TSUKUBA J. MATH.
Vol. 24 No. 1 (2000), 171–188

AREA-MINIMIZING OF THE CONE OVER SYMMETRIC *R*-SPACES

By

Daigo HIROHASHI, Takahiro KANNO and Hiroyuki TASAKI

Introduction

Let B denote a submanifold of the unit sphere in $\boldsymbol{R}^n$ and C_B the cone over B, which is the union of rays starting from the origin and passing through B.

A cone is called area-minimizing if the truncated cone C_B^1 inside the unit ball is area-minimizing among all surfaces with boundary B. The surfaces we will use are integral currents. A tangent cone to surface S at a point $p \in S$ can be thought of as the union of rays starting from p and tangent to S at p. This is the generalization of the notion of tangent plane. If the tangent cone at p is not a plane, then p is a singular point of S. If S is area-minimizing, then each tangent cone to S is area-minimizing. Thus in order to study area-minimizing surface with singularities, we need to know which cones are area-minimizing.

G. R. Lawlor proposed a criterion for area-minimization in [5]. His principal idea is to construct an area-nonincreasing retraction $\Pi : \boldsymbol{R}^n \to C$. If S is another surface which has the same boundary as C_B^1, it will follow that

$$\mathrm{vol}(S) \geq \mathrm{vol}(\Pi(S)) \geq \mathrm{vol}(C_B^1)$$

since $\Pi(S)$ must cover all of C_B^1. Using this method, he gave a complete classification of area-minimizing cones C over products of spheres and the first example of minimizing cone over a nonorientable manifold. In order to construct the retraction he solved a differential equation with numerical analysis.

In this paper, we consider the canonical imbeddings of symmetric R-spaces which are linear isotropy orbits of symmetric pairs. Using root systems, we construct area-nonincreasing retractions concretely.

In section 1 we prepare some notation and terminology, and prove an essential theorem (Theorem 1.6) for construction of the retractions. In section 2 we describe the canonical imbeddings of symmetric R-spaces, and construct

Received March 8, 1999
Revised May 21, 1999

retractions onto the cones over them. In section 3 we apply the result of section 2 to symmetric R-spaces associated with symmetric pairs of type B_l.

Concerning the cones over symmetric R-spaces, B. N. Cheng [1] proved the cone over $U(n)/O(n)$ and $U(n)$ are area-minimizing in $\boldsymbol{R}^{n^2+n}$ for $n \geq 7$ and $\boldsymbol{R}^{2n^2}$, respectively, by calibration. G. R. Lawlor [5] proved the cone over $SO(n)$ are area-minimizing in $\boldsymbol{R}^{n^2}$. Using the criterion of Lawlor in [5], M. Kerckhove [4] proved the cone over an isolated orbit of the action of $SU(n)$ on the unit sphere in the vector space of traceless n-by-n Hermitian symmetric matrices is area-minimizing for $n > 2$ and the cone over an isolated orbit of the adjoint action of $SO(n)$ is area-minimizing for $n > 3$.

The authors would like to thank the referee for reading carefully the manuscript and for pointing out some mistakes in it.

1. Preliminaries

Let G be a compact connected Lie group and θ an involutive automorphism of G. We denote by G_θ the closed subgroup of all fixed points of θ in G. For a closed subgroup K of G which lies between G_θ and the identity component of G_θ, (G, K) is a Riemannian symmetric pair. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K respectively. The involutive automorphism θ of G induces an involutive automorphism of $\mathfrak{g}$, which is also denoted by θ. Since K lies between G_θ and the identity component of G_θ, we have

$$\mathfrak{k} = \{X \in \mathfrak{g} \mid \theta(X) = X\}.$$

An inner product $\langle\, , \rangle$ on $\mathfrak{g}$ which is invariant under the actions of $\mathrm{Ad}(G)$ and θ induces a bi-invariant Riemannian metric on G and G-invariant Riemannian metric on the homogeneous space $M = G/K$, which are also denoted by the same symbol $\langle\, , \rangle$. Then M is a compact Riemannian symmetric space with respect to $\langle\, , \rangle$. Conversely any compact symmetric space is constructed in this way. Put

$$\mathfrak{m} = \{X \in \mathfrak{g} \mid \theta(X) = -X\}.$$

Since θ is involutive, we have an orthogonal direct sum decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{m}.$$

This decomposition is called a canonical decomposition of the orthogonal symmetric Lie algebra $(\mathfrak{g}, \theta)$.

Take and fix a maximal Abelian subspace $\mathfrak{a}$ in $\mathfrak{m}$ and a maximal Abelian subalgebra $\mathfrak{t}$ in $\mathfrak{g}$ including $\mathfrak{a}$. Let $\mathfrak{c}$ be the center of $\mathfrak{g}$ and $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$. We have an orthogonal direct sum decomposition:

$$\mathfrak{g} = \mathfrak{c} + \mathfrak{g}'.$$

We set

$$\mathfrak{a}' = \mathfrak{a} \cap \mathfrak{g}', \quad \mathfrak{c}_{\mathfrak{m}} = \mathfrak{c} \cap \mathfrak{m}.$$

We have an orthogonal direct sum decomposition:

$$\mathfrak{a} = \mathfrak{c}_{\mathfrak{m}} + \mathfrak{a}'.$$

Put

$$\mathfrak{b} = \mathfrak{t} \cap \mathfrak{k}.$$

Since $\mathfrak{t}$ is θ-invariant we get an orthogonal direct sum decomposition of $\mathfrak{t}$:

$$\mathfrak{t} = \mathfrak{b} + \mathfrak{a}.$$

For $\alpha \in \mathfrak{t}$ we put

$$\tilde{\mathfrak{g}}_{\alpha} = \{X \in \mathfrak{g}^{C} \mid [H, X] = \sqrt{-1}\langle \alpha, H \rangle X (H \in \mathfrak{t})\}$$

and define the root system $\tilde{R}(\mathfrak{g})$ of $\mathfrak{g}$ by

$$\tilde{R}(\mathfrak{g}) = \{\alpha \in \mathfrak{t} - \{0\} \mid \tilde{\mathfrak{g}}_{\alpha} \neq \{0\}\}.$$

We also denote $\tilde{R}$ instead of $\tilde{R}(\mathfrak{g})$. For $\alpha \in \mathfrak{a}$ we put

$$\mathfrak{g}_{\alpha} = \{X \in \mathfrak{g}^{C} \mid [H, X] = \sqrt{-1}\langle \alpha, H \rangle X (H \in \mathfrak{a})\}$$

and define the root system $R(\mathfrak{g}, \mathfrak{k})$ of $(\mathfrak{g}, \mathfrak{k})$ by

$$R(\mathfrak{g}, \mathfrak{k}) = \{\alpha \in \mathfrak{a} - \{0\} \mid \mathfrak{g}_{\alpha} \neq \{0\}\}.$$

