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THE INJECTIVITY RADIUS AND THE FUNDAMENTAL GROUP OF COMPACT HOMOGENEOUS RIEMANNIAN MANIFOLDS OF POSITIVE CURVATURE

By

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1. Introduction

In this paper we characterize a homogeneous spherical space form whose fundamental group is a binary dihedral group by means of the injectivity radius of the exponential map. As an application of this result we show that the fundamental group of a non-simply connected homogeneous 1/4-pinched Riemannian manifold is isomorphic to a finite subgroup of the special unitary group $SU(2)$ of degree 2. Moreover we show that the fundamental group of a non-simply connected homogeneous 1/4-pinched Riemannian manifold whose dimension is of $4j+1$ $(j \geq 1)$ is a cyclic group. As is well known, every nontrivial finite subgroup of $SU(2)$ is a cyclic, binary dihedral or binary polyhedral group ([21]). Therefore the fundamental group of a non-simply connected homogeneous 1/4-pinched Riemannian manifold is isomorphic to one of the finite groups stated above.

Let M be an m-dimensional $(m \geq 3)$ complete, connected Riemannian manifold and N an n-dimensional $(0 \leq n \leq m-1)$ connected, compact submanifold (without boundary) embedded in M. Let $Exp_N : \nu(N) \to M$ denote the normal exponential map of N. Here $\nu(N)$ is the total space of the normal bundle of N in M. In the case where N is a point of M, $\nu(N)$ stands for the tangent space to M at that point. We denote by $i(N)$ the injectivity radius of Exp_N. It is defined as the supremum of the set of all $r > 0$ for which $Exp_N : \nu_r(N) \to M$ is an embedding, where $\nu_r(N)$ is the set of all normal vectors to N of length less than r. The injectivity radius $Inj(M)$ of M is then defined as $Inj(M) = \inf\{i(x) \mid x \in M\}$. If M is compact, then $Inj(M)$ is positive.

A binary dihedral group D_k^* is a finite group generated by two elements a

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and b with fundamental relations $a^{2k} = e$, $a^k = b^2$ and $aba = b$, where $2k$ is the order of a $(k \geq 2)$ and e denotes the unit element of D_k^*. In case of $k = 2$, D_2^* is isomorphic to the quaternion group $Q8$. As stated above, $SU(2)$ contains binary dihedral groups.

There are homogeneous spherical space forms whose fundamental groups are binary dihedral groups. We can characterize such spherical space forms in terms of the injectivity radius of the exponential map.

Let M denote an m-dimensional $(m \geq 3)$ connected, compact, non-simply connected Riemannian manifold with sectional curvature $K_M \geq 1$. An upper bound of $Inj(M)$ is closely related to the fundamental group $\pi_1(M)$ of M. The diameter sphere theorem due to Grove and Shiohama ([8]) shows that $Inj(M) \leq \pi/2$. Toponogov's diameter theorem implies that if $Inj(M) = \pi/2$, then M is isometric to the m-dimensional real projective space RP^m with constant curvature 1. Shiohama showed in [18] the following result. If $\pi_1(M) \cong Z_3$, then $Inj(M) \leq \pi/3$ and equality holds if and only if M is isometric to the lens space of constant curvature 1. We showed in [11] that if the order of $\pi_1(M)$ is not a prime, then $Inj(M) \leq \pi/4$ and equality holds if and only if M is a homogeneous Riemannian manifold of constant curvature 1 and $\pi_1(M)$ is isomorphic to either Z_4 or $Q8$. Here if $\pi_1(M) \cong Q8$, then we have $m = 4j - 1$ $(j \geq 1)$.

Let M be as above, and let N be an n-dimensional $(n \geq 1)$ connected, compact, totally geodesic submanifold embedded in M such that $2n \leq m - 1$. We note here that if $2n \geq m$, then the first relative homotopy class $\pi_1(M, N)$ is trivial, i.e., the homomorphism $\imath_\sharp : \pi_1(N) \to \pi_1(M)$ induced from the inclusion $\imath : N \to M$ is surjective ([6]). In the case where $\pi_1(M, N)$ is nontrivial, i.e., $\imath_\sharp(\pi_1(N)) \neq \pi_1(M)$, an upper bound of $i(N)$ is closely related to $\pi_1(M)$ and $\pi_1(M, N)$. We now assume that M is a homogeneous Riemannian manifold. Then we showed in [11] that if N is a simple closed geodesic of M which is homotopically nontrivial and if $\pi_1(M)$ is not a cyclic group, then $i(N) \leq \pi/4$. Here if equality holds, then M is of constant curvature 1, $m = 4j - 1$ $(j \geq 1)$ and $\pi_1(M)$ is a binary dihedral group. In this result we can eliminate the assumption that N is homotopically nontrivial. Moreover this result is also true for the case $n \geq 2$. In this paper we show the following.

THEOREM A. *Let M be an m-dimensional $(m \geq 3)$ connected, compact, non-simply connected homogeneous Riemannian manifold with sectional curvature $K_M \geq 1$. Let N be an n-dimensional $(n \geq 1)$ compact, connected, totally geodesic submanifold embedded in M such that $2n \leq m - 1$. Assume that $\pi_1(M)$ is not a cyclic group and that $\pi_1(M, N)$ is nontrivial. Then $i(N) \leq \pi/4$. Here if equality* *holds, then M is of constant curvature* 1, $m = 4j - 1$ $(j \geq 1)$ *and* $\pi_1(M)$ *is isomorphic to a binary dihedral group.*

As applications of Theorem A we have the following two results.

THEOREM B. *Let M be an m-dimensional* $(m \geq 2)$ *connected, compact, non-simply connected homogeneous Riemannian manifold whose sectional curvature* K_M *satisfies* $1 \leq K_M \leq 4$. *Then* $\pi_1(M)$ *is isomorphic to a finite subgroup of* $SU(2)$.

THEOREM C. *Let M be an m-dimensional connected, compact, non-simply connected homogeneous Riemannian manifold whose sectional curvature* K_M *satisfies* $1 \leq K_M \leq 4$. *If* $m = 4j + 1$ $(j \geq 1)$, *then* $\pi_1(M)$ *is a cyclic group.*

In the case where M is a non-simply connected homogeneous spherical space form, Theorems B and C are classical results ([21]; p. 229). However, even if M is such a spherical space form, our proof for these theorems is different from one given in [21].

The proof of the theorems stated above will be given in Sections 3 and 4. We give in Section 5 examples of connected, compact, non-simply connected homogeneous Riemannian manifolds with $1 \leq K \leq 4$.

2. Preliminaries

Throughout this paper we always assume that all geodesics on Riemannian manifolds are parameterized by arc-length, unless otherwise stated.

In this section we prepare lemmas which will be used in the proof of the theorems stated in Section 1.

Throughout this section let M be an m-dimensional $(m \geq 3)$ connected, compact, non-simply connected Riemannian manifold whose sectional curvature K_M satisfies $K_M \geq 1$ and let $p : V \to M$ denote the universal Riemannian covering. V is a complete Riemannian manifold with sectional curvature $K_V \geq 1$. We will denote by d the distance function on V which is induced from the Riemannian metric of V. Let Γ denote the deck transformation group of V corresponding to the fundamental group $\pi_1(M)$ of M. Γ acts freely on V.

