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ANALYTIC SMOOTHING EFFECTS FOR SOME DERIVATIVE NONLINEAR SCHRÖDINGER EQUATIONS

By

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§ 1. Introduction

In this paper we study an analytic smoothing property of solutions to the Cauchy problem for the derivative nonlinear Schrödinger equation:

$$\begin{cases} iu_t + u_{xx} = \mathcal{N}(u, \bar{u}, u_x, \bar{u}_x), & x \in \boldsymbol{R}, \quad t \in \boldsymbol{R}, \\ u(0, x) = u_0(x), & x \in \boldsymbol{R}, \end{cases} \tag{1.1}$$

where the nonlinearity has a form

$$\mathcal{N} = \sum_{k+l-m-n=1} C_{klmn} u^k u_x^l \bar{u}^m \bar{u}_x^n, \quad k, l, m, n \in \boldsymbol{N} \cup \{0\}$$

satisfying the gauge condition such that

$$\omega \sum_{k+l-m-n=1} C_{klmn} u^k u_x^l \bar{u}^m \bar{u}_x^n = \sum_{k+l-m-n=1} C_{klmn} (\omega u)^k (\omega u_x)^l (\overline{\omega u})^m (\overline{\omega u_x})^n$$

for any $\omega \in \boldsymbol{C}$ and the coefficients $C_{klmn} = C_{klmn}(|u|^2) = C_{klmn}(f)$ are analytic and have analytic continuations $C_{klmn}(z)$ with $z = f + ig$ in the circle $|z| < \rho$, so that we can write the Taylor expansions

$$C_{klmn}(z) = \sum_{j=0}^{\infty} a_{j,klmn} z^j, \quad a_{j,klmn} = \frac{1}{j!} C_{klmn}^{(j)}(0) = \frac{1}{j!} \frac{d^j}{dz^j} C_{klmn}(0)$$

for $|z| < \rho$. We also assume that $\sum_{j=0}^{\infty} |a_{j,klmn}||z|^j \leq C(\rho)$ for $|z| < \rho$, where $C(\rho)$ is a continuous function on ρ. When $\rho = 1$ equation (1.1) involves the case of the nonlinearity $\mathcal{N} = \dfrac{\bar{u}u_x^2}{1 + |u|^2}$, which appears in the classical pseudospin magnet model, see [14].

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Smoothing effects of solutions to the nonlinear Schrödinger equation (1.1) with $\mathcal{N} = \sum_{k-m=1} C_{km} u^k \bar{u}^m$ was studied in [5], [8] by using the operator $\mathcal{J} = x + 2it\partial_x$, which commutes with the linear Schrödinger operator $\mathcal{L} = i\partial_t + \partial_x^2$. Recently in [6] we have shown the $\boldsymbol{C}^\infty$ smoothing effect for equation (1.1) by making use of a smoothing property of solutions to the linear Schödinger equations [Lemma 2.2, 6] (see also [4]). For generalized KdV-type equations similar $\boldsymbol{C}^\infty$ smoothing effect was shown in [2].

Our purpose is to extend the result of paper [6] to the analytical case. In this paper we will show that if the initial data u_0 satisfies the condition $(\cosh \beta x)u_0 \in \boldsymbol{H}^{3,0}$ and the norm $\|(\cosh \beta x)u_0\|_{3,0} < \rho$, when the nonlinearity $\mathcal{N}$ does not depend on $\bar{u}_x$, and the norm $\|(\cosh \beta x)u_0\|_{3,0}$, is sufficiently small when the nonlinearity $\mathcal{N}$ depends on $\bar{u}_x$, then there exist a positive time T depending on the size of the initial function $\|(\cosh \beta x)u_0\|_{3,0}$ and a unique solution u of the Cauchy problem (1.1) which is analytic with respect to x and has an analytic continuation on the complex plane $z = x + iy$ with $|y| < 2|\beta t|$, for all $t \in [-T, T] \backslash \{0\}$. Here the weighted Sobolev space $\boldsymbol{H}^{m,s} = \{\phi \in \boldsymbol{L}^2; \|\phi\|_{m,s} = \|(1+x^2)^{s/2}(1-\partial_x^2)^{m/2}\phi\| < \infty\}$, $m, s \in \boldsymbol{R}^+$.

Analytic smoothing effects of solutions to nonlinear dispersive equations were studied in [1], [12] for generalized KdV equations and in [1], [7], [8], [13] for nonlinear Schrödinger equations. However there are no result on analytic smoothing effects of solutions to nonlinear Schrödinger equations of derivative type except for the following derivative nonlinear Schrödinger equation

$$\begin{cases} iu_t + u_{xx} = i(|u|^2 u)_x, & x \in \boldsymbol{R},\ t \in \boldsymbol{R}, \\ u(0,x) = u_0(x), & x \in \boldsymbol{R}. \end{cases} \tag{1.2}$$

By using some gauge transformation technique the derivative nonlinear Schrödinger equation (1.2) can be translated to a system of nonlinear Schrödinger equations without derivatives of unknown functions. So in paper [7] the results similar to that of Theorem 1.1 stated below were shown for the Cauchy problem (1.2).

For linear Schrödinger-type equations with variable coefficients, $\boldsymbol{C}^\infty$ smoothing effects were studied in [3], [11], [16] and their results were extended to analytic cases in [10], [15] (see also [9]).

Before stating our results precisely, we give some notations and function spaces. We let $\partial_x = \partial/\partial x$ and $\mathcal{F}\phi$ or $\hat{\phi}$ be the Fourier transform of ϕ defined by $\mathcal{F}\phi(\chi) = 1/\sqrt{2\pi}\int e^{-ix\chi}\phi(x)\,dx$ and $\mathcal{F}^{-1}\phi(x)$ or $\check{\phi}(x)$ be the inverse Fourier transform of ϕ, i.e. $\mathcal{F}^{-1}\phi(x) = 1/\sqrt{2\pi}\int e^{ix\chi}\phi(\chi)\,d\chi$. We introduce some function spaces.