We also denote R instead of $R(\mathfrak{g}, \mathfrak{k})$. Put

$$\tilde{R}_0(\mathfrak{g}) = \tilde{R}(\mathfrak{g}) \cap \mathfrak{b}$$

and denote the orthogonal projection from $\mathfrak{t}$ to $\mathfrak{a}$ by $H \mapsto \bar{H}$. Then we have

$$R(\mathfrak{g}, \mathfrak{k}) = \{\bar{\alpha} \mid \alpha \in \tilde{R}(\mathfrak{g}) - \tilde{R}_0(\mathfrak{g})\}.$$

We extend a basis of $\mathfrak{a}$ to that of $\mathfrak{t}$ and define the lexicographic orderings $>$ on $\mathfrak{a}$ and $\mathfrak{t}$ with respect to these bases. Then for $H \in \mathfrak{t}$, $\bar{H} > 0$ implies $H > 0$. We denote by $\tilde{F}(\mathfrak{g})$ the fundamental system of $\tilde{R}(\mathfrak{g})$ with respect to the ordering $>$. We also denote $\tilde{F}$ instead of $\tilde{F}(\mathfrak{g})$. Put

$$\tilde{F}_0(\mathfrak{g}) = \tilde{F}(\mathfrak{g}) \cap \tilde{R}_0(\mathfrak{g}).$$

Then the fundamental system $F(\mathfrak{g},\mathfrak{k})$ of $R(\mathfrak{g},\mathfrak{k})$ with respect to the ordering $>$ is given by

$$F(\mathfrak{g},\mathfrak{k}) = \{\bar{\alpha} | \alpha \in \tilde{F}(\mathfrak{g}) - \tilde{F}_0(\mathfrak{g})\}.$$

We define positive root systems by

$$\tilde{R}_+(\mathfrak{g}) = \{\alpha \in \tilde{R}(\mathfrak{g}) | \alpha > 0\}$$

$$R_+(\mathfrak{g},\mathfrak{k}) = \{\alpha \in R(\mathfrak{g},\mathfrak{k}) | \alpha > 0\}.$$

We also denote $\tilde{R}_+$ and R_+ instead of $\tilde{R}_+(\mathfrak{g})$ and $R_+(\mathfrak{g},\mathfrak{k})$. Then

$$R_+(\mathfrak{g},\mathfrak{k}) = \{\bar{\alpha} | \alpha \in \tilde{R}_+(\mathfrak{g}) - \tilde{R}_0(\mathfrak{g})\}$$

holds. We set

$$\mathfrak{k}_0 = \{X \in \mathfrak{k} \mid [X,H] = 0 (H \in \mathfrak{a})\}$$

and

$$\mathfrak{k}_\alpha = \mathfrak{k} \cap (\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha})$$

$$\mathfrak{m}_\alpha = \mathfrak{m} \cap (\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha})$$

for $\alpha \in R_+(\mathfrak{g},\mathfrak{k})$. We have the following lemma ([2]).

LEMMA 1.1. (1) *We have orthogonal direct sum decompositions*:

$$\mathfrak{k} = \mathfrak{k}_0 + \sum_{\alpha \in R_+} \mathfrak{k}_\alpha, \quad \mathfrak{m} = \mathfrak{a} + \sum_{\alpha \in R_+} \mathfrak{m}_\alpha.$$

(2) *For each* $\alpha \in \tilde{R}_+ - \tilde{R}_0$ *there exist* $S_\alpha \in \mathfrak{k}$ *and* $T_\alpha \in \mathfrak{m}$ *such that*

$$\{S_\alpha | \alpha \in \tilde{R}_+, \bar{\alpha} = \lambda\}, \quad \{T_\alpha | \alpha \in \tilde{R}_+, \bar{\alpha} = \lambda\}$$

are respectively orthonormal bases of $\mathfrak{k}_\lambda$, $\mathfrak{m}_\lambda$ *and that for* $H \in \mathfrak{a}$

$$[H, S_\alpha] = \langle \alpha, H \rangle T_\alpha, \quad [H, T_\alpha] = -\langle \alpha, H \rangle S_\alpha.$$

We denote $m_\lambda = \dim \mathfrak{m}_\lambda = \dim \mathfrak{k}_\lambda$ and call it the multiplicity of λ.

We define a subset D of $\mathfrak{a}$ by

$$D = \bigcup_{\alpha \in R} \{H \in \mathfrak{a} \mid \langle \alpha, H \rangle = 0\}.$$

Each connected component of $\mathfrak{a} - D$ is called a Weyl chamber. We define the fundamental Weyl chamber by

$$C = \{H \in \mathfrak{a} \mid \langle \alpha, H \rangle > 0 \ (\alpha \in F(\mathfrak{g}, \mathfrak{k}))\}.$$

Its closure is given by

$$\bar{C} = \{H \in \mathfrak{a} \mid \langle \alpha, H \rangle \geq 0 \ (\alpha \in F(\mathfrak{g}, \mathfrak{k}))\}.$$

For each subset $\Delta \subset F = F(\mathfrak{g}, \mathfrak{k})$ we define a subset C^Δ of $\bar{C}$ by

$$C^\Delta = \{H \in \bar{C} \mid \langle \alpha, H \rangle > 0 \ (\alpha \in \Delta), \langle \beta, H \rangle = 0 \ (\beta \in F - \Delta)\}.$$

We easily get the following lemma.

LEMMA 1.2. (1) *For* $\Delta_1 \subset F$

$$\overline{C^{\Delta_1}} = \bigcup_{\Delta \subset \Delta_1} C^\Delta$$

is a disjoint union. In particular $\bar{C} = \bigcup_{\Delta \subset F} C^\Delta$ *is a disjoint union.*
(2) *For* $\Delta_1, \Delta_2 \subset F$, $\Delta_1 \subset \Delta_2$ *if and only if* $C^{\Delta_1} \subset \overline{C^{\Delta_2}}$.