By the theorem of Bonnet-Myers, the diameter $d(V)$ of V is not greater than π. Hence V is compact and Γ is a finite group. Toponogov's diameter theorem shows that $d(V) = \pi$ holds if and only if V is isometric to the Euclidean unit m-sphere S^m. By the diameter sphere theorem of Grove and Shiohama ([8]), the diameter $d(M)$ of M is not greater than $\pi/2$. Rigidity theorem due to Gromoll and Grove ([7]) implies that if m is odd and $d(M) = \pi/2$, then M is of constant curvature 1.

A nonempty subset C of V is called *totally r-convex* in V $(r > 0)$ if for every geodesic $\gamma : [0, a] \to V$ with $\gamma(0)$, $\gamma(a) \in C$ and $0 < a < r$ we have $\gamma([0, a]) \subset C$.

Let C be a connected, compact, totally r-convex set in V whose boundary ∂C is nonempty. The interior of C is a totally geodesic submanifold embedded in V. We set $C^a = \{x \in C \mid d(x, \partial C) \geq a\}$ $(a \geq 0)$ and $\rho = \max\{d(x, \partial C) \mid x \in C\}$. Then the set $\bigcap_{0 \leq a \leq \rho} C^a$ consists of one point s_C, which is called the soul of C ([3]). If C is invariant under an isometry φ of V, then s_C is a fixed point of φ because φ leaves ∂C invariant. Hence we have

LEMMA 2.1. *Let C be a connected, compact, totally r-convex proper subset in V. If C is invariant under a fixed point free isometry of V, then* $\dim C \geq 1$ *and* $\partial C = \varnothing$.

Let C be a compact totally π-convex proper subset in V. If C is not arcwise connected, then there exist two points $x, y \in C$ such that $d(x, y) \geq \pi$. Then by the theorem due to Bonnet-Myers we get $d(x, y) = \pi$. Hence V is isometric to S^m and C consists of exactly two points. Thus we have

LEMMA 2.2. *Let C be a compact totally π-convex proper subset in V. If C contains at least three points, then C is arcwise connected and* $\dim C \geq 1$.

Let C be a connected, compact, totally π-convex proper subset in V with $\dim C \geq 1$. Then any two points of C can be connected by a minimizing geodesic in V which is contained in C.

Let A be a nonempty compact proper subset in V. We set

$$B = \{x \in V \mid d(x, A) \geq \pi/2\}, \quad C = \{x \in V \mid d(x, B) \geq \pi/2\}.$$

Then we shall show the following lemma.

LEMMA 2.3. *Let A, B and C be as above. Let Γ_1 be a subgroup of Γ such that $\Gamma_1 \neq \{I_V\}$. Assume that $m (\geq 3)$ is odd. Suppose that A is invariant under Γ_1 and that C and B contain connected, compact submanifolds N_1 and N_2 with $1 \leq$* $\dim N_1$, $\dim N_2 \leq m - 2$, *respectively. Then we have*

(1) *B and C are totally π-convex in V and $\partial B = \partial C = \varnothing$.*

(2) *If $x \in B$ and $y \in C$, then $d(x, y) = \pi/2$.*

(3) *V is isometric to S^m.*

PROOF. By using the comparison theorem of Toponogov, we can show that both B and C are totally π-convex in V ([9], [10]). Since $N_1 \subset C$ and $N_2 \subset B$, Lemma 2.2 shows that both B and C are arcwise connected. Since A is invariant under Γ_1, B and C are also invariant under Γ_1. Then Lemma 2.1 implies that $\partial B = \partial C = \emptyset$. By using again the comparison theorem of Toponogov, we obtain that $d(x, y) = \pi/2$ for any $x \in B$ and $y \in C$. Let $p_1 : V \to V/\Gamma_1$ be the Riemannian covering of the quotient Riemannian manifold V/Γ_1. Since B and C are invariant under Γ_1, the distance between $p_1(B)$ and $p_1(C)$ in V/Γ_1 is equal to $\pi/2$. Hence we have $d(V/\Gamma_1) = \pi/2$. Since m is odd, by the rigidity theorem ([7]) V/Γ_1 is of constant curvature 1, and hence V is isometric to S^m.

LEMMA 2.4. *Let N_0 be an n-dimensional connected, compact, totally geodesic submanifold (without boundary) embedded in V with $1 \le n \le m-2$. Let Γ_1 be a subgroup of Γ such that $\Gamma_1 \neq \{I_V\}$. Assume that m (≥ 3) is odd and that N_0 is invariant under Γ_1. If there exists a point $x_0 \in V$ such that $d(x_0, N_0) \ge \pi/2$, then V is isometric to S^m.*

PROOF. We set

$$A_1 = N_0, \quad B_1 = \{x \in V \mid d(x, A_1) \ge \pi/2\}, \quad C_1 = \{x \in V \mid d(x, B_1) \ge \pi/2\}.$$

Then $x_0 \in B_1$ and $A_1 \subset C_1$. Both B_1 and C_1 are invariant under Γ_1 because Γ_1 leaves A_1 invariant. By using the comparison theorem of Toponogov, we conclude that both B_1 and C_1 are totally π-convex in V ([9], [10]). We shall show that B_1 is arcwise connected. To do that, we assume that B_1 is not arcwise connected. Since B_1 is totally π-convex, V is isometric to S^m and B_1 consists of exactly two points. Hence C_1 is isometric to a great $(m-1)$-sphere S_1 in S^m. Then A_1 is isometric to a great n-sphere in S^m which is contained in S_1 because A_1 is totally geodesic in V and is contained in C_1. Since $n \le m-2$, there exists a point $x \in C_1$ such that $d(x, A_1) = \pi/2$, which shows $x \in B_1 \cap C_1$. This is a contradiction. Thus B_1 is arcwise connected. Since B_1 is invariant under Γ_1, by Lemma 2.1 B_1 has no boundary and $\dim B_1 \ge 1$. Similarly C_1 has no boundary. By Frankel's theorem ([5]), we have $\dim B_1 + \dim C_1 \le m-1$. Since $\dim B_1$, $\dim C_1 \ge 1$, we obtain that $\dim B_1$, $\dim C_1 \le m-2$. Applying Lemma 2.3 to the present situation, we conclude that V is isometric to S^m.

For each $\varphi \in \Gamma$ we set $T(\varphi) = \min\{d(x, \varphi(x)) \mid x \in V\}$. Let $\varphi \in \Gamma \backslash \{I_V\}$. Suppose that the displacement function $d(\cdot, \varphi(\cdot)) : V \to R$ takes the minimum at $x_0 \in V$. Let σ be a minimizing geodesic segment from x_0 to $\varphi(x_0)$ and $\tilde{\sigma} : R \to V$ the geodesic extension of σ in the both directions. Then φ translates $\tilde{\sigma}$, i.e., $\varphi(\tilde{\sigma}(t)) = \tilde{\sigma}(t + T(\varphi))$ for all $t \in R$. Furthermore $\tilde{\sigma} : [0, kT(\varphi)] \to V$ is a closed geodesic where k is the order of φ.