We let L^p be the Lebesgue space with the norm $\|\phi\|_p = (\int |\phi(x)|^p\,dx)^{1/p}$ if $1 \le p < \infty$ and $\|\phi\|_\infty = \text{ess.sup}\{|\phi(x)|; x \in \boldsymbol{R}\}$ if $p = \infty$. For simplicity we let $\|\phi\| = \|\phi\|_2$. Weighted Sobolev space $\boldsymbol{H}_p^{m,s} = \{\phi \in \boldsymbol{L}^2; \|\phi\|_{m,s,p} = \|(1+x^2)^{s/2}(1-\partial_x^2)^{m/2}\phi\|_p < \infty\}$, $m, s \in \boldsymbol{R}^+$, $1 \le p \le \infty$. For simplicity we write $\boldsymbol{H}^{m,s} = \boldsymbol{H}_2^{m,s}$ and let $\|\cdot\|_{m,s} = \|\cdot\|_{m,s,2}$. Also we define the analytic function space $\boldsymbol{A}^{\beta,m} = \{\phi \in \boldsymbol{L}^2; \|(\cosh\beta\chi)\hat{\phi}(\chi)\|_{m,0} < \infty\}$, $\beta > 0$, $m \in \boldsymbol{R}^+$ with the following norm $\|\phi\|_{\boldsymbol{A}^{\beta,m}} = \|(\cosh\beta\chi)\hat{\phi}(\chi)\|_{m,0}$, which can be expressed in the x-representation in terms of the analyticity in the strip $\{z = x + iy; -\infty < x < \infty, -\beta < y < \beta\}$ via the following norm $\|\phi(\cdot + i\beta)\|_{0,m} + \|\phi(\cdot - i\beta)\|_{0,m}$. Indeed we have the inequality $\|\phi\|_{\boldsymbol{A}^{\beta,m}} \le \|\phi(\cdot + i\beta)\|_{0,m} + \|\phi(\cdot - i\beta)\|_{0,m} \le 2\|\phi\|_{\boldsymbol{A}^{\beta,m}}$. We denote $(\psi, \varphi) = \int \psi(x) \cdot \bar{\varphi}(x)\,dx$. By $\boldsymbol{C}(\boldsymbol{I}; \boldsymbol{E})$ we denote the space of continuous functions from an interval $\boldsymbol{I}$ to a Banach space $\boldsymbol{E}$. We also use the following relations $|\partial_x| = \mathscr{F}^{-1}|\xi|\mathscr{F} = -\mathscr{H}\partial_x$. The Hilbert transformation $\mathscr{H}$ with respect to the variable x is defined as follows

$$\mathscr{H}\phi(x) = \frac{1}{\pi}\mathrm{Pv}\int_{\boldsymbol{R}} \frac{\phi(z)}{x - z}\,dz = -i\mathscr{F}^{-1}\frac{\xi}{|\xi|}\mathscr{F}\phi,$$

where Pv means the principal value of the singular integral. Let $\mathscr{J} = \mathscr{J}(t) = x + 2it\partial_x = \mathscr{U}(t)x\mathscr{U}(-t) = M(t)(2it\partial_x)M(-t)$, where $M = M(t) = \exp(ix^2/4t)$. We also freely use the following identities $[\mathscr{J}, \partial_x] = -1$, $[\mathscr{L}, \mathscr{J}] = 0$, where $\mathscr{L} = i\partial_t + \partial_x^2$. Different positive constants might be denoted by the same letter C, when it does not cause any confusion.

We now state our results in this paper.

THEOREM 1.1. *We assume that the nonlinear term $\mathscr{N}$ does not depend on $\bar{u}_x$, and the initial data u_0 are such that $u_0 \cosh\beta x \in \boldsymbol{H}^{3,0}$, where $\beta \in \boldsymbol{R}$ and the norm $\|u_0 \cosh\beta x\|_{3,0} < \rho$. Then there exist a time $T > 0$ depending on $\|u_0 \cosh\beta x\|_{3,0}$ and a unique solution u of the Cauchy problem* (1.1) *such that $u \in \boldsymbol{C}([-T, T]; \boldsymbol{H}^{2,0}) \cap \boldsymbol{L}^\infty(-T, T; \boldsymbol{H}^{3,0})$ and the solution u has an analytic continuation $u(t, z)$ to the strip $\{z = x + iy; -\infty < x < \infty, -2|t\beta| < y < 2|t\beta|, t \in [-T, T]\backslash\{0\}\}$ satisfying the estimate*

$$\sup_{-2|t\beta| < y < 2|t\beta|} |u(t, x + iy)| \le C\cosh\beta x \|u_0 \cosh\beta x\|_{3,0}$$

$$\textit{for all } (t, x) \in [-T, T]\backslash\{0\} \times \boldsymbol{R}.$$

For the case of the nonlinearities depending on $\bar{u}_x$ we have to assume the additional smallness condition on the initial data. We prove the following result.

THEOREM 1.2. *We assume that the nonlinear term $\mathscr{N}$ depends on $\bar{u}_x$, the initial data u_0 are such that $u_0 \cosh\beta x \in \boldsymbol{H}^{3,0}, \beta \in \boldsymbol{R}$ and the norm $\|u_0 \cosh\beta x\|_{3,0}$ is sufficiently small. Then the same results as in Theorem* 1.1 *are true.*

REMARK 1.1. In the case of C^∞ smoothing effect of nonlinear Schrödinger equation, existence time T depends on the size of the initial data such that $\|u_0\|_{[n/2]+3,0}$, where $[s]$ denotes the largest integer less than or equal to s and n denotes the spatial dimensions. However in the case of an analytic smoothing effect, situation is completely different from C^∞ case as the reader can see in our main theorems. We try to explain the reason why: the difference arises by considering the simple nonlinearity $\mathscr{N} = |u|^2 u$. As in [5] we have the following estimates of solution to (1.1) by a classical energy estimate

$$\sum_{0\le j\le m} \|x^j \mathscr{U}(-t)u(t)\| \le \sum_{0\le j\le m} \|x^j u_0\|_{1,0} + \int_0^t \|u(t)\|_{1,0}^2 \sum_{0\le j\le m} \|x^j \mathscr{U}(-t)u(t)\|\, dt$$

which yields C^∞ smoothing effect since $\mathscr{U}(t)x^j\mathscr{U}(-t) = M(it\partial_x)^j\overline{M}$ and T depends on $\|u_0\|_{1,0}^2$ (for derivative nonlinearities, T depends on $\|u_0\|_{3,0}^2$). For an analytic case we have as in [8]