For $\beta \in F$ we take $H_\beta \in \mathfrak{a}'$ satisfying the following condition.

$$\langle \alpha, H_\beta \rangle = \begin{cases} 1 & (\alpha = \beta) \\ 0 & (\alpha \neq \beta). \end{cases}$$

We have

$$\bar{C} = \mathfrak{c}_\mathfrak{m} \times \left\{ \sum_{\alpha \in F} t_\alpha H_\alpha \,\middle|\, t_\alpha \geq 0 \right\}$$

and for $\Delta \subset F$

$$C^\Delta = \mathfrak{c}_\mathfrak{m} \times \left\{ \sum_{\alpha \in \Delta} t_\alpha H_\alpha \,\middle|\, t_\alpha > 0 \right\}.$$

For $H \in \mathfrak{m}$ we put

$$Z^H = \{g \in G \mid \mathrm{Ad}(g)H = H\},$$

$$Z_K^H = \{k \in K \mid \mathrm{Ad}(k)H = H\}.$$

$Z_K^H = Z^H \cap K$ holds. Z^H is a closed subgroup of G and Z_K^H is a closed subgroup of K. We can prove the following lemma by the standard argument of compact Lie groups, so we omit its proof.

LEMMA 1.3. *Z^H is connected.*

For $\Delta \subset F$ we put

$$N^\Delta = \{g \in G \mid \mathrm{Ad}(g)C^\Delta = C^\Delta\}$$

$$Z^\Delta = \{g \in G \mid \mathrm{Ad}(g)|_{C^\Delta} = 1\}$$

$$N_K^\Delta = \{k \in K \mid \mathrm{Ad}(k)C^\Delta = C^\Delta\}$$

$$Z_K^\Delta = \{k \in K \mid \mathrm{Ad}(k)|_{C^\Delta} = 1\}.$$

By the above definitions we have $N_K^\Delta = N^\Delta \cap K$ and $Z_K^\Delta = Z^\Delta \cap K$. Z^Δ is a closed subgroup of G and Z_K^Δ is a closed subgroup of K. If $H \in C^\Delta$, then

$$Z^\Delta \subset Z^H, \quad Z_K^\Delta \subset Z_K^H.$$

We put

$$R^\Delta = R \cap (F - \Delta)_Z$$

$$R_+^\Delta = R^\Delta \cap R_+$$

$$\mathfrak{g}^\Delta = \mathfrak{k}_0 + \mathfrak{a} + \sum_{\alpha \in R_+^\Delta} (\mathfrak{k}_\alpha + \mathfrak{m}_\alpha)$$

and

$$\mathfrak{k}^\Delta = \mathfrak{g}^\Delta \cap \mathfrak{k} = \mathfrak{k}_0 + \sum_{\alpha \in R_+^\Delta} \mathfrak{k}_\alpha$$

$$\mathfrak{m}^\Delta = \mathfrak{g}^\Delta \cap \mathfrak{m} = \mathfrak{a} + \sum_{\alpha \in R_+^\Delta} \mathfrak{m}_\alpha.$$

We have an orthogonal direct sum decomposition:

$$\mathfrak{g}^\Delta = \mathfrak{k}^\Delta + \mathfrak{m}^\Delta.$$

LEMMA 1.4. *For $\Delta \subset F$ and $H \in C^\Delta$, we obtain the following equations.*
(1) $R_+^\Delta = \{\alpha \in R_+ \mid \langle \alpha, H \rangle = 0\}$
(2) $R^\Delta = \{\alpha \in R \mid \langle \alpha, H \rangle = 0\}$
(3) $\mathfrak{g}^\Delta = \{X \in \mathfrak{g} \mid [H, X] = 0\}$

PROOF. Any $\alpha \in R_+$ can be written as follows:

$$\alpha \in \sum_{\gamma \in F} n_\gamma \gamma \quad (n_\gamma \in \boldsymbol{Z},\ n_\gamma \geq 0).$$

So we obtain

$$\langle \alpha, H \rangle = \sum_{\gamma \in F} n_\gamma \langle \gamma, H \rangle = \sum_{\gamma \in \Delta} n_\gamma \langle \gamma, H \rangle.$$

From this $\langle \alpha, H \rangle = 0$ if and only if $\alpha \in R_+^\Delta$. Therefore we obtain

$$R_+^\Delta = \{\alpha \in R_+ \mid \langle \alpha, H \rangle = 0\}.$$

This implies

$$R^\Delta = \{\alpha \in R \mid \langle \alpha, H \rangle = 0\}.$$

Any $X \in \mathfrak{g}$ can be written as follows:

$$X = S_0 + \sum_{\alpha \in R_+} a_\alpha S_\alpha + T_0 + \sum_{\alpha \in R_+} b_\alpha T_\alpha,$$

where $S_0 \in \mathfrak{k}_0$ and $T_0 \in \mathfrak{a}$. It follows from Lemma 1.1 and (1) that

$$[H, X] = \sum_{\alpha \notin R_+^\Delta} a_\alpha \langle \alpha, H \rangle T_\alpha - \sum_{\alpha \notin R_+^\Delta} b_\alpha \langle \alpha, H \rangle S_\alpha.$$

From this $[H, X] = 0$ if and only if $X \in \mathfrak{g}^\Delta$. Therefore we obtain

$$\mathfrak{g}^\Delta = \{X \in \mathfrak{g} \mid [H, X] = 0\}.$$

Lemma 1.5. (1) *Take* $\Delta_1, \Delta_2 \subset F$, $H_1 \in C^{\Delta_1}$, $H_2 \in C^{\Delta_2}$ *and* $g \in G$. *If* $\mathrm{Ad}(g)H_1 = H_2$, *then* $\mathrm{Ad}(g)\mathfrak{g}^{\Delta_1} = \mathfrak{g}^{\Delta_2}$.

(2) *If* $\Delta \subset F$, *then* $N^\Delta \subset N(\mathfrak{g}^\Delta)$. *For any* $H \in C^\Delta$, *all of* Z^H, Z^Δ, N^Δ *and* $N(\mathfrak{g}^\Delta)$ *are compact subgroups of* G *and all of their Lie subalgebras coincide with* $\mathfrak{g}^\Delta$.

Proof. (1) By Lemma 1.4 $\mathfrak{g}^{\Delta_1}$ is the centralizer of H_1. $\mathrm{Ad}(g)\mathfrak{g}^{\Delta_1}$ is the centralizer of $\mathrm{Ad}(g)H_1 = H_2$, so is equal to $\mathfrak{g}^{\Delta_2}$.

(2) (1) with $\Delta_1 = \Delta_2 = \Delta$ implies $N^\Delta \subset N(\mathfrak{g}^\Delta)$. Z^H and $N(\mathfrak{g}^\Delta)$ are compact subgroups in G.