Let $Z(\Gamma)$ be the centralizer of Γ in the full isometry group of V. M is a homogeneous Riemannian manifold if and only if $Z(\Gamma)$ acts transitively on V ([21]; p. 73). We now assume that M is a homogeneous Riemannian manifold. Then V is also a homogeneous Riemannian manifold. Each $\varphi \in \Gamma$ is a Clifford transformation of V, i.e., the displacement function $d(\cdot, \varphi(\cdot)) : V \to R$ is a constant function ([21]). Hence for any $\varphi \in \Gamma$ we have $T(\varphi) = d(y, \varphi(y))$, $y \in V$. Therefore for any $x \in M$ and for any $[\gamma] \in \pi_1(M, x) \backslash \{e\}$ we can choose a closed geodesic as a representation of $[\gamma]$.

LEMMA 2.5. *Let M and V be as above. Assume that M is a homogeneous Riemannian manifold. Let $\varphi \in \Gamma \backslash \{I_V\}$. Let $\sigma : [0, a] \to V$ be a simple closed geodesic satisfying the conditions*: (1) $\sigma(0) = \sigma(a)$, $T(\varphi) < a$; (2) *σ is invariant under φ*; (3) *$\sigma : [0, T(\varphi)] \to V$ is a minimizing geodesic segment between $\sigma(0)$ and $\varphi(\sigma(0))$. Let Γ_σ be the subgroup of Γ whose any element leaves σ invariant. If $T(\varphi) \leq T(\psi)$ for any $\psi \in \Gamma_\sigma \backslash \{I_V\}$, then Γ_σ is the cyclic group generated by φ.*

PROOF. Let Γ_1 be the cyclic group generated by φ and k its order where $k \geq 2$. We put $x = \sigma(0)$. Then φ translates σ and we have $\varphi^j(x) = \sigma(jT(\varphi))$ for each j $(0 \leq j \leq k-1)$ where $\varphi^j = \varphi \circ \cdots \circ \varphi$ (j times). Hence we have $a = kT(\varphi)$. For each j $(0 \leq j \leq k-1)$ $\sigma : [jT(\varphi), (j+1)T(\varphi)] \to V$ is a minimizing geodesic segment between $\varphi^j(x)$ and $\varphi^{j+1}(x)$. Suppose that there is a $\psi \in \Gamma_\sigma \backslash \Gamma_1$. Since σ is invariant under ψ and Γ acts freely on V, we have $\psi(x) \in \sigma((jT(\varphi), (j+1)T(\varphi)))$ for some j $(0 \leq j \leq k-1)$. Since $\varphi_1 = \varphi^{-j} \circ \psi$ leaves σ invariant and $\varphi_1(x) \in \sigma((0, T(\varphi)))$, we get $T(\varphi_1) < T(\varphi)$. This is a contradiction. Hence we have $\Gamma_\sigma = \Gamma_1$.

The following lemmas are well known results (for the proof, see [1], [2]).

LEMMA 2.6. *Let W be an m-dimensional $(m \geq 3)$ connected, complete Riemannian manifold with sectional curvature $K_W \leq \lambda^2$ $(\lambda > 0)$ and N a connected, compact, totally geodesic submanifold embedded in W such that $1 \leq \dim N \leq m - 2$. Let $\gamma : [0, \infty) \to W$ be a geodesic. Then we have*

(1) *If $\gamma(a)$ is the first conjugate point to $\gamma(0)$ along γ, then $a\lambda \geq \pi$.*

(2) *If $\gamma(0) \in N$ and the tangent vector $\gamma'(0)$ is orthogonal to N and if $\gamma(a)$ is the first focal point of N along γ, then $2a\lambda \geq \pi$.*

LEMMA 2.7. *Let W be as in Lemma 2.6. Let x and y be distinct points of W. Suppose that there exist distinct minimizing geodesics $\sigma_1, \sigma_2 : [0,a] \to W$ from x to y. If $a = i(x)$ and $a\lambda < \pi$, then $\sigma_1'(a) = -\sigma_2'(a)$.*

LEMMA 2.8. *Let W and N be as in Lemma 2.6. Let $x \in W \backslash N$. Suppose that there exist distinct minimizing geodesics $\sigma_1, \sigma_2 : [0,a] \to W$ from x to N. If $a = i(N)$ and $2a\lambda < \pi$, then $\sigma_1'(0) = -\sigma_2'(0)$.*

The following theorem will be used in the proof of Theorems B and C.

THEOREM 2.1 ([2], [4], [12]). *Let W be an m-dimensional ($m \geq 2$) connected, complete, simply connected Riemannian manifold with $1 \leq K_W \leq 4$. Then we have*

(1) *$Inj(W) \geq \pi/2$.*

(2) *If $d(W) = \pi/2$ and m (≥ 3) is odd, then W is isometric to the Euclidean m-sphere $S^m(4)$ with constant curvature 4.*

3. Proof of Theorem A

Throughout this section let M be an m-dimensional ($m \geq 3$) connected, compact, non-simply connected homogeneous Riemannian manifold whose sectional curvature K_M satisfies $K_M \geq 1$ and N an n-dimensional ($n \geq 1$) connected, compact, totally geodesic submanifold (without boundary) embedded in M.

Let $\imath_\sharp : \pi_1(N) \to \pi_1(M)$ be the homomorphism which is induced from the inclusion $\imath : N \to M$. Let $\pi_1(M,N)$ denote the first relative homotopy class. For the sake of convenience we write $\pi_1(M,N) = 0$ if $\imath_\sharp$ is surjective and $\pi_1(M,N) \neq 0$ otherwise. As stated in Section 1, if $2n \geq m$, then we have $\pi_1(M,N) = 0$.

Let $p : V \to M$ denote the universal Riemannian covering. V is also compact homogeneous Riemannian manifold with $K_V \geq 1$. We denote by Γ the deck transformation group of V corresponding to $\pi_1(M)$. Let Γ_0 be the subgroup of Γ which corresponds to $\imath_\sharp(\pi_1(N))$. If $\pi_1(M,N) \neq 0$, then $p^{-1}(N)$ has at least two connected components and we have $2i(N) \leq d(N_1, N_2)$ for any distinct connected components N_1 and N_2 of $p^{-1}(N)$. Let N_0 be a connected component of $p^{-1}(N)$. Let $\varphi \in \Gamma$. Then φ is contained in Γ_0 if and only if N_0 is invariant under φ.

For $\varphi_1, \dots, \varphi_k \in \Gamma$ we will denote by $\Gamma(\varphi_1, \dots, \varphi_k)$ the subgroup of Γ generated by $\varphi_1, \dots, \varphi_k$.

In order to prove Theorem A, we prepare several lemmas.

Lemma 3.1. *Assume that* Γ *is not cyclic and* $\Gamma_0 = \{I_V\}$. *Then* $i(N) \le \pi/4$. *Here if equality holds, then* M *is of constant curvature* 1 *and* $\Gamma \cong Q8$.