$$\begin{aligned}\|\cosh(\beta x)\mathscr{U}(-t)u(t)\| &\le \|\cosh(\beta x)u_0\|_{1,0}\\ &\quad + \int_0^t \|\cosh(\beta x)\mathscr{U}(-t)u(t)\|_{1,0}^2 \|\cosh(\beta x)\mathscr{U}(-t)u(t)\|\, dt\end{aligned}$$

which yields an analytic smoothing effect and therefore T depends on $\|\cosh(\beta x)u_0\|_{1,0}^2$. We can not expect the estimate

$$\|\cosh(\beta x)\mathscr{U}(-t)u(t)\| \le \|\cosh(\beta x)u_0\|_{1,0} + \int_0^t \|u(t)\|_{1,0}^2 \|\cosh(\beta x)\mathscr{U}(-t)u(t)\|\, dt$$

since the solution becomes analytic for $t \ne 0$.

REMARK 1.2. Roughly speaking, in order to prove Theorems we introduce the function space

$$X = \left\{ f \in C([0,T]; L^2);\ \sup_{t\in[0,T]} \|(\cosh \beta x)\mathscr{U}(-t)u(t)\|_{3,0} < \infty \right\}$$

We make X into a complete metric space by the distance function

$$d(f,g) = \|(\cosh \beta x)\mathscr{U}(-t)(f-g)\|_{2,0}.$$

We must use this metric in a sub-space of X (defined precisely in Section 3) since we use a contraction mapping method to prove this existence of u with analytic properties. That is the reason why even if the data $u_0 \in \boldsymbol{H}^{3,0}$, the solutions of (1.1) are not continuous in time in $\boldsymbol{H}^{3,0}$ space. Expected result from a-priori estimates of solutions obtained in Section 3 is $u \in \boldsymbol{C}([0,T];\boldsymbol{H}^{s,0})$, where $2 < s < 3$. Indeed we have this result by

$$\|u(t) - u(\tau)\|_{s,0} \leq C\|u(t) - u(\tau)\|_{2,0}^{3-s}\|u(t) - u(\tau)\|_{3,0}^{s-2}$$

which is obtained by Sobolev's inequality.

The rest of the paper is organized as follows. In Section 2 we describe a smoothing property of the linear Schrödinger equation and some estimates of nonlinearities. Then in Section 3 we prove in Lemma 3.1 the local existence of solutions to the Cauchy problem (1.1) in the functional space $\{u \in \boldsymbol{C}([-T,T];\boldsymbol{L}^2);$ $\|(\cosh \beta x)\mathscr{U}(-t)u(t)\|_{3,0} < \infty\}$, where $\mathscr{U}(t)$ is the free Schrödinger evolution group. And as a simple consequence we obtain the result of Theorem 1.1. Section 4 is devoted to the outline of the proof of Theorem 1.2.

§ 2. Linear smoothing effect

The aim of this section is to present the smoothing effect for solutions to the Cauchy problem for the linear Schrödinger equations

$$\begin{cases} iu_t + u_{xx} = f, & x \in \boldsymbol{R}, \ t \in \boldsymbol{R}, \\ u(0,x) = u_0(x), & x \in \boldsymbol{R}, \end{cases} \tag{2.1}$$

where the function $f(t,x)$ is a force. In order to state Lemma 2.1 and Lemma 2.2 which have been shown in paper [6], we define a pseudo-differential operator $\mathscr{S}(\varphi) = \cosh(\varphi) + i\sinh(\varphi)\mathscr{H}$ which yields a smoothing effect of solutions to (2.1), where the real-valued function $\varphi(t,x) \in \boldsymbol{L}^\infty(0,T;\boldsymbol{H}_\infty^{2,0}) \cap \boldsymbol{C}^1([0,T];\boldsymbol{L}^\infty)$ and is positive. From its definition we easily see that the operator $\mathscr{S}$ acts continuously from $\boldsymbol{L}^2$ to $\boldsymbol{L}^2$ with the following estimate $\|\mathscr{S}(\varphi)\psi\| \leq 2\exp(\|\varphi\|_\infty)\|\varphi\|$. The inverse operator $\mathscr{S}^{-1}(\varphi) = (1 + i\tanh(\varphi)\mathscr{H})^{-1}1/\cosh(\varphi)$ also exists and is continuous

$$\|\mathscr{S}^{-1}(\varphi)\psi\| \leq (1 - \tanh(\|\varphi\|_\infty))^{-1}\|\psi\| \leq \exp(\|\varphi\|_\infty)\|\psi\|. \tag{2.2}$$

The operator $\mathscr{S}$ helps us to obtain a smoothing property of the Schrödinger-type equation (2.1) by virtue of the usual energy estimates. In the next lemma we prepare an energy estimate, involving the operator $\mathscr{S}$, in which we have an additional positive term giving us the norm of the half derivative of the unknown function u. We also assume that $\varphi(x)$ is written as $\varphi(x) = \partial_x^{-1}(\omega^2)$, so that $\omega(x) = \sqrt{(\partial_x\varphi)}$.

LEMMA 2.1. *The following inequality*

$$\frac{d}{dt}\|\mathscr{S}u\|^2 + \|\omega\mathscr{S}\sqrt{|\partial_x|}u\|^2 \le 2|Im(\mathscr{S}u, \mathscr{S}f)|$$

$$+ C\|u\|^2 e^{2\|\varphi\|_\infty}(\|\omega\|_\infty^4 + \|\omega\|_\infty^6 + \|\omega\|_{1,0,\infty}\|\omega\|_\infty + \|\varphi_t\|_\infty)$$

is valid for the solution u of the Cauchy problem (2.1).