Since $\mathfrak{g}^\Delta$ is the centralizer of H by Lemma 1.4, the Lie algebra $\mathscr{L}(Z^H)$ of Z^H is equal to $\mathfrak{g}^\Delta$.

We show that the Lie algebras $\mathscr{L}(Z^\Delta)$, $\mathscr{L}(N^\Delta)$ and $\mathscr{L}(N(\mathfrak{g}^\Delta))$ of Z^Δ, N^Δ and $N(\mathfrak{g}^\Delta)$ are all equal to $\mathfrak{g}^\Delta$. By their definitions $Z^\Delta \subset N^\Delta \subset N(\mathfrak{g}^\Delta)$. Z^Δ, $N(\mathfrak{g}^\Delta)$ is a closed subgroup of G, so we have $\mathscr{L}(Z^\Delta) \subset \mathscr{L}(N(\mathfrak{g}^\Delta))$. Any element X of $\mathfrak{g}^\Delta$ can be written as follows:

$$X = S_0 + \sum_{\alpha \in R_+^\Delta} a_\alpha S_\alpha + T_0 + \sum_{\alpha \in R_+^\Delta} b_\alpha T_\alpha \quad (S_0 \in \mathfrak{k}_0, T_0 \in \mathfrak{a}).$$

From this we get $X \in \mathscr{L}(Z^\Delta)$, hence $\mathfrak{g}^\Delta \subset \mathscr{L}(Z^\Delta)$. Therefore

$$\mathfrak{g}^\Delta \subset \mathscr{L}(Z^\Delta) \subset \mathscr{L}(N(\mathfrak{g}^\Delta)).$$

Conversely we assume an element

$$X = S_0 + \sum_{\alpha \in R_+} a_\alpha S_\alpha + T_0 + \sum_{\alpha \in R_+} b_\alpha T_\alpha \quad (S_0 \in \mathfrak{k}_0, T_0 \in \mathfrak{a})$$

in $\mathfrak{g}$ is contained in $\mathscr{L}(N(\mathfrak{g}^\Delta))$. Since

$$[H, X] = \sum_{\alpha \in R_+} a_\alpha \langle \alpha, H \rangle T_\alpha - \sum_{\alpha \in R_+} b_\alpha \langle \alpha, H \rangle S_\alpha$$

we obtain $X \in \mathfrak{g}^\Delta$. Therefore we get

$$\mathfrak{g}^\Delta = \mathscr{L}(Z^\Delta) = \mathscr{L}(N(\mathfrak{g}^\Delta)).$$

N^Δ satisfies $Z^\Delta \subset N^\Delta \subset N(\mathfrak{g}^\Delta)$ and $\mathscr{L}(Z^\Delta) = \mathscr{L}(N(\mathfrak{g}^\Delta))$. So N^Δ is also a compact subgroup of G and $\mathscr{L}(N^\Delta) = \mathfrak{g}^\Delta$ holds.

THEOREM 1.6. *For any $\Delta \subset F$ and $H \in C^\Delta$*

$$Z^\Delta = Z^H = N^\Delta, \quad Z_K^\Delta = Z_K^H = N_K^\Delta.$$

PROOF. By the definition we have $Z^\Delta \subset Z^H$ and by the above lemma their Lie algebras coincides. Moreover Z^H is connected by Lemma 1.3, so we obtain $Z^\Delta = Z^H$.

Z^Δ and N^Δ are compact and have the same Lie algebra $\mathfrak{g}^\Delta$. Since Z^Δ is the kernel of the homomorphism

$$\mathrm{Ad} : N^\Delta \to \text{The permutation group of } C^\Delta,$$

Z^Δ is a normal subgroup of N^Δ and N^Δ / Z^Δ is a finite group. For any $g \in N^\Delta$, the action of $\mathrm{Ad}(g)$ on C^Δ has a finite order, that is, there is an integer N satisfying $\mathrm{Ad}(g)^N|_{C^\Delta} = 1$. Take $H_0 \in C^\Delta$ and put

$$H_1 = \frac{1}{N}(H_0 + \mathrm{Ad}(g)H_0 + \cdots + \mathrm{Ad}(g)^{N-1}H_0).$$

Each $\mathrm{Ad}(g)^i H_0$ is contained in C^Δ and C^Δ is convex, so we get $H_1 \in C^\Delta$. $\mathrm{Ad}(g)H_1 = H_1$ holds and $g \in Z^{H_1} = Z^\Delta$. Hence $N^\Delta \subset Z^\Delta$ and we obtain $Z^\Delta = N^\Delta$.

The second equation follows from the first one.

2. Construction of Retractions

The notation of the preceding section will be preserved. Let B be a compact submanifold of the unit sphere in $\boldsymbol{R}^n$. We call $C_B = \{tx | x \in B, t \geq 0\}$ the *cone* over B. C_B is said to be *area-minimizing* if $C_B^1 = \{tx | x \in B, 0 \leq t \leq 1\}$ has the least area among all surfaces with boundary B.

For an unit vector $H \in \bar{C}$, the orbit $\mathrm{Ad}(K)H$ is a submanifold of the unit sphere in $\mathfrak{m}$. Then the mapping

$$f : kZ_K^H \mapsto \mathrm{Ad}(k)H,$$

is a diffeomorphism of the homogeneous space K/Z_K^H to $\mathrm{Ad}(K)H$.

Proposition 2.1. *The orbit* $\mathrm{Ad}(K)H$ *is connected.*

Proof. Since $\mathfrak{m} = \bigcup_{k \in K_0} \mathrm{Ad}(k) \cdot \mathfrak{a}$, where K_0 be the identity component of K, for any $\mathrm{Ad}(k)H \in \mathrm{Ad}(K)H$ there exists an element $k_1 \in K_0$ such that $\mathrm{Ad}(k_1 k)H \in \mathfrak{a}$. From Proposition 2.2 (p. 285) of [2], there exists a member of *Weyl group* whose action on $\mathfrak{a}$ is represented by $\mathrm{Ad}(k_2)$ for some $k_2 \in K_0$ such that $\mathrm{Ad}(k_1 k)H = \mathrm{Ad}(k_2)H$. If we put $k_0 = k_1^{-1} k_2 \in K_0$, then $\mathrm{Ad}(k)H = \mathrm{Ad}(k_0)H$ holds. Thus we get $\mathrm{Ad}(K)H = \mathrm{Ad}(K_0)H$ and it is connected.

From now on we assume that (G, K) is irreducible.

