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A POWER SERIES SATISFYING A CERTAIN FUNCTIONAL EQUATION

By

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1. Introduction

In this paper, we are concerned with an enumeration of rooted trees. We consider isomers of chain saturated mono-hydroxy alcohols, that is to say, having no double, triple bonds and cyclic structure. Since the