LEMMA 2.2. *We have the following estimates*

$$|(\mathscr{S}u, \mathscr{S}\phi\psi\partial_x v)| \le \|\,|\phi|\mathscr{S}\sqrt{|\partial_x|}u\|^2 + \|\,|\psi|\mathscr{S}\sqrt{|\partial_x|}v\|^2$$

$$+ C(\|u\|^2 + \|v\|^2)e^{6\|\varphi\|_\infty}(\|\phi\|_{1,0,\infty}^2 + \|\psi\|_{1,0,\infty}^2)(1 + \|\varphi\|_{1,0,\infty}^2),$$

and

$$|(\mathscr{S}u, \mathscr{S}\phi\psi\partial_x \bar{v})| \le \|\,|\phi|\mathscr{S}\sqrt{|\partial_x|}u\|^2 + e^{4\|\varphi\|_\infty}\|\,|\psi|\mathscr{S}\sqrt{|\partial_x|}v\|^2$$

$$+ C(\|u\|^2 + \|v\|^2)e^{6\|\varphi\|_\infty}(\|\phi\|_{1,0,\infty}^2 + \|\psi\|_{1,0,\infty}^2)(1 + \|\varphi\|_{1,0,\infty}^2),$$

provided that the right hand sides are bounded.

For the proofs of Lemma 2.1 and Lemma 2.2, see [6].
The following lemma is the analytic version of Lemma 2.2.

LEMMA 2.3. *We have the following estimates*

$$|(\mathscr{S}\mathscr{F}^{-1}e^{\beta\chi}\mathscr{F}u, \mathscr{S}\mathscr{F}^{-1}e^{\beta\chi}\mathscr{F}(\phi\psi\partial_x v))|$$

$$\le \|\,|\phi(\cdot + i\beta)|\mathscr{S}\sqrt{|\partial_x|}u(\cdot + i\beta)\|^2 + \|\,|\psi(\cdot + i\beta)|\mathscr{S}\sqrt{|\partial_x|}v(\cdot + i\beta)\|^2$$

$$+ C(\|u(\cdot + i\beta)\|^2 + \|v(\cdot + i\beta)\|^2)e^{6\|\varphi\|_\infty}$$

$$\times (\|\phi(\cdot + i\beta)\|_{1,0,\infty}^2 + \|\psi(\cdot + i\beta)\|_{1,0,\infty}^2)(1 + \|\varphi\|_{1,0,\infty}^2),$$

and

$$|(\mathscr{S}\mathscr{F}^{-1}e^{\beta\chi}\mathscr{F}u, \mathscr{S}\mathscr{F}^{-1}e^{\beta\chi}\mathscr{F}(\phi\psi\partial_x \bar{v}))|$$

$$\le \|\,|\phi(\cdot + i\beta)|\mathscr{S}\sqrt{|\partial_x|}u(\cdot + i\beta)\|^2 + e^{4\|\varphi\|_\infty}\|\,|\psi(\cdot + i\beta)|\mathscr{S}\sqrt{|\partial_x|}v(\cdot - i\beta)\|^2$$

$$+ C(\|u(\cdot + i\beta)\|^2 + \|v(\cdot + i\beta)\|^2)e^{6\|\varphi\|_\infty}$$

$$\times (\|\phi(\cdot + i\beta)\|_{1,0,\infty}^2 + \|\psi(\cdot + i\beta)\|_{1,0,\infty}^2)(1 + \|\varphi\|_{1,0,\infty}^2),$$

provided that the right hand sides are bounded.

PROOF. The lemma follows from Lemma 2.2 and the identity $\mathscr{F}^{-1}e^{\beta\chi}\mathscr{F}(\phi\psi\partial_x v) = \phi(\cdot+i\beta)\psi(\cdot+i\beta)\partial_x v(\cdot+i\beta)$. □

Finally we show the estimates of the nonlinear terms in the analytic function space.

LEMMA 2.4. *We have the estimate*

$$\|(\cosh\beta x)\mathscr{U}(-t)(\phi\psi\bar{v})\|$$
$$\leq C\|(\cosh\beta x)\mathscr{U}(-t)\phi\|_{1,0}\|(\cosh\beta x)\mathscr{U}(-t)\psi\|_{1,0}\|(\cosh\beta x)\mathscr{U}(-t)v\|,$$

provided that the right hand side is finite and if $\|(\cosh\beta x)\mathscr{U}(-t)u\|_{1,0} < \rho$ *we have the estimate*

$$\|(\cosh\beta x)\mathscr{U}(-t)(C_{klmn}(|u|^2)v)\| \leq C(\rho)\|(\cosh\beta x)\mathscr{U}(-t)v\|$$

provided that the right hand side is finite.

PROOF. By the identity $\mathscr{U}(t)e^{\beta x}\mathscr{U}(-t) = M\mathscr{F}^{-1}e^{2t\beta\chi}\mathscr{F}\overline{M}$ we have

$$\begin{aligned}
\mathscr{U}(t)e^{\beta x}\mathscr{U}(-t)(\phi\psi\bar{v}) &= M\mathscr{F}^{-1}e^{2t\beta\chi}\mathscr{F}(\overline{M}\phi)(\overline{M}\psi)(\overline{\overline{M}v}) \\
&= M((\overline{M}\phi)(x+2it\beta))((\overline{M}\psi)(x+2it\beta))(\overline{(\overline{M}v)(x-2it\beta)}) \\
&= (M\mathscr{F}^{-1}e^{2t\beta\chi}\mathscr{F}\overline{M}\phi)(M\mathscr{F}^{-1}e^{2t\beta\chi}\mathscr{F}\overline{M}\psi)(\overline{M\mathscr{F}^{-1}e^{-2t\beta\chi}\mathscr{F}\overline{M}v}) \\
&= (\mathscr{U}(t)e^{\beta x}\mathscr{U}(-t)\phi)(\mathscr{U}(t)e^{\beta x}\mathscr{U}(-t)\psi)(\overline{\mathscr{U}(t)e^{-\beta x}\mathscr{U}(-t)v}). \quad (2.3)
\end{aligned}$$

We take the L^2 norm, and apply the Sobolev's inequality and the identity

$$\|(\cosh\beta x)\mathscr{U}(-t)f\|^2 = \frac{1}{4}(\|e^{\beta x}\mathscr{U}(-t)f\|^2 + 2\|u\|^2 + \|e^{-\beta x}\mathscr{U}(-t)f\|^2)$$

to (2.3) to see that the norm $\|e^{\beta x}\mathscr{U}(-t)(\phi\psi\bar{v})\|$ is bounded from above by the right hand side of the first estimate of the lemma. The value $\|e^{-\beta x}\mathscr{U}(-t)(\phi\psi\bar{v})\|$ is estimated in the same way. Thus we obtain the first estimate of the lemma.