Let $F(\mathfrak{g}, \mathfrak{k}) = \{\alpha_1, \ldots, \alpha_l\}$ be the fundamental root system and $\tilde{\alpha} = n_1\alpha_1 + \cdots + n_l\alpha_l$ be the highest root of $R(\mathfrak{g}, \mathfrak{k})$. Select $\alpha_i \in F(\mathfrak{g}, \mathfrak{k})$ such that $n_i = 1$, we put $\alpha_0 = \alpha_i$ and $A_0 = H_{\alpha_0}/|H_{\alpha_0}|$. It is known that f is an isometry of $K/Z_K^{A_0}$ with the normal homogeneous Riemannian metric multiplied some constant onto $\mathrm{Ad}(K)A_0$ and that $K/Z_K^{A_0}$ is a symmetric space. We call this space a *symmetric R-space*, and f its *canonical imbedding*. Because $\mathrm{Ad}(K)A_0$ is an isolated orbit, $\mathrm{Ad}(K)A_0$ is a minimal submanifold of the unit sphere in $\mathfrak{m}$ by a result of Hsiang [3]. Hence the cone $C_{\mathrm{Ad}(K)A_0}$ is also a minimal submanifold of $\mathfrak{m}$ ([6] p. 97, Prop. 6.1.1). The purpose of this article is to prove $C_{\mathrm{Ad}(K)A_0}$ is an area-minimizing cone.

Proposition 2.2. *Let V and W be two vector spaces with inner products. Suppose $n = \dim W \leq \dim V$. For a linear mapping F of V to W we put*

$$JF = \sup\{|F(u_1) \wedge \cdots \wedge F(u_n)|\},$$

where $u_1, \ldots, u_n$ runs over all orthonormal vectors of V. If F is not surjective, then

$JF = 0$. *If F is surjective, then JF coincides with*

$$|F(v_1) \wedge \cdots \wedge F(v_n)|,$$

for an orthonormal base $v_1, \dots, v_n$ of $(\ker F)^{\perp}$.

Let B be a compact submanifold of the unit sphere in $\mathfrak{m}$. We call a differentiable retraction $\Phi : \mathfrak{m} \to C_B$ a *area-nonincreasing* if

$$J(d\Phi_x) \leq 1. \tag{1}$$

for $x \in \mathfrak{m}$.

PROPOSITION 2.3. *The cone C_B over a compact submanifold B of the unit sphere in $\mathfrak{m}$ is area-minimizing if there exists an area-nonincreasing retraction $\Phi : \mathfrak{m} \to C_B$.*

PROOF. Let S be a surface in $\mathfrak{m}$ with boundary B. Since $C_B^1 \subset \Phi(S)$, we have $\mathrm{vol}(C_B^1) \leq \mathrm{vol}(\Phi(S))$. Let $e_1, \dots, e_n$ be an orthonormal frame of S, then

$$\begin{aligned} \mathrm{vol}(\Phi(S)) &= \int_S |d\Phi(e_1 \wedge \cdots \wedge e_n)| \, d\mu_S \\ &\leq \int_S J(d\Phi_x) \, d\mu_S \\ &\leq \int_S d\mu_S \\ &= \mathrm{vol}(S). \end{aligned}$$

Consequently,

$$\mathrm{vol}(C_B^1) \leq \mathrm{vol}(\Phi(S)) \leq \mathrm{vol}(S).$$

This proves the proposition.

We shall now consider a way to construct area-nonincreasing retractions.

LEMMA 2.4. *Suppose ϕ is a mapping of $\bar{C}$ into itself such that $\phi(C^{\Delta}) \subset \overline{C^{\Delta}}$ for each $\Delta \subset F(\mathfrak{g}, \mathfrak{k})$. Then ϕ extends to a mapping Φ of $\mathfrak{m}$ as*

$$\Phi(X) = \mathrm{Ad}(k)\phi(H),$$

for each $X = \mathrm{Ad}(k)H \in \mathfrak{m}$ $(k \in K, H \in \bar{C})$.

PROOF. Suppose $k_1, k_2 \in K$ and $H_1, H_2 \in \bar{C}$ satisfy $\mathrm{Ad}(k_1)H_1 = \mathrm{Ad}(k_2)H_2$. Then $\mathrm{Ad}(k_2^{-1}k_1)H_1 = H_2 \in \mathfrak{a}$ and we have $H_1 = H_2$, because each orbit of the Weyl group on $\mathfrak{a}$ intersects $\bar{C}$ in exactly one point ([3], p. 293, Th. 2.22). Let

$$\Delta = \{\alpha \in F \mid \langle \alpha, H_1 \rangle > 0\}.$$

We have $H_1 \in C^\Delta$ and $\phi(H_1) \in \overline{C^\Delta}$ by the assumption of ϕ. Thus Theorem 1.6 implies

$$k_2^{-1}k_1 \in Z_K^{H_1} = Z_K^\Delta,$$

therefore $\mathrm{Ad}(k_2^{-1}k_1)\phi(H_1) = \phi(H_1)$.

From Lemma 2.4, we have the following.

PROPOSITION 2.5. *Let $\phi : \bar{C} \to \{tA_0 | t \geq 0\}$ be a differentiable mapping. Denote $\phi(x) = f(x)A_0$. If f satisfies $f(tA_0) = t$ $(t \geq 0)$ and $f|_{\{\alpha_0\}^\perp} \equiv 0$, then ϕ extends to a differentiable retraction $\Phi : \mathfrak{m} \to C_{\mathrm{Ad}(K)A_0}$.*

In this case Φ is area-nonincreasing if and only if (1) *holds for each $x \in C$.*

We will compute $J(d\Phi_x)$ of Φ in Proposition 2.5 for $x \in C$.