Proof. Assuming that $i(N) \ge \pi/4$, we shall show that $i(N) = \pi/4$ and $K_M \equiv 1$. Let N_0 be a connected component of $p^{-1}(N)$ and fix it. By the assumption on Γ_0, we have $\varphi(N_0) \cap N_0 = \emptyset$ for all $\varphi \in \Gamma \backslash \{I_V\}$. Since Γ is not cyclic, m is odd by Synge's theorem and Γ contains a proper subgroup. The assumption that $i(N) \ge \pi/4$ implies that $d(N_0, \psi(N_0)) \ge \pi/2$ for all $\psi \in \Gamma \backslash \{I_V\}$. Let Γ_1 be an arbitrary proper subgroup of Γ. We set

$$A = \bigcup_{\varphi \in \Gamma_1} \varphi(N_0), \quad B = \{x \in V \mid d(x, A) \ge \pi/2\}, \quad C = \{x \in V \mid d(x, B) \ge \pi/2\}.$$

Then A is invariant under Γ_1 and we have $\psi(N_0) \subset B$ for all $\psi \in \Gamma \backslash \Gamma_1$. We can apply Lemma 2.3 to the present situation. Hence the assertions (1), (2) and (3) in Lemma 2.3 hold for the present situation. Thus M is of constant curvature 1. Let $\psi \in \Gamma \backslash \Gamma_1$. By Lemma 2.3 (2), we have $d(x, y) = \pi/2$ for any $x \in N_0$ and $y \in \psi(N_0)$. This shows that $T(\psi) = \pi/2$ and $d(N_0, \psi(N_0)) = \pi/2$. Hence we have $i(N) = \pi/4$.

In the following we assume that $i(N) = \pi/4$. We shall show that $\Gamma \cong Q8$. It follows from the argument above that there exists a $\varphi \in \Gamma$ with $T(\varphi) = \pi/2$ and $T(\psi) = \pi/2$ holds for all $\psi \in \Gamma \backslash \Gamma(\varphi)$. From now on we identify V with S^m and view Γ as a finite subgroup of the orthogonal group $O(m+1)$. By homogeneity of V, Γ is a Clifford transformation group of S^m. We take a $\varphi_1 \in \Gamma$ with $T(\varphi_1) = \pi/2$ and fix it. For each $x \in S^m$, φ_1 translates the great circle in S^m passing through x and $\varphi_1(x)$. Hence φ_1 has the properties that ${\varphi_1}^2 = -I$ and $\Gamma(\varphi_1) \cong Z_4$, where I denotes the unit $(m+1)$-matrix. Each $\psi \in \Gamma \backslash \Gamma(\varphi_1)$ has the properties that $T(\psi) = \pi/2$, $\psi^2 = -I$ and $\Gamma(\psi) \cong Z_4$. Let $\varphi_2 \in \Gamma \backslash \Gamma(\varphi_1)$ and fix it. We have the relations $(\varphi_1\varphi_2)^2 = {\varphi_1}^2 = {\varphi_2}^2 = -I$ since $\varphi_1\varphi_2 \notin \Gamma(\varphi_1)$. By using these relations, we obtain that $\varphi_1\varphi_2\varphi_1 = \varphi_2$ and $\varphi_2\varphi_1\varphi_2 = \varphi_1$. This shows that $\Gamma(\varphi_1, \varphi_2) \cong Q8$. We put $\Gamma_2 = \Gamma(\varphi_1, \varphi_2)$. We now assume that $\Gamma \ne \Gamma_2$. Take a $\varphi \in \Gamma \backslash \Gamma_2$. Since $\varphi_1\varphi, \varphi_2\varphi$ and $\varphi_1\varphi_2\varphi$ are not contained in Γ_2, we obtain that $(\varphi_1\varphi_2\varphi)^2 = (\varphi_1\varphi)^2 = (\varphi_2\varphi)^2 = {\varphi_1}^2 = \varphi^2 = -I$. The relation $(\varphi_1\varphi)^2 = \varphi^2$ implies $\varphi_1\varphi\varphi_1 = \varphi$. By using the relations that $(\varphi_1\varphi_2\varphi)^2 = {\varphi_1}^2$, $\varphi_1\varphi\varphi_1 = \varphi$ and $\varphi_1\varphi_2\varphi_1 = \varphi_2$, we get $(\varphi_2\varphi)^2 = I$. This is a contradiction. Hence we have $\Gamma = \Gamma_2$, which shows that $\Gamma \cong Q8$.

Lemma 3.2. *Suppose that* Γ *is not cyclic and that* Γ_0 *is a proper subgroup of* Γ. *Then* $i(N) \le \pi/4$. *Here if equality holds, then* M *is of constant curvature* 1 *and furthermore, identifying* V *with* S^m *and viewing* Γ *as a finite subgroup of* $O(m+1)$, *we have*

(1) *If* $\psi \in \Gamma \backslash \Gamma_0$, *then* $\psi^2 = -I \in \Gamma_0$ *and* $\Gamma(\psi) \cong Z_4$.
(2) *If* $\psi \in \Gamma \backslash \Gamma_0$ *and* $\varphi \in \Gamma_0$, *then* $\varphi\psi\varphi = \psi$.

PROOF. As in the proof of Lemma 3.1 we fix a connected component N_0 of $p^{-1}(N)$. Then N_0 is invariant under Γ_0. Suppose that $i(N) \geq \pi/4$. We shall show that $i(N) = \pi/4$ and $K_M \equiv 1$. We set

$$A = N_0, \quad B = \{x \in V \mid d(x, A) \geq \pi/2\}, \quad C = \{x \in V \mid d(x, B) \geq \pi/2\}.$$

Let $\psi \in \Gamma \backslash \Gamma_0$. We have $d(N_0, \psi(N_0)) \geq \pi/2$ because $i(N) \geq \pi/4$. Thus we have $\psi(N_0) \subset B$ for all $\psi \in \Gamma \backslash \Gamma_0$. The order of Γ is greater than 2 since Γ_0 is a proper subgroup of Γ. Hence m (≥ 3) is odd by Synge's theorem. By applying Lemma 2.3 to the present situation, we conclude that $i(N) = \pi/4$ and $K_M \equiv 1$. From now on we assume that $i(N) = \pi/4$. By Lemma 2.3, both B and C are totally geodesic submanifolds of V without boundary and we obtain $d(x, y) = \pi/2$ for any $x \in B$ and $y \in C$. We identify V with S^m and view Γ as a finite subgroup of $O(m+1)$. Then N_0 is a great n-sphere in S^m and B is a great $(m-n-1)$-sphere in S^m. Hence we have $N_0 = C$ by the definition of C. Let $\psi \in \Gamma \backslash \Gamma_0$. Since $\psi(N_0) \subset B$ and ψ is a Clifford transformation, we have $T(\psi) = \pi/2$. Let $x \in N_0$. Then ψ translates the great circle in S^m passing through x and $\psi(x)$. Hence $\psi^2(x) = -x \in N_0$. Since Γ acts freely on S^m and ψ^2 leaves N_0 invariant, we obtain that $\psi^2 = -I \in \Gamma_0$ and $\Gamma(\psi) \cong Z_4$. This shows (1). Next we shall show (2). Let $\psi \in \Gamma \backslash \Gamma_0$ and $\varphi \in \Gamma_0$. We may assume that $\varphi \neq \pm I$. Since $\psi\varphi \in \Gamma \backslash \Gamma_0$, we have $(\psi\varphi)^2 = \psi^2 = -I$. From the relation $(\psi\varphi)^2 = \psi^2$, we get $\varphi\psi\varphi = \psi$.