From the analyticity condition on the functions $C_{klmn}(|u|^2)$ and by the first estimate of Lemma 2.4 we have

$$\|(\cosh\beta x)\mathscr{U}(-t)(C_{klmn}(|u|^2)v)\|$$
$$\leq \sum_{j=0}^{\infty}|a_{j,klmn}|\,\|(\cosh\beta x)\mathscr{U}(-t)u\|_{1,0}^{2j}\|(\cosh\beta x)\mathscr{U}(-t)v\|$$

which implies the second estimate of the lemma. Lemma 2.4 is proved. □

§3. Proof of Theorem 1.1

In what follows we consider the case $t > 0$ only since the case $t < 0$ can be treated similarly. First we prove the local existence of solutions.

LEMMA 3.1. *We assume that the nonlinear term* $\mathcal{N}$ *does not depend on* $\bar{u}_x$, *and the initial data are such that* $u_0 \cosh \beta x \in \boldsymbol{H}^{3,0}$. *Then there exists a time* $T > 0$ *depending on* $\|u_0 \cosh \beta x\|_{3,0}$ *and a unique solution* u *of the Cauchy problem* (1.1) *such that*

$$\sup_{t \in [0,T]} \|(\cosh \beta x)\mathcal{U}(-t)u\|_{3,0} < \infty.$$

PROOF. Applying the operator $(1 - \partial_x^2)$ to the equation (1.1), we get for the function $v = (1 - \partial_x^2)u$

$$\begin{cases} \mathcal{L}v = \mathcal{G}(u, u_x)v_x + \mathcal{R}(v) \\ v(0, x) = (1 - \partial_x^2)u_0(x), \end{cases} \tag{3.1}$$

where the coefficient at the main term is $\mathcal{G}(u, u_x) = \partial_{u_x}\mathcal{N}$ and $\mathcal{R}(v)$ is the remainder term. It is easy to see that

$$\mathcal{G}(u, u_x) = \sum_{\substack{k+l-m-n=1 \\ l \geq 1}} lC_{klmn}u^k u_x^{l-1}\bar{u}^m \bar{u}_x^n + \sum_{\substack{k+l-m-n=1 \\ n \geq 1}} nC_{klmn}u^k u_x^l \bar{u}^m \bar{u}_x^{n-1}$$

when the nonlinearity $\mathcal{N}$ depends on $\bar{u}_x$ and we have

$$\mathcal{G}(u, u_x) = \sum_{\substack{k+l-m-n=1 \\ l \geq 1}} lC_{klm0}u^k u_x^{l-1}\bar{u}^m$$

when the nonlinearity $\mathcal{N}$ does not depend on $\bar{u}_x$.

We now consider the linearized version of equation (3.1)

$$\begin{cases} \mathcal{L}v = \mathcal{G}(\tilde{u}, \tilde{u}_x)v_x + \mathcal{R}(\tilde{v}) \\ v(0, x) = (1 - \partial_x^2)u_0(x), \end{cases} \tag{3.2}$$

where the function $\tilde{u} = (1 - \partial_x^2)^{-1}\tilde{v}$ is defined by the known function $\tilde{v}$ which is in the ball

$$\boldsymbol{B} = \Big\{ \tilde{v} \in \boldsymbol{C}^1([0, T]; \boldsymbol{L}^2) : \sup_{t \in [0,T]} \|(\cosh \beta x)\mathcal{U}(-t)\tilde{v}\| \leq 2\rho,$$

$$\sup_{t \in [0,T]} \|(\cosh \beta x)\mathcal{U}(-t)\tilde{v}\|_{1,0} \leq \mu,$$

$$\sup_{t\in[0,T]} \|\partial_t\partial_x^{-1}(|\mathcal{U}(t)e^{\beta x}\mathcal{U}(-t)\tilde{u}|^2 + |\mathcal{U}(t)e^{\beta x}\mathcal{U}(-t)\tilde{u}_x|^2)\|_\infty \le 2\nu,$$

$$\left.\sup_{t\in[0,T]} \|\partial_t\partial_x^{-1}(|\mathcal{U}(t)e^{-\beta x}\mathcal{U}(-t)\tilde{u}|^2 + |\mathcal{U}(t)e^{-\beta x}\mathcal{U}(-t)\tilde{u}_x|^2)\|_\infty \le 2\nu\right\},$$

where $\rho = \|(\cosh\beta x)u_0\|_{3,0}$, and μ, ν are some positive constants depending on ρ. Thus the Cauchy problem (3.2) defines a mapping $\mathcal{A} : v = \mathcal{A}(\tilde{v})$. First let us show that there exists a time $T > 0$, such that the mapping $\mathcal{A}$ transforms the closed ball $\boldsymbol{B}$ into itself. Then we prove that there exists a time $T > 0$ such that $\mathcal{A}$ is a contraction mapping in the norm $\sup_{t\in[0,T]}\|(\cosh\beta x)\mathcal{U}(-t)\cdot\|$ under the constraint that it acts on the subspace $\boldsymbol{B}$. By the classical energy method and Lemma 2.4 we have from equation (3.2)

$$\frac{d}{dt}\|(\cosh\beta x)\mathcal{U}(-t)v(t)\| \le C + C\|(\cosh\beta x)\mathcal{U}(-t)v_x(t)\|,$$

hence we get

$$\sup_{t\in[0,T]}\|(\cosh\beta x)\mathcal{U}(-t)v(t)\| \le \rho + \sqrt{T}\sup_{t\in[0,T]}\|(\cosh\beta x)\mathcal{U}(-t)v_x(t)\|, \tag{3.3}$$

if we choose a time $T > 0$ to be sufficiently small.