PROPOSITION 2.6. *We denote $R_+(A_0) = \{\lambda \in R_+ | \langle \lambda, A_0 \rangle > 0\}$.*

$$J(d\Phi_x) = |\mathrm{grad}(f)| \prod_{\lambda \in R_+(A_0)} \left(\frac{\langle \lambda, A_0 \rangle}{\langle \lambda, x \rangle} f(x) \right)^{m_\lambda}. \tag{2}$$

PROOF. If $f(x) = 0$, then the both sides of the equation are 0. So we consider the case $f(x) \neq 0$. By the definition of Φ, $d\Phi_x(\mathfrak{a}) \subset \boldsymbol{R}A_0$. By using the equation

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}(\exp tS_\alpha)x = -\langle \alpha, x \rangle T_\alpha,$$

for $\alpha \in \tilde{R}_+(\mathfrak{g})$, $\bar{\alpha} \neq 0$, we have

$$d\Phi_x(T_\alpha) = \frac{\langle \alpha, \phi(x) \rangle}{\langle \alpha, x \rangle} T_\alpha.$$

From this we get

$$d\Phi_x \left(\sum_{\lambda \in R_+} \mathfrak{m}_\lambda \right) \subset \sum_{\lambda \in R_+(A_0)} \mathfrak{m}_\lambda,$$

so we can write

$$J(d\Phi_x) = J_1(x) \cdot J_2(x),$$
$$J_1(x) = J(d\Phi_x|\mathfrak{a}),$$
$$J_2(x) = J(d\Phi_x|_{\sum \mathfrak{m}_\lambda}).$$

Take a unit vector v in $\mathfrak{a}$.

$$\begin{aligned} d\Phi_x(v) &= d\phi_x(v) \\ &= df_x(v)A_0 \\ &= \langle \mathrm{grad}(f), v\rangle A_0. \end{aligned}$$

Therefore

$$\begin{aligned} J_1(x) &= \max\{|d\Phi_x(v)| \,|v \in \mathfrak{a}, |v| = 1\} \\ &= |\mathrm{grad}(f)|. \end{aligned}$$

Secondly for $J_2(x)$, we see the kernel of $d\Phi_x|_{\sum \mathfrak{m}_\lambda}$. By the above expression of $d\Phi_x(T_\alpha)$ we get

$$\ker\left(d\Phi_x|_{\sum \mathfrak{m}_\lambda}\right) = \sum_{\substack{\lambda \in R_+ \\ \langle \lambda, A_0\rangle = 0}} \mathfrak{m}_\lambda.$$

We can take $\{T_\alpha | \alpha \in \tilde{R}_+(\mathfrak{g}), \langle \alpha, A_0\rangle > 0\}$ as an orthonormal base of $\ker\left(d\Phi_x|_{\sum \mathfrak{m}_\lambda}\right)^\perp$. It follows that

$$J_2(x) = \left| \bigwedge_{\substack{\alpha \in \tilde{R}_+ \\ \langle \alpha, A_0\rangle > 0}} d\Phi_x(T_\alpha) \right| = \prod_{\lambda \in R_+(A_0)} \left(\frac{\langle \lambda, A_0\rangle}{\langle \lambda, x\rangle} f(x) \right)^{m_\lambda},$$

where m_λ is the multiplicity of λ. So we have

$$J(d\Phi_x) = |\mathrm{grad}(f)| \prod_{\lambda \in R_+(A_0)} \left(\frac{\langle \lambda, A_0\rangle}{\langle \lambda, x\rangle} f(x) \right)^{m_\lambda}.$$

3. Construction of Area-nonincreasing Retractions

THEOREM 3.1. *The cones over*

$$\frac{SO(2l+1)}{SO(2) \times SO(2l-1)}, \quad \frac{SO(l) \times SO(l+n)}{S'(O(l-1) \times O(l+n-1))}$$

corresponding to symmetric pairs

$$(SO(2l+1)^2, SO(2l+1)), \quad (SO(2l+n), SO(l)\times SO(l+n)) \quad (n\geq 2)$$

respectively are area-minimizing, where

$$S'(O(l-1)\times O(l+n-1))$$
$$=\left\{\begin{pmatrix} \varepsilon & & & \\ & A & & \\ & & \varepsilon & \\ & & & B\end{pmatrix}\in SO(l)\times SO(l+n) \;\middle|\; \begin{matrix}\varepsilon=\pm 1, A\in O(l-1),\\ B\in O(l+n-1)\end{matrix}\right\}.$$

Proof. We consider symmetric pairs of type B_l. Let $\varepsilon_1,\dots,\varepsilon_l$ be an orthonormal basis of the maximal Abelian subspace $\mathfrak{a}$ such that all roots are

$$\pm\varepsilon_i\pm\varepsilon_j \ (1\leq i<j\leq l), \quad \pm\varepsilon_i \ (1\leq i\leq l).$$

Then for a suitable ordering

$$F(\mathfrak{g},\mathfrak{k})=\{\alpha_1,\alpha_2,\dots,\alpha_{l-1},\alpha_l\},$$
$$\alpha_i=\varepsilon_i-\varepsilon_{i+1} \ (1\leq i<l), \quad \alpha_l=\varepsilon_l,$$
$$H_{\alpha_i}=\varepsilon_1+\cdots+\varepsilon_i \ (1\leq i\leq l),$$
$$\tilde{\alpha}=\alpha_1+2\alpha_2+\cdots+2\alpha_l=\varepsilon_1+\varepsilon_2$$

and we put

$$A_0=\frac{H_{\alpha_1}}{|H_{\alpha_1}|}=\varepsilon_1.$$

We have

$$R_+(A_0)=\left\{\sum_{i=1}^{l}\alpha_i\right\}\cup\left\{\sum_{i=1}^{k-1}\alpha_i \;\middle|\; 2\leq k\leq l\right\}$$
$$\cup\left\{\tilde{\alpha}-\sum_{i=2}^{k}\alpha_i \;\middle|\; 2\leq k\leq l-1\right\}\cup\{\tilde{\alpha}\}$$

Because the multiplicities of roots of same length coincide with each other, we can denote by m_1 the multiplicity of the $\sum_{i=1}^{l}\alpha_i$ and by m_2 the multiplicity of the rest. For $x=\sum_{i=1}^{l}x_iH_{\alpha_i}\in\bar{C}$ we define

$$f(x) = \sqrt{\langle \alpha_1, x \rangle \langle \tilde{\alpha}, x \rangle} = \sqrt{x_1(x_1 + 2x_2 + \cdots + 2x_l)}$$

then it satisfies Proposition 2.5. Using (2), we calculate $J(d\Phi_x)$. Since

$$\frac{\partial f}{\partial x_1} = \frac{x_1 + \cdots + x_l}{f},$$

$$\frac{\partial f}{\partial x_i} = \frac{x_1}{f} \quad (2 \leq i \leq l),$$

we get

$$J_1(x) = |\mathrm{grad}(f)| = \left| \sum_{i=1}^{l} \frac{\partial f}{\partial x_i} \alpha_i \right| = \sqrt{\frac{\langle \alpha_1, x \rangle^2 + \langle \tilde{\alpha}, x \rangle^2}{2\langle \alpha_1, x \rangle \langle \tilde{\alpha}, x \rangle}}.$$