LEMMA 3.3. *Suppose that* Γ *and* Γ_0 *satisfy the same hypotheses as in Lemma* 3.2 *and that* $i(N) = \pi/4$. *Then* Γ_0 *is a cyclic group of order* $2k$ ($k \geq 1$).

PROOF. By Lemma 3.2 we can identify V with S^m. Then Γ can be viewed as a finite subgroup of $O(m+1)$. Let N_0 be a connected component of $p^{-1}(N)$ and fix it. Then N_0 is a great n-sphere in S^m and is invariant under Γ_0. We may assume that $N_0 = S^n = S^m \cap R^{n+1}$. We take a $\psi_1 \in \Gamma_0 \backslash \{I\}$ such that $T(\psi_1) \leq T(\varphi)$ for all $\varphi \in \Gamma_0 \backslash \{I\}$. We shall show that $\Gamma_0 = \Gamma(\psi_1)$. We first assume that $\psi_1 = -I$. Since $T(\psi_1) = \pi$, we have $T(\varphi) = \pi$ for all $\varphi \in \Gamma_0 \backslash \{I\}$. This implies that $\Gamma_0 = \{I, \psi_1\} = \Gamma(\psi_1)$. We next assume that $\psi_1 \neq -I$. In case of $n = 1$, by Lemma 2.5 Γ_0 is the cyclic group generated by ψ_1. From now on, let $n \geq 2$. Let C_1 be the great circle in S^n which contains x_0 and $\psi_1(x_0)$. Then C_1 is invariant under ψ_1. Since $T(\psi_1) < \pi$, the order of Γ_0 is greater than 2. Hence n (≥ 3) is odd by Synge's theorem. Let $n = 2q + 1$, $q \geq 1$. Since $-I \in \Gamma_0$ and $(-I)(C_1) = C_1$, by

Lemma 2.5 there exists the smallest positive integer $k \geq 2$ such that $(\psi_1)^k = -I$. Hence the order of $\Gamma(\psi_1)$ is equal to $2k$. Let $x_0 \in N_0$ and $\psi_2 \in \Gamma \backslash \Gamma_0$, and fix them. Lemma 3.2 shows that $(\psi_2)^2 = -I \in \Gamma_0$ and $\varphi\psi_2\varphi = \psi_2$ for all $\varphi \in \Gamma_0$. Let $\varphi \in \Gamma_0 \backslash \{\pm I, \psi_1\}$. Since $\psi_1\psi_2\varphi$ and $\psi_1\psi_2$ are not contained in Γ_0, by Lemma 3.2 (1) we have the relation $(\psi_1\psi_2\varphi)^2 = (\psi_1\psi_2)^2$. By combining this with the relation $\varphi\psi_2\varphi = \psi_2$, we obtain $\varphi\psi_1 = \psi_1\varphi$. Then there exists a complex vector $\xi \in S^n \subset C^{q+1}$ which is a common eigenvector of φ and ψ_1. Let C_2 be the great circle in S^n determined by ξ and $\bar{\xi}$ where $\bar{\xi}$ is the conjugate vector of ξ in C^{q+1}. Then C_2 is invariant under φ and ψ_1. By Lemma 2.5 we have $\varphi \in \Gamma(\psi_1)$. Hence we have $\Gamma_0 = \Gamma(\psi_1)$. Thus Γ_0 is the cyclic group of order $2k$ generated by ψ_1 $(k \geq 1)$.

LEMMA 3.4. *Suppose that Γ and Γ_0 satisfy the same hypotheses as in Lemma 3.2 and that $i(N) = \pi/4$. Then $\Gamma \cong D_s^*$ $(s \geq 2)$.*

PROOF. By Lemma 3.2 we may identify V with S^m and view Γ as a finite subgroup of $O(m+1)$. We take a $\psi_1 \in \Gamma_0 \backslash \{I_V\}$ such that $T(\psi_1) \leq T(\varphi)$ for all $\varphi \in \Gamma_0 \backslash \{I_V\}$. As we have shown in Lemma 3.3, Γ_0 is a cyclic group generated by ψ_1 with order $2k$ $(k \geq 1)$ and $(\psi_1)^k = -I$. Let $\psi_2 \in \Gamma \backslash \Gamma_0$. It follows from Lemma 3.2 that $(\psi_2)^2 = -I$, $\psi_1\psi_2\psi_1 = \psi_2$ and $\Gamma(\psi_2) \cong Z_4$. We first consider the case $k \geq 2$. Then we have $\Gamma(\psi_1, \psi_2) \cong D_k^*$. We shall show that $\Gamma = \Gamma(\psi_1, \psi_2)$. To do that, we assume that there exists a $\varphi \in \Gamma \backslash \Gamma(\psi_1, \psi_2)$. Since $\psi_2 \notin \Gamma(\psi_1)$ and $\psi_2\varphi \notin \Gamma(\psi_1, \psi_2)$, by Lemma 3.2 we obtain that $\psi_1\psi_2\psi_1 = \psi_2$, $\psi_1\varphi\psi_1 = \varphi$ and $\psi_1\psi_2\varphi\psi_1 = \psi_2\varphi$. By using these relations, we get $(\psi_1)^2 = I$. This is a contradiction because $(\psi_1)^{2k} = I$ and $k \geq 2$. Thus we have $\Gamma = \Gamma(\psi_1, \psi_2)$. Next let us consider the case $k = 1$. Then $\psi_1 = -I$ and $\Gamma_0 \subset \Gamma(\psi_2)$. We take a $\psi_3 \in \Gamma \backslash \Gamma(\psi_2)$. By Lemma 3.2 (1) we obtain that $(\psi_2\psi_3)^2 = (\psi_2)^2 = (\psi_3)^2 = -I$. These relations yield that $\psi_2\psi_3\psi_2 = \psi_3$ and $\psi_3\psi_2\psi_3 = \psi_2$. Hence we have $\Gamma(\psi_2, \psi_3) \cong Q8$. By the same way as in the proof of Lemma 3.1, we can show that $\Gamma = \Gamma(\psi_2, \psi_3)$. Thus we have $\Gamma \cong D_s^*$ $(s \geq 2)$.