In order to obtain the estimates of the norm $\sup_{t\in[0,T]}\|(\cosh\beta x)\mathcal{U}(-t)v_x(t)\|$ we use the operator $\mathcal{S}(\varphi) = \cosh(\varphi) + i\sinh(\varphi)\mathcal{H}$ introduced in Section 2, where the function $\varphi(t,x) = \varphi_\beta(t,x) + \varphi_{-\beta}(t,x)$ and

$$\varphi_\beta(t,x) = \frac{1}{\delta}\partial_x^{-1}\Bigg(|\mathcal{U}(t)e^{\beta x}\mathcal{U}(-t)\tilde{u}(t,x)|^2 + |\mathcal{U}(t)e^{\beta x}\mathcal{U}(-t)\tilde{u}_x(t,x)|^2$$
$$+ \sum_{\substack{k+l-m-n=1\\ k+m\neq 0}} |\mathcal{U}(t)e^{\beta x}\mathcal{U}(-t)C_{klmn}\tilde{u}|^2 + \sum_{l-n=1} |\mathcal{U}(t)e^{\beta x}\mathcal{U}(-t)C_{0l0n}\tilde{u}_x|^2\Bigg)$$

is in the space $\boldsymbol{L}^\infty(0,T;\boldsymbol{C}^2(\boldsymbol{R})) \cap \boldsymbol{C}^1([0,T];\boldsymbol{L}^\infty)$. As in Section 2 we denote $\omega(t,x) = (\partial_x\varphi(t,x))^{1/2}$. Therefore applying Lemma 2.2 we obtain the energy type inequality for the function $h = \mathcal{U}(t)e^{\beta x}\mathcal{U}(-t)\partial_x v$

$$\frac{d}{dt}\|\mathcal{S}h\|^2 + \|\omega\mathcal{S}\sqrt{|\partial_x|}h\|^2$$
$$\le 2|\mathrm{Im}(\mathcal{S}h, \mathcal{S}\mathcal{U}(t)e^{\beta x}\mathcal{U}(-t)\partial_x(\mathcal{G}(\tilde{u},\tilde{u}_x)v_x))|$$
$$+ 2|\mathrm{Im}(\mathcal{S}h, \mathcal{S}\mathcal{U}(t)e^{\beta x}\mathcal{U}(-t)\partial_x\mathcal{R}(\tilde{v}))|$$
$$+ Ce^{\|\varphi\|_\infty}(\|\omega\|_\infty^4 + \|\omega\|_\infty^6 + \|\omega\|_{1,0,\infty}\|\omega\|_\infty + \|\varphi_t\|_\infty)\|h\|^2. \tag{3.4}$$

By Sobolev's inequality, Lemma 2.4 and the identity (2.1) we get $\|\omega\|_\infty \le C(\rho)/\sqrt{\delta}$, $\|\omega\|_{1,0,\infty} \le C(\mu)/\sqrt{\delta}$, $\|\varphi\|_\infty \le C(\rho)/\delta$ and $\|\varphi_t\|_\infty \le C(v)/\delta$, where $C(\rho)$, $C(\mu)$ and $C(v)$ are some positive constants depending on ρ, μ and v respectively. Via the Schwarz inequality, (2.1) and Lemma 2.4, we obtain

$$|\mathrm{Im}(\mathscr{S}h, \mathscr{S}\mathscr{U}(t)e^{\beta x}\mathscr{U}(-t)\partial_x\mathscr{R}(\tilde{v}))| \le e^{2\|\varphi\|_\infty}\|h\|\,\|\mathscr{U}(t)e^{\beta x}\mathscr{U}(-t)\mathscr{R}(\tilde{v})\|_{1,0}$$
$$\le C(\mu)\|(\cosh\beta x)\mathscr{U}(-t)u\|_{3,0}. \tag{3.5}$$

In the same way as in the proof of (3.5)

$$\|e^{\beta x}\mathscr{U}(-t)(\partial_x(\mathscr{G}(\tilde{u},\tilde{u}_x)v_x) - (\mathscr{G}(\tilde{u},\tilde{u}_x)v_{xx}))\| \le C(\mu)\|(\cosh\beta x)\mathscr{U}(-t)u\|_{3,0}.$$

Thus we have

$$|\mathrm{Im}(\mathscr{S}h, \mathscr{S}\mathscr{U}(t)e^{\beta x}\mathscr{U}(-t)\partial_x\mathscr{G}(\tilde{u},\tilde{u}_x)v_x)|$$
$$\le |\mathrm{Im}(\mathscr{S}h, \mathscr{S}\mathscr{U}(t)e^{\beta x}\mathscr{U}(-t)(\mathscr{G}(\tilde{u},\tilde{u}_x)v_{xx}))|$$
$$+ C(\mu)\|(\cosh\beta x)\mathscr{U}(-t)u\|_{3,0}^2. \tag{3.6}$$

To estimate the main term $\mathrm{Im}(\mathscr{S}h, \mathscr{S}\mathscr{U}(t)e^{\beta x}\mathscr{U}(-t)(\mathscr{G}(\tilde{u},\tilde{u}_x)v_{xx}))$ in the left hand side of (3.6) we apply Lemma 2.3 to get

$$|\mathrm{Im}(\mathscr{S}h, \mathscr{S}\mathscr{U}(t)e^{\beta x}\mathscr{U}(-t)(\mathscr{G}(\tilde{u},\tilde{u}_x)v_{xx}))|$$
$$= |\mathrm{Im}(\mathscr{S}h, \mathscr{S}M\mathscr{F}^{-1}e^{2it\beta x}\mathscr{F}\overline{M}(\mathscr{G}(\tilde{u},\tilde{u}_x)v_{xx}))|$$
$$\le C\delta\|\omega\mathscr{S}\sqrt{|\partial_x|}\mathscr{U}(t)e^{\beta x}\mathscr{U}(-t)v_x\|^2 + C(\mu)\|(\cosh\beta x)\mathscr{U}(-t)u\|_{3,0}^2.$$

Hence we get by (3.6)

$$|\mathrm{Im}(\mathscr{S}h, \mathscr{S}\mathscr{U}(t)e^{\beta x}\mathscr{U}(-t)\partial_x(\mathscr{G}(\tilde{u},\tilde{u}_x)v_x))|$$
$$\le C\delta\|\omega\mathscr{S}\sqrt{|\partial_x|}\mathscr{U}(t)e^{\beta x}\mathscr{U}(-t)v_x\|^2 + C(\mu)\|(\cosh\beta x)\mathscr{U}(-t)u\|_{3,0}^2. \tag{3.7}$$