We also obtain

$$J_2(x) = \left(\frac{f}{\langle \sum_{i=1}^{l} \alpha_i, x \rangle} \right)^{m_1} \times \prod_{k=2}^{l-1} \left(\frac{f^2}{\langle \sum_{i=1}^{k} \alpha_i, x \rangle \langle \tilde{\alpha} - \sum_{i=2}^{k} \alpha_i, x \rangle} \right)^{m_2}.$$

Therefore

$$J(d\Phi_x) = \sqrt{\frac{\langle \alpha_1, x \rangle^2 + \langle \tilde{\alpha}, x \rangle^2}{2\langle \alpha_1, x \rangle \langle \tilde{\alpha}, x \rangle}} \left(\frac{\sqrt{\langle \alpha_1, x \rangle \langle \tilde{\alpha}, x \rangle}}{\langle \sum_{i=1}^{l} \alpha_i, x \rangle} \right)^{m_1}$$

$$\times \prod_{k=2}^{l-1} \left(\frac{\langle \alpha_1, x \rangle \langle \tilde{\alpha}, x \rangle}{\langle \sum_{i=1}^{k} \alpha_i, x \rangle \langle \tilde{\alpha} - \sum_{i=2}^{k} \alpha_i, x \rangle} \right)^{m_2}.$$

For any k,

$$\begin{aligned}
&\left\langle \sum_{i=1}^{k} \alpha_i, x \right\rangle \left\langle \tilde{\alpha} - \sum_{i=2}^{k} \alpha_i, x \right\rangle \\
&= \left\langle \sum_{i=1}^{k} \alpha_i, x \right\rangle \langle \tilde{\alpha}, x \rangle - \left\langle \sum_{i=1}^{k} \alpha_i, x \right\rangle \left\langle \sum_{i=2}^{k} \alpha_i, x \right\rangle \\
&= \langle \alpha_1, x \rangle \langle \tilde{\alpha}, x \rangle + \left\langle \sum_{i=2}^{k} \alpha_i, x \right\rangle \langle \tilde{\alpha}, x \rangle - \left\langle \sum_{i=1}^{k} \alpha_i, x \right\rangle \left\langle \sum_{i=2}^{k} \alpha_i, x \right\rangle \\
&= \langle \alpha_1, x \rangle \langle \tilde{\alpha}, x \rangle + \left\langle \sum_{i=2}^{k} \alpha_i, x \right\rangle \left(\langle \tilde{\alpha}, x \rangle - \left\langle \sum_{i=1}^{k} \alpha_i, x \right\rangle \right) \\
&\geq \langle \alpha_1, x \rangle \langle \tilde{\alpha}, x \rangle,
\end{aligned}$$

hence

$$\frac{\langle \alpha_1, x\rangle\langle \tilde{\alpha}, x\rangle}{\langle \sum_{i=1}^{k} \alpha_i, x\rangle\langle \tilde{\alpha} - \sum_{i=2}^{k} \alpha_i, x\rangle} \leq 1.$$

On the other hand

$$\begin{aligned}
&\left\langle \sum_{i=1}^{l} \alpha_i, x \right\rangle^2 - \langle \alpha_1, x\rangle\langle \tilde{\alpha}, x\rangle \\
&= \left\langle \sum_{i=1}^{l} \alpha_i, x \right\rangle^2 - \langle \alpha_1, x\rangle \left\langle \sum_{i=1}^{l} \alpha_i + \sum_{i=2}^{l} \alpha_i, x \right\rangle \\
&= \left\langle \sum_{i=1}^{l} \alpha_i, x \right\rangle \left\langle \sum_{i=2}^{l} \alpha_i, x \right\rangle - \langle \alpha_1, x\rangle \left\langle \sum_{i=2}^{l} \alpha_i, x \right\rangle \\
&= \left\langle \sum_{i=2}^{l} \alpha_i, x \right\rangle^2 \geq 0.
\end{aligned}$$

Therefore

$$\frac{\sqrt{\langle \alpha_1, x\rangle\langle \tilde{\alpha}, x\rangle}}{\langle \sum_{i=1}^{l} \alpha_i, x\rangle} \leq 1.$$

We consider the case $m_1 \geq 2$ and let

$$A = J_1(x)\left(\frac{\sqrt{\langle \alpha_1, x\rangle\langle \tilde{\alpha}, x\rangle}}{\langle \sum_{i=1}^{l} \alpha_i, x\rangle}\right)^2.$$

If $A \leq 1$ then $J(d\Phi_x) \leq 1$.

$$\begin{aligned}
A^2 &= \frac{\langle \alpha_1, x\rangle^2 + \langle \tilde{\alpha}, x\rangle^2}{2\langle \alpha_1, x\rangle\langle \tilde{\alpha}, x\rangle} \frac{\langle \alpha_1, x\rangle^2\langle \tilde{\alpha}, x\rangle^2}{\langle \sum_{i=1}^{l} \alpha_i, x\rangle^4} \\
&= \frac{8(\langle \alpha_1, x\rangle^3\langle \tilde{\alpha}, x\rangle + \langle \alpha_1, x\rangle\langle \tilde{\alpha}, x\rangle^3)}{(\langle \tilde{\alpha}, x\rangle + \langle \alpha_1, x\rangle)^4}.
\end{aligned}$$

Here subtracting the numerator from the denominator we have

$$\begin{aligned}
&(\langle \tilde{\alpha}, x\rangle + \langle \alpha_1, x\rangle)^4 - 8(\langle \alpha_1, x\rangle^3\langle \tilde{\alpha}, x\rangle + \langle \alpha_1, x\rangle\langle \tilde{\alpha}, x\rangle^3) \\
&= (\langle \tilde{\alpha}, x\rangle - \langle \alpha_1, x\rangle)^4 \geq 0.
\end{aligned}$$

Therefore if $m_1 \geq 2$, we get $J(d\Phi_x) \leq 1$.

There are two kind of symmetric pairs: (i) $m_1 = n$ $(n \geq 2)$, $m_2 = 1$ and (ii) $m_1 = m_2 = 2$.