LEMMA 3.5. *Suppose that Γ is not a cyclic group and that $\Gamma_0 \neq \Gamma$. If $i(N) = \pi/4$, then $m = 4j - 1$ $(j \geq 1)$.*

PROOF. By Lemmas 3.1 and 3.2, V is isometric to S^m and Γ is isomorphic to D_s^* $(s \geq 2)$. In the case where Γ_0 is trivial, Γ is isomorphic to $Q8$. If Γ_0 is nontrivial, then Γ_0 is a cyclic group of order $2k$ $(k \geq 1)$ by Lemma 3.3. We identify V with S^m and view Γ as a finite subgroup of $O(m+1)$. As we have shown in the proofs of Lemmas 3.1 and 3.4, we can choose a generator $\{\varphi_1, \varphi_2\}$

of Γ as follows. In the case where $\Gamma_0 = \{I\}$ or $\Gamma_0 = \{I, -I\}$, φ_1 and φ_2 have the properties that $T(\varphi_1) = T(\varphi_2) = \pi/2$ and $\varphi_1\varphi_2\varphi_1 = \varphi_2$, $\varphi_2\varphi_1\varphi_2 = \varphi_1$. If the order of Γ_0 is greater that 2, then φ_1 is a generator of Γ_0 and $T(\varphi_1) = \pi/k$ $(k \geq 2)$, $T(\varphi_2) = \pi/2$. In this case φ_1 and φ_2 satisfy the relations $\varphi_1\varphi_2\varphi_1 = \varphi_2$, $(\varphi_1)^k = (\varphi_2)^2 = -I$. Let $x \in S^m$. For φ_i $(i = 1, 2)$ let C_i be the great circle in S^m passing through x and $\varphi_i(x)$. Then C_i is invariant under φ_i, $i = 1, 2$. Let $C_3 = \varphi_2(C_1)$. Since $T(\varphi_2) = \pi/2$ and $\varphi_2 \notin \Gamma(\varphi_1)$, we have $C_1 \cap C_3 = \emptyset$. The relations $\varphi_1\varphi_2\varphi_1 = \varphi_2$ and $(\varphi_2)^2 = -I$ imply that $\varphi_1(C_3) = C_3$ and $\varphi_2(C_3) = (-I)(C_1) = C_1$. Let W_i be the 2-dimensional subspace in R^{m+1} such that $C_i = W_i \cap S^m$ $(i = 1, 2, 3)$. We set $W_4 = W_1 \oplus W_3$. Then W_2 is contained in W_4 and both φ_1 and φ_2 leave W_4 invariant. Since Γ is generated by φ_1 and φ_2, W_4 is Γ-invariant. Hence for any $x \in S^m$ there exists a Γ-invariant 4-dimensional subspace of R^{m+1} containing x. Thus R^{m+1} can be expressed as a direct sum of Γ-invariant 4-dimensional subspaces, which implies that $m = 4j - 1$ $(j \geq 1)$.

Proof of Theorem A. Lemmas 3.1 and 3.2 show that $i(N) \leq \pi/4$. Suppose $i(N) = \pi/4$. Then M is of constant curvature 1. Moreover Lemmas 3.1, 3.4 and 3.5 imply that $\pi_1(M) \cong D_s^*$ $(s \geq 2)$ and $m = 4j - 1$ $(j \geq 1)$.

4. Proof of Theorems B and C

First of all we state a theorem which will be used in the proof of Theorem C.

Theorem 4.1 ([11]). *Let M be an m-dimensional $(m \geq 3)$ connected, compact, non-simply connected Riemannian manifold with sectional curvature $K_M \geq 1$. Suppose that the order of $\pi_1(M)$ is not a prime. Then $Inj(M) \leq \pi/4$. If equality holds, then M is of constant curvature 1 and $\pi_1(M)$ is isomorphic to either Z_4 or $Q8$. Here if $\pi_1(M) \cong Q8$, then $m = 4j - 1$ $(j \geq 1)$.*

Throughout this section let M denote an m-dimensional $(m \geq 3)$ connected, compact, non-simply connected homogeneous Riemannian manifold whose sectional curvature K_M satisfies $1 \leq K_M \leq 4$. If m is even, then $\pi_1(M)$ is isomorphic to Z_2 by Synge's theorem. In the following we assume that m (≥ 3) is odd, unless otherwise stated. Then M is orientable. Let $p : V \to M$ be the universal Riemannian covering and Γ the deck transformation group corresponding to $\pi_1(M)$. Let G denote the identity connected component of the full isometry group of M. G is a compact Lie group with respect to the compact open topology. G also acts on M transitively. We take an $x_0 \in M$ and fix it in the following. Let H be the isotropy subgroup of G at x_0. The action $\Psi : G \times M \to M$ $((\varphi, x) \mapsto \varphi(x))$ on M on the left induces a diffeomorphism $\hat{\Psi} : G/H \to M$ $(\varphi H \mapsto \varphi(x_0))$.

LEMMA 4.1. *Let G and H be as above. If* $\dim H = 0$, *then* S^3 *is a covering space of M and Γ is isomorphic to a finite subgroup of* $SU(2)$.

PROOF. We identify M with G/H. By assumption, H is a finite subgroup of G. Hence the natural projection $p_1 : G \to G/H$ is a covering map. Let $\hat{G}$ be the universal covering Lie group of G with covering homomorphism p_2. Then $\hat{p} := p_1 \circ p_2 : \hat{G} \to G/H$ is a universal covering map and Γ is isomorphic to $p_2^{-1}(H)$. Hence $\hat{G}$ is compact. Let $\hat{g}$ be the Riemannian metric on $\hat{G}$ induced from that of G/H by $\hat{p}$. Then $\hat{g}$ is a left invariant metric on $\hat{G}$ and each sectional curvature K of $(\hat{G}, \hat{g})$ satisfies $1 \le K \le 4$. By a theorem due to Wallach ([20]; Theorem 2.1), $\hat{G}$ is isomorphic to $SU(2)$ as a Lie group. This completes the proof.

In what follows we assume that $\dim H \ge 1$. Any nontrivial one-parameter subgroup of H induces a nontrivial Killing vector field on M which vanishes at x_0. Let X be a nontrivial Killing vector field on M vanishing at x_0. Let L be the set of all points of M at which X vanishes. Each connected component of L is a compact totally geodesic submanifold (without boundary) embedded in M whose codimension is even ([13]; p. 59). Hence the dimension of each connected component of L is odd since m is odd.

Under the condition that $1 \le K_M \le 4$, L has the following properties.

LEMMA 4.2. *Let M and L be as above. Then*
(1) *L is connected.*
(2) *L is totally $\pi/2$-convex in M.*
(3) $i(L) \ge \pi/4$.

PROOF. Suppose that L is disconnected. Let $L_1, \dots, L_s$ $(s \ge 2)$ be the distinct connected components of L. By exchanging indices if necessary, we may assume that $d(L_1, L_2) \le d(L_i, L_j)$, $1 \le i < j \le s$. Let $\sigma : [0, a] \to M$ be a minimizing geodesic segment between L_1 and L_2 such that $\sigma(0) \in L_1$ and $\sigma(a) \in L_2$ where $a = d(L_1, L_2)$. Then X is a Jacobi field along σ which vanishes at $\sigma(0)$ and $\sigma(a)$. We note here that X does not vanish at $\sigma(t)$, $0 < t < a$. Since $K_M \le 4$, by Lemma 2.6 (1) we get $a \ge \pi/2$. Hence we have $d(M) = a = \pi/2$ because $d(M) \le \pi/2$. By the rigidity theorem ([7], [11]), M is of constant curvature 1 because $K_M \geq 1$ and m is odd. Since $\sigma(a)$ is the first conjugate point to $\sigma(0)$ along σ, it must be $a = \pi$. This is a contradiction, which implies (1).

Let $\gamma : [0, b] \to M$ be a geodesic segment such that $\gamma(0)$, $\gamma(b) \in L$ and $\gamma([0, b]) \not\subset L$. Since X is a nontrivial Jacobi field along γ, we have $b \geq \pi/2$ by Lemma 2.6 (1). This proves (2).