Substitution of (3.5)–(3.7) into (3.4) yields

$$\frac{d}{dt}\|\mathscr{S}\mathscr{U}(t)e^{\beta x}\mathscr{U}(-t)v_x\|^2 + (1 - C\delta)\|\omega\mathscr{S}\sqrt{|\partial_x|}\mathscr{U}(t)e^{\beta x}\mathscr{U}(-t)v_x\|^2$$
$$\le (C(\mu) + C(v))\|(\cosh\beta x)\mathscr{U}(-t)u\|_{3,0}^2. \tag{3.8}$$

If we now choose $\delta = 1/C$, then integration of (3.8) and (3.3) give us the estimate $\|e^{\beta x}\mathscr{U}(-t)u\|_{3,0} \le \mu/C$. In the same way $\|e^{-\beta x}\mathscr{U}(-t)u\|_{3,0} \le \mu/C$. Therefore we have $\|(\cosh\beta x)\mathscr{U}(-t)u\|_{3,0} \le \mu$ by (2.1). By virtue of this estimate and (3.3) we get $\|(\cosh\beta x)\mathscr{U}(-t)u\|_{2,0} \le 2\rho$, if the time interval $T > 0$ is sufficiently small.

Now directly from the system (3.2) we see that the function u satisfy the equation $\mathscr{L}u = (1-\partial_x^2)^{-1}(\mathscr{G}(\tilde{u},\tilde{u}_x)v_x + \mathscr{R}(\tilde{v}))$. Hence we get by Lemma 2.4

$$
\begin{aligned}
&\|\partial_t\partial_x^{-1}|\mathscr{U}(t)e^{\beta x}\mathscr{U}(-t)u|^2\|_\infty\\
&\quad= 2\|\partial_x^{-1}\mathrm{Re}(\overline{\mathscr{U}(t)e^{\beta x}\mathscr{U}(-t)u}\cdot(\mathscr{U}(t)e^{\beta x}\mathscr{U}(-t)u)_t)\|_\infty\\
&\quad\le \|\overline{\mathscr{U}(t)e^{\beta x}\mathscr{U}(-t)u}(\mathscr{U}(t)e^{\beta x}\mathscr{U}(-t)u)_x)\|_\infty\\
&\qquad+2\|e^{\beta x}\mathscr{U}(-t)u\|\,\|e^{\beta x}\mathscr{U}(-t)(1-\partial_x^2)^{-1}\mathscr{G}(\tilde{u},\tilde{u}_x)v_x\|\\
&\qquad+2\|e^{\beta x}\mathscr{U}(-t)u\|\,\|e^{\beta x}\mathscr{U}(-t)(1-\partial_x^2)^{-1}\mathscr{R}(\tilde{v})\|\\
&\quad\le \|(\cosh\beta x)\mathscr{U}(-t)u\|_{1,0}\|(\cosh\beta x)\mathscr{U}(-t)u_x\|_{1,0}\\
&\qquad+C(\rho)\|(\cosh\beta x)\mathscr{U}(-t)u\|_{2,0}+C(\mu)\|(\cosh\beta x)\mathscr{U}(-t)\tilde{u}\|_{2,0}^2\le\nu.
\end{aligned}
$$

In the same manner we have the estimates

$$
\begin{aligned}
&\|\partial_t\partial_x^{-1}|\mathscr{U}(t)e^{\beta x}\mathscr{U}(-t)u_x|^2\|_\infty\le\nu,\\
&\|\partial_t\partial_x^{-1}|\mathscr{U}(t)e^{-\beta x}\mathscr{U}(-t)u|^2\|_\infty\le\nu
\end{aligned}
$$

and

$$
\|\partial_t\partial_x^{-1}|\mathscr{U}(t)e^{-\beta x}\mathscr{U}(-t)u_x|^2\|_\infty\le\nu.
$$

Thus the mapping $\mathscr{A}$ transforms the ball $\boldsymbol{B}$ into itself. Let us show now that $\mathscr{A}$ is a contraction mapping in the norm $\sup_{t\in[0,T]}\|(\cosh\beta x)\mathscr{U}(-t)\cdot\|$. Let $v^\dagger$ satisfy the equation (3.2) with the known function $\tilde{v}^\dagger\in\boldsymbol{B}$ instead of $\tilde{v}$. Then for the difference $g=v^\dagger-v$ we get

$$
\begin{cases}
\mathscr{L}g=\mathscr{G}(\tilde{u}^\dagger,\tilde{u}_x^\dagger)g_x+(\mathscr{G}(\tilde{u}^\dagger,\tilde{u}_x^\dagger)-\mathscr{G}(\tilde{u},\tilde{u}_x))v_x & \\
\qquad+\mathscr{R}(\tilde{v}^\dagger)-\mathscr{R}(\tilde{v}), & x\in\boldsymbol{R},\quad t\in[0,T]\\
g(0,x)=0, & x\in\boldsymbol{R}.
\end{cases}\tag{3.9}
$$

Denoting $\tilde{g}=\tilde{v}^\dagger-\tilde{v}$ we get by Lemma 2.4

$$
\|e^{\beta x}\mathscr{U}(-t)(\mathscr{G}(\tilde{u}^\dagger,\tilde{u}_x^\dagger)v_x-\mathscr{G}(\tilde{u},\tilde{u}_x)v_x)\|\le C\|(\cosh\beta x)\mathscr{U}(-t)\tilde{g}\|
$$

and $\|e^{\beta x}\mathscr{U}(-t)(\mathscr{R}(\tilde{v}^\dagger)-\mathscr{R}(\tilde{v}))\|\le C\|(\cosh\beta x)\mathscr{U}(-t)\tilde{g}\|$. Considering the value g similarly to the function h we get from (3.9) the estimate analogous to (3.8)

$$
\begin{aligned}
&\frac{d}{dt}\|\mathscr{S}\mathscr{U}(t)e^{\beta x}\mathscr{U}(-t)g\|^2+(1-C\delta)\|\omega\mathscr{S}\sqrt{|\partial_x|}\mathscr{U}(t)e^{\beta x}\mathscr{U}(-t)g\|^2\\
&\qquad\le(C(\mu)+C(\nu))(\|(\cosh\beta x)\mathscr{U}(-t)\tilde{g}\|+\|(\cosh\beta x)\mathscr{U}(-t)g\|),
\end{aligned}
$$

therefore integrating with respect to time t, we have

$$\|e^{\beta x}\mathscr{U}(-t)g\|^2 \le CT(\|(\cosh \beta x)\mathscr{U}(-t)\tilde{g}\| + \|(\cosh \beta x)\mathscr{U}(-t)g\|).$$