(i) $(G, K) = (SO(2l+n), SO(l) \times SO(l+n))$. It is defined by the involution:

$$\theta(g) = I_{l,l+n} g I_{l,l+n} \quad (g \in SO(2l+n)), \quad I_{l,l+n} = \begin{pmatrix} -I_l & 0 \\ 0 & I_{l+n} \end{pmatrix}.$$

Then

$$\mathfrak{g} = \mathfrak{so}(2l+n) = \{X \in M_{2l+n}(\boldsymbol{R}) \mid X + {}^tX = 0\},$$

$$\mathfrak{k} = \mathfrak{so}(l) \times \mathfrak{so}(l+n) = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \middle| x \in \mathfrak{so}(l), y \in \mathfrak{so}(l+n) \right\},$$

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & x \\ -{}^tx & 0 \end{pmatrix} \middle| x \in M_{l,l+n}(\boldsymbol{R}) \right\} \cong M_{l,l+n}(\boldsymbol{R}).$$

Since $\mathfrak{m}$ is isomorphic to $M_{l,l+n}(\boldsymbol{R})$, we identify them. The action of K on $\mathfrak{m}$ through this identification is

$$\mathrm{Ad}\begin{pmatrix} k & 0 \\ 0 & k' \end{pmatrix} \cdot x = kx{}^tk'.$$

We define an $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{m}$ by

$$\langle X, Y \rangle = \mathrm{Trace}({}^tXY) \quad (X, Y \in \mathfrak{m}).$$

Let

$$\mathfrak{a} = \left\{ \begin{pmatrix} 0 & t \\ -{}^tt & 0 \end{pmatrix} \middle| t = \begin{pmatrix} t_1 & & & \\ & \ddots & & \mathbf{0} \\ & & t_l & \end{pmatrix}, t_i \in \boldsymbol{R} \right\}$$

$$\cong \left\{ t = \begin{pmatrix} t_1 & & & \\ & \ddots & & \mathbf{0} \\ & & t_l & \end{pmatrix} \right\}.$$

Then $\mathfrak{a}$ is a maximal Abelian subspace in $\mathfrak{m}$. Next, we consider a root space decomposition of $\mathfrak{m}$ with respect to $\mathfrak{a}$. Let E_{pq} be a matrix whose (p,q)-entry is 1 and all other entries are 0. Then

$$\varepsilon_p = E_{pp},$$

$$\mathfrak{m}_{\varepsilon_p} = \sum_{q=l+1}^{l+n} \boldsymbol{R} E_{pq},$$

$$\mathfrak{m}_{\varepsilon_p - \varepsilon_q} = \boldsymbol{R}(E_{pq} + E_{qp}),$$

$$\mathfrak{m}_{\varepsilon_p + \varepsilon_q} = \boldsymbol{R}(E_{pq} - E_{qp}).$$

Since $A_0 = \varepsilon_1 = E_{11}$, we have

$$Z_K^{A_0} = S'(O(l-1) \times O(l+n-1)).$$

Hence the corresponding symmetric R-space is

$$\frac{SO(l) \times SO(l+n)}{S'(O(l-1) \times O(l+n-1))}.$$

(ii) $(G, K) = (SO(2l+1)^2, SO(2l+1))$. It is defined by the involution such that

$$\theta(g_1, g_2) = (g_2, g_1) \quad ((g_1, g_2) \in SO(2l+1)).$$

Then

$$\mathfrak{g} = \mathfrak{so}(2l+1) \times \mathfrak{so}(2l+1),$$

$$\mathfrak{k} = \{(x, x) | x \in \mathfrak{so}(2l+1)\},$$

$$\mathfrak{m} = \{(x, -x) | x \in \mathfrak{so}(2l+1)\} \cong \mathfrak{so}(2l+1).$$

Since $\mathfrak{m}$ is isomorphic to $\mathfrak{so}(2l+1)$, we identify them. The action of K on $\mathfrak{m}$ through this identification is

$$\mathrm{Ad}(k) \cdot x = kx{}^tk \quad (k \in SO(2l+1), x \in \mathfrak{so}(2l+1)).$$

We define an $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{m}$ by

$$\langle X, Y \rangle = \frac{1}{2}\mathrm{Trace}({}^tXY) \quad (X, Y \in \mathfrak{m}).$$

Let

$$\mathfrak{a} = \left\{ t = \begin{pmatrix} t_1' & & & \\ & \ddots & & \\ & & t_l' & \\ & & & 0 \end{pmatrix} \middle| \, t_i' = \begin{pmatrix} 0 & t_i \\ -t_i & 0 \end{pmatrix}, \, t_i \in \boldsymbol{R} \right\} \subset \mathfrak{m} \cong \mathfrak{so}(2l+1).$$

Then $\mathfrak{a}$ is a maximal Abelian subspace in $\mathfrak{m}$. Next, we consider a root space decomposition of $\mathfrak{m}$ with respect to $\mathfrak{a}$. Let $G_{pq} = E_{pq} - E_{qp}$. Then

$$\varepsilon_p = G_{2p-1,2p},$$

$$\mathfrak{m}_{\varepsilon_p} = \boldsymbol{R}G_{2p-1,2l+1} + \boldsymbol{R}G_{2p,2l+1},$$

$$\mathfrak{m}_{\varepsilon_p-\varepsilon_q} = \boldsymbol{R}(G_{2p-1,2q-1} - G_{2p,2q}) + \boldsymbol{R}(G_{2p-1,2q} + G_{2p,2q-1}),$$

$$\mathfrak{m}_{\varepsilon_p+\varepsilon_q} = \boldsymbol{R}(G_{2p-1,2q-1} + G_{2p,2q}) + \boldsymbol{R}(G_{2p-1,2q} - G_{2p,2q-1}).$$

Since $A_0 = \varepsilon_1 = G_{12}$, we have

$$Z_K^{A_0} = SO(2) \times SO(2l-1).$$

Hence the corresponding symmetric R-space is

$$\frac{SO(2l+1)}{SO(2) \times SO(2l-1)}.$$

References

[1] B. N. Cheng, Area-minimizing cone-type surfaces and coflat calibrations, Indiana Math. Univ. J. **37** (1988), 505–535.

[2] S. Helgason, Differential geometry, Lie groups, and symmetric spaces, Academic Press, (1978).

[3] W. Y. Hsiang, On compact homogeneous submanifolds. Proc. Nat. Acad. Sci. U.S.A. **56** (1966), 5–6.

[4] M. Kerckhove, Isolated orbits of the adjoint action and area-minimizing cones, Proc. Amer. Math. Soc. **121** (1994), 497–503.

[5] G. R. Lawlor, A sufficient criterion for a cone to be area-minimizing, Mem. Amer. Math. Soc. **91** no. 446, 1991.

[6] J. Simons, Minimal varieties in riemannian manifolds, Ann. of Math. **88** (1968), 62–105.

Institute of Mathematics
University of Tsukuba
Tsukuba, Ibaraki
305-8571
Japan
daigo@math.tsukuba.ac.jp
kanno@math.tsukuba.ac.jp
tasaki@math.tsukuba.ac.jp