To show (3), we suppose that $r := i(L) < \pi/4$. Let x be a cut point of L with $d(x, L) = r$. It follows from Lemma 2.6 (2) that for each geodesic $\gamma : [0, \infty) \to M$ emanating orthogonally from L $\gamma(r)$ is not a focal point of L along γ. Hence there exist distinct minimizing geodesics $\sigma_1, \sigma_2 : [0, r] \to M$ from x to L. By Lemma 2.8 we have $\sigma_2'(0) = -\sigma_1'(0)$. Thus there exists a geodesic $\sigma : [0, 2r] \to M$ such that $\sigma(0)$, $\sigma(2r) \in L$ and $\sigma((0, 2r)) \cap L = \emptyset$. Since L is totally $\pi/2$-convex in M, we have $2r \geq \pi/2$, which is a contradiction. This shows (3).

Let L be as above. By homogeneity of M, L is a homogeneous Riemannian manifold ([14]; p. 60).

LEMMA 4.3. *M contains an embedded, connected, compact, totally geodesic submanifold N (without boundary) with the following properties*:

(1) $\dim N$ *is either* 1 *or* 3.

(2) *It is totally $\pi/2$-convex in M.*

(3) $i(N) \geq \pi/4$.

(4) *If* $\dim N = 3$, *then any nontrivial Killing vector field on N nowhere vanishes.*

PROOF. Let L be as above. As stated above, $\dim L$ is odd and codim L is even. In the case where $\dim L = 1$, we let $N = L$. Then the claim follows from Lemma 4.2. In the following we assume that $\dim L \geq 3$. We first consider the case where any nontrivial Killing field on L nowhere vanishes. Then the isotropy subgroup of the isometry group of L at x_0 is a discrete group. Lemma 4.1 shows $\dim L = 3$. Setting $N = L$, we obtain a submanifold with the required properties. Next let us consider the case where there exists a nontrivial Killing vector field X_1 on L vanishing at some point. Let L_1 be the set of all points of L at which X_1 vanishes. Then $\dim L_1$ is odd and $\dim L_1 \geq 1$. L_1 has the properties (1), (2) and (3) in Lemma 4.2 as a submanifold of L. Since L is totally $\pi/2$-convex in M, so is L_1. Moreover L_1 is a connected, compact, totally geodesic submanifold (without boundary) embedded in M. By the same way as in the proof of Lemma 4.2 (3), we obtain $i(L_1) \geq \pi/4$ as a submanifold of M. If $\dim L_1 \geq 3$, then in L_1 we can carry out the same argument as above. By repeating the argument above, we obtain a submanifold N of M which has the required properties.

From now on let N denote a connected, compact, totally geodesic submanifold (without boundary) embedded in M with the properties stated in Lemma 4.3. Since M is homogeneous, we may assume that $x_0 \in N$. N is also a homogeneous Riemannian manifold. Let G_1 be the identity connected component of the isometry group of N and H_1 the isotropy subgroup of G_1 at x_0. G_1 is a compact Lie group and acts transitively on N.

With the notations stated above, we have

LEMMA 4.4. *Assume that* $\dim N = 3$. *Then*
(1) H_1 *is a finite group.*
(2) N *is covered by* S^3.
(3) $\pi_1(N)$ *is isomorphic to a finite subgroup of* $SU(2)$.

PROOF. If $\dim H_1 \geq 1$, then each nontrivial one-parameter subgroup of H induces a nontrivial Killing vector field which vanishes at x_0. This contradicts Lemma 4.3 (4). Hence $\dim H_1 = 0$ and H_1 is a finite group. Then (2) and (3) follow from (1) and Lemma 4.1.

The following is evident.

LEMMA 4.5. *If* $\dim N = 1$ *and* $\pi_1(M,N) = 0$, *then* Γ *is a cyclic group.*

From Theorem A and Lemma 4.3 we have

LEMMA 4.6. *If* $\pi_1(M,N) \neq 0$, *then* Γ *is isomorphic to either a cyclic group or a binary dihedral group. Moreover if* $\pi_1(M,N) \neq 0$ *and* Γ *is a binary dihedral group, then* $m = 4j - 1$ $(j \geq 1)$.

PROOF. Suppose that Γ is not cyclic. Theorem A and Lemma 4.3 (3) imply that $i(N) = \pi/4$. Then Theorem A shows that $\Gamma \cong D_s^*$ $(s \geq 2)$ and $m = 4j - 1$ $(j \geq 1)$.

LEMMA 4.7. *Suppose that* $\dim N = 3$ *and* $\pi_1(M,N) = 0$. *Then* $\pi_1(M) \cong \pi_1(N)$.

PROOF. Let $\iota_\sharp : \pi_1(N,x_0) \to \pi_1(M,x_0)$ be the homomorphism induced from the inclusion $\iota : N \to M$. We take an $x_1 \in p^{-1}(x_0)$ and fix it. By assumption, it suffices to show that $\iota_\sharp$ is injective. To do that, we suppose that $\ker \iota_\sharp \neq \{e\}$. Let $[\gamma] \in \ker \iota_\sharp \backslash \{e\}$. By assumption, $\hat{N} := p^{-1}(N)$ is connected and Γ-invariant. Let $\hat{\gamma}$:

$[0,a] \to V$ be the lift of γ emanating from x_1. Since γ is homotopic to the point curve x_0 in M, $\hat{\gamma}$ is a loop in $\hat{N}$. Hence $\hat{N}$ is not simply connected since $\hat{\gamma}$ is homotopically nontrivial in $\hat{N}$. Thus the intrinsic diameter of $\hat{N}$ is not greater than $\pi/2$ ([8]). Therefore we have $d(z,w) \leq \pi/2$ for any z, $w \in \hat{N}$. Let x, y be points of V such that $d(x,y) = d(V)$. Then we have $d(x,y) \geq \pi/2$ because $1 \leq K_M \leq 4$ (Theorem 2.1 (1)). We shall show that $d(x,y) = \pi/2$. By homogeneity of V we may assume that $x \in \hat{N}$. If $y \in \hat{N}$, then $d(x,y) = \pi/2$. Let $y \notin \hat{N}$. If $d(y,\hat{N}) \geq \pi/2$, then by applying Lemma 2.4 to the present situation we conclude that V is isometric to S^m. Since $\hat{N}$ is totally geodesic, it is isometric to S^3, which contradicts that $\hat{N}$ is non-simply connected. Thus we have $d(y,\hat{N}) < \pi/2$. Let $\sigma_1 : [0,a] \to V$ be a minimizing geodesic between $\hat{N}$ and y such that $\sigma_1(0) \in \hat{N}$ and $\sigma_1(a) = y$ where $0 < a < \pi/2$. Let $\sigma_2 : [0,b] \to \hat{N}$ be a minimizing geodesic in $\hat{N}$ from $\sigma_1(0)$ to x where $0 < b \leq \pi/2$. Then $\sigma_1'(0)$ is orthogonal to $\sigma_2'(0)$. By applying Toponogov's comparison theorem to the hinge $(\sigma_1, \sigma_2, \pi/2)$, we obtain $d(x,y) \leq \pi/2$. Hence it must be $d(x,y) = \pi/2$. Thus we have $d(V) = \pi/2$. By Berger's minimal diameter theorem (Theorem 2.1 (2)), V is isometric to m-sphere $S^m(4)$ with constant curvature 4. Then $\hat{N}$ is isometric to 3-sphere $S^3(4)$ with constant curvature 4 because $\hat{N}$ is totally geodesic in V. This is a contradiction. Thus we have $\ker \imath_\sharp = \{e\}$, which shows that $\pi_1(M) \cong \pi_1(N)$.