Similarly, the value $\|e^{-\beta x}\mathscr{U}(-t)g\|^2$ is estimated by the right hand side of the above inequality. On a sufficiently small interval $T > 0$, we get by (2.1) the desired estimate

$$\sup_{t \in [0,T]} \|(\cosh \beta x)\mathscr{U}(-t)g\| \le \frac{1}{2} \sup_{t \in [0,T]} \|(\cosh \beta x)\mathscr{U}(-t)\tilde{g}\|.$$

Thus the transformation $\mathscr{A}$ is a contraction mapping. Therefore there exists a unique solution $u \in \boldsymbol{C}([0,T]; \boldsymbol{H}^{2,0})$ of the Cauchy problem (1.1) such that $(\cosh \beta x)\mathscr{U}(-t)u \in \boldsymbol{L}^\infty(0,T;\boldsymbol{H}^{3,0})$ for all $0 < t \le T$. This completes the proof of Lemma 3.1. □

PROOF OF THEOREM 1.1. Using the identity $\mathscr{U}(t)e^{\beta x}\mathscr{U}(-t) = M\mathscr{F}^{-1}e^{2t\beta\chi}\mathscr{F}\overline{M}$ and equality (2.1), we get

$$\begin{aligned}\|\mathscr{U}(t)(\cosh \beta x)\mathscr{U}(-t)u\|^2 &= \frac{1}{4}(\|e^{2it\beta\partial_x}\overline{M}u\|^2 + 2\|u\|^2 + \|e^{-2it\beta\partial_x}\overline{M}u\|^2) \\ &= \frac{1}{4}(\|e^{\beta x}u(t,x+2it\beta)\|^2 + 2\|u\|^2 + \|e^{-\beta x}u(t,x-2it\beta)\|^2).\end{aligned}$$

Therefore we have by Lemma 3.1

$$\|e^{\beta x}u(t,x+2it\beta)\|_{3,0} + \|e^{-\beta x}u(t,x-2it\beta)\|_{3,0} \le C\|(\cosh \beta x)u_0\|_{3,0}.$$

Hence the result of Theorem 1.1 follows. □

§ 4. Proof of Theorem 1.2

In the same way as in the proof of (3.8) we have by the Sobolev embedding inequality, Lemma 2.2 and the second estimate of Lemma 2.3

$$\begin{aligned}&\frac{d}{dt}\|\mathscr{S}\mathscr{U}(t)e^{\beta x}\mathscr{U}(-t)v_x\|^2 + (1 - C\delta e^{4\|\varphi\|_\infty})\|\omega\mathscr{S}\sqrt{|\partial_x|}\mathscr{U}(t)e^{\beta x}\mathscr{U}(-t)v_x\|^2 \\ &\quad \le C\|(\cosh \beta x)\mathscr{U}(-t)u\|_{3,0}^2.\end{aligned}$$

So to treat the growing with $\delta \to 0$ coefficient $e^{2\|\varphi\|_\infty}$ we now have to choose $\rho = \delta > 0$ to be a small constant. The rest of the proof is the same as in Theorem 1.1, so we leave it to the reader. □

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References

[1] A. de Bouard, N. Hayashi and K. Kato, Gevrey regularizing effect for the (generalized) Korteweg—de Vries equation and nonlinear Schrödinger equations, Ann. Henri Inst. Poincare. Analyse non linéaire **12** (1995), 673–725.

[2] W. Craig, T. Kappeler and W. Strauss, Gain of regularity for equations of KdV type, Ann. Inst. Henri Poincaré, Analyse non linéaire **9** (1992), 147–186.

[3] W. Craig, T. Kappeler and W. Strauss, Microlocal dispersive smoothing for the Schrödinger equation, Commun. Pure Appl. Math. **48** (1995), 769–860.

[4] S. Doi, On the Cauchy problem for Schrödinger type equations and regularity of solutions, J. Math. Kyoto Univ. **34** (1994), 319–328.

[5] N. Hayashi, K. Nakamitsu and M. Tsutsumi, On solutions of the initial value problem for nonlinear Schrödinger equations in one space dimension, Math. Z. **192** (1986), 637–650.

[6] N. Hayashi P. I. Naumkin and P.-N. Pipolo, Smoothing effects for some derivative nonlinear Schrödinger equations, Discrete and Continuous Dynamical Systems, **5** (1999), 685–695.

[7] N. Hayashi and T. Ozawa, On the derivative nonlinear Schrödinger equations, Physica D **55** (1992), 14–36.

[8] N. Hayashi and S. Saitoh, Analyticity and global existence of small solutions to some nonlinear Schrödinger equations, Commun. Math. Phys. **129** (1990), 27–42.

[9] K. Kajitani, Analytically smoothing effect for Schrödinger equations, Proceeding of the International Conference on Dynamical and Systems and Differential Equations, Springfield, Missouri **1** (1996).

[10] K. Kajitani and S. Wakabayashi, Analytically smoothing effect for Schrödinger type equations with variable coefficients, Proceeding of Symposium of P.D.E. at University of Delaware (1997).

[11] L. Kapitanski and Y. Safarov, Dispersive smoothing effect for Schrödinger equations, Math. Research Letters **3** (1996), 77–91.

[12] T. Kato, On the Cauchy problem for the (generalized) Korteweg-de Vries equation, Stud. Appl. Math. Adv. in Math. Supplementary Studies **18** (1983), 93–128.

[13] K. Kato and K. Taniguchi, Gevrey regularizing effect for nonlinear Schrödinger equations, Osaka J. Math. **33** (1996), 863–880.

[14] V. G. Makhankov and O. K. Pashaev, Integrable pseudospin models in condensed matter, Sov. Sci. Rev. C. Math. Phys. **9** (1992), 1–152.

[15] L. Robbiano and C. Zuily, Microlocal analytic smoothing effect for the Schrödinger equation, Prépublication d'Orsay **98–37** (1998).

[16] M. Yamazaki, On the microlocal smoothing effect of dispersive partial differential equations I; second order linear equations, Algebraic Analysis **11** (1988), 911–926.

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