phi AX - \eta(X)\phi AY - g(\phi AX, Y)\xi$$

defines a naturally reductive homogeneous structure of M.

Here T is said to be naturally reductive if $T_X X = 0$ is satisfied for any tangent vector $X \in TM$.

According to Theorem 3.1, Theorem 4.3 and the results of [6], we have

COROLLARY 4.4. *For $S^{2p+1}(r_1) \times S^{2q+1}(r_2)$ $(r_1^2 + r_2^2 = 1)$ the following tensor T defines a homogeneous structure on it*:

$$T_X Y = (T^A_{\pi(X)}\pi(Y))^* - g'(X, V)(\phi\pi(Y))^* - g'(Y, V)(\phi\pi(X))^* + g(\phi\pi(X), \pi(Y))V.$$

REMARK 4.5. The above tensor T^B is not naturally reductive because $T_X^B X$ does not vanish. Indeed, substituting $X = v_x + \xi$, $v_x \in T_x$ into (4.1), we get $T_X^B X = (x(x^2-3))/(x^2-1)\phi v_x \neq 0$, since $0 < x < 1$ (see Theorem 2.4). In the paper [7] the author proves that the only naturally reductive homogeneous real hypersurfaces in $\boldsymbol{C}P_n$ are of type (A).

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References

[1] W. Ambrose and I. M. Singer, On homogeneous Riemannian manifolds, Duke Math. J. **25** (1958), 647–669.

[2] S. Ishihara and M. Konishi, *Differential geometry of fibred space*, Study Group of Differential Geometry, Japan, 1973.

[3] U-H. Ki, H. S. Kim and H. Nakagawa, A characterization of a real hypersurface of type B, Tsukuba J. Math. Vol. 14 No. 1 (1990), 9–26.

[4] M. Kimura and S. Maeda, On real hypersurfaces of a complex projective space, Math. Z. **202** (1989), 299–311.

[5] Y. Maeda, On real hypersurfaces of a complex projective space, J. Math. Soc. Japan **28** (1976), 529–540.

[6] S. Nagai, Naturally reductive Riemannian homogeneous structure on a homogeneous real hypersurface in a complex space form, Bollettino U. M. I. (7) **9-A** (1995), 391–400.

[7] S. Nagai, The classification of naturally reductive homogeneous real hypersurfaces in complex projective space, Arch. Math. **69** (1997), 523–528.

[8] K. Nomizu, Some results in E. Cartan's theory of isoparametric families of hypersurfaces, Bull. Amer. Math. Soc. **79** (1973), 1184–1188.

[9] B. O'Neill, The fundamental equations of a submersion, Michigan Math. J. **13** (1966), 459–469.

[10] M. Okumura, Submanifolds of real codimension of a complex projective space, Atti Accad. Naz. Lincei **58** (1975), 544–555.

[11] R. Takagi, On homogeneous real hypersurfaces in a complex projective space, Osaka J. Math. **10** (1973), 495–506.

[12] R. Takagi, Real hypersurfaces in a complex projective space with constant principal curvatures, J. Math. Soc. Japan **27** (1975), 43–53.

[13] R. Takagi, A class of hypersurfaces with constant principal curvatures in a sphere, J. Differential Geometry **11** (1976), 225–233.

[14] F. Tricerri and L. Vanhecke, Homogeneous structures on Riemannian manifolds, London Math. Soc. Lecture Note Ser. **83** (1983).

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