Lemma 4.8. *Suppose that* $\dim N = 3$ *and that there exists a* $\varphi \in G$ *such that* $\varphi(N) \neq N$ *and* $\varphi(N) \cap N \neq \emptyset$. *Moreover assume that* Γ *is not cyclic. Then* $m = 4j - 1$ $(j \geq 2)$.

Proof. Let N_1 be a connected component of $\varphi(N) \cap N$. From the property of N (Lemma 4.3 (2)), N_1 is totally $\pi/2$-convex in M. Moreover N_1 is a compact, totally geodesic submanifold (without boundary) embedded in M. Since $\dim N = 3$ and $\varphi(N) \neq N$, we have $\dim N_1 \leq 2$. By the same way as in the proof of Lemma 4.2 (3), the inequality $i(N_1) \geq \pi/4$ holds as a submanifold of M. We first assume that $\dim N_1 = 0$. By homogeneity of M we obtain that $Inj(M) \geq \pi/4$. Since Γ is not cyclic, Theorem 4.1 shows that $Inj(M) = \pi/4$ and $m = 4j - 1$ $(j \geq 2)$. If $\dim N_1 = 1$, then $\pi_1(M, N_1) \neq 0$ because Γ is not cyclic. If $\dim N_1 = 2$, then the order of $\pi_1(N_1)$ is at most two, which implies $\pi_1(M, N_1) \neq 0$. Hence we have $\pi_1(M, N_1) \neq 0$ if $1 \leq \dim N_1 \leq 2$. Then Theorem A shows that $m = 4j - 1$ $(j \geq 2)$.

As a consequence of Lemma 4.8, we have

Lemma 4.9. *Assume that* $m = 5$ *and* $\dim N = 3$. *Then* Γ *is a cyclic group.*

PROOF. Suppose that Γ is not cyclic. Let $x \in M \backslash N$. By homogeneity of M, there exists a $\varphi \in G$ such that $\varphi(x_0) = x$. Clearly, we have $\varphi(N) \neq N$. It follows from our assumption and Frankel's theorem ([5]) that $\varphi(N) \cap N \neq \emptyset$. Then Lemma 4.8 shows that $m = 4j - 1$ $(j \geq 2)$, which is a contradiction. Thus Γ is a cyclic group.

We shall prove Theorems B and C. We use the same notations as above.

PROOF OF THEOREM B. Let N be as above. By Lemmas 4.5 and 4.6 it suffices to consider the case where $\dim N = 3$ and $\pi_1(M, N) = 0$. It follows from Lemmas 4.4 (3) and 4.7 that Γ is isomorphic to a finite subgroup of $SU(2)$.

PROOF OF THEOREM C. We suppose that Γ is not cyclic. Let N be as above. Since $m = 4j + 1$ $(j \geq 1)$, Lemmas 4.5 and 4.6 imply that $\dim N = 3$ and $\pi_1(M, N) = 0$. By Lemma 4.9 we may assume that $m = 4j + 1 \geq 9$. It follows from Lemma 4.8 that $\varphi(N) \cap N = \emptyset$ or $\varphi(N) = N$ for all $\varphi \in G$. Thus we have $\varphi(N) = N$ for all $\varphi \in H$. Let T $(\subset T_{x_0} M)$ be the tangent space to N at x_0. Let $\varphi, \psi \in G$ be such that $\varphi(x_0) = \psi(x_0)$. Since $\varphi(N) = \psi(N)$, we have $(d\varphi)_{x_0}(T) = (d\psi)_{x_0}(T)$. Hence the action $\Psi : G \times M \to M$ $((\varphi, x) \mapsto \varphi(x))$ induces a smooth field of 3-planes on M. This field of 3-planes can be lifted to V. Since V is homeomorphic to S^{4j+1} by the sphere theorem ([2], [8]), there exists a continuous field of 3-planes on S^{4j+1}. But this is a contradiction because S^{4j+1} does not admit a continuous field of 3-planes ([19]; p. 144). Therefore Γ is a cyclic group.

5. Examples

We give examples of connected, compact, non-simply connected homogeneous Riemannian manifolds whose sectional curvature K satisfies $\delta A \leq K \leq A$, where A and δ are positive constants and $1/4 \leq \delta < 1$. These manifolds are obtained as quotient spaces of Berger spheres.

5.1. By using the formula given in [15] we see that $SU(2)$ admits a left invariant Riemannian metric whose sectional curvature K satisfies $\delta A \leq K \leq A$. Let Γ be a nontrivial finite subgroup of $SU(2)$. Then the quotient space $M := SU(2)/\Gamma$ is a homogeneous Riemannian manifold with sectional curvature $\delta A \leq K_M \leq A$.

5.2. Let HP^m be the quaternion projective space with the standard Riemannian metric whose sectional curvature K satisfies $1 \leq K \leq 4$ where $m \geq 2$.

The symplectic group $Sp(m+1)$ acts transitively on HP^m as an isometry group. Fix an $x \in HP^m$. The isotropy subgroup of $Sp(m+1)$ at x is $Sp(m) \times Sp(1)$. Let V_r denote the geodesic hypersphere in HP^m with radius r and center x, $0 < r < \pi/2$. V_r is diffeomorphic to S^{4m-1}. The principal curvatures of V_r with respect to the inner unit normal are $2\cot 2r$ and $\cot r$ whose multiplicity are 3 and $4m-4$ respectively. Let K_σ be an arbitrary sectional curvature of V_r with the metric induced from HP^m. By using the equation of Gauss, we obtain $1 + 4\cot^2 2r \leq K_\sigma \leq 4 + \cot^2 r$. Thus there exists an r such that $0 < r < \pi/2$ and $4(1 + 4\cot^2 r) \geq 4 + \cot^2 r$. Let r be such a positive. Since HP^m is a two point homogeneous Riemannian manifold, $Sp(m) \times Sp(1)$ acts transitively on V_r as an isometry group. Let Γ_0 be a nontrivial finite subgroup of $Sp(1)$. Then $\Gamma := \{I\} \times \Gamma_0$ acts freely on V_r. Since $Sp(m) \times \{I\}$ acts on V_r transitively and $Sp(m) \times \{I\} \subset Z(\Gamma)$, the quotient space $M = V_r/\Gamma$ is a homogeneous Riemannian manifold ([21]; p. 73). Then all sectional curvature K_M of M satisfy $\delta A \leq K_M \leq A$, where $A = 4 + \cot^2 r$ and $\delta = (1 + 4\cot^2 2r)/(4 + \cot^2 r)$.

5.3. For the complex projective space CP^m with $1 \leq K \leq 4$ the same method as in 5.2 gives us non-simply connected homogeneous Riemannian manifolds M with $\delta A \leq K_M \leq A$ whose fundamental groups are cyclic groups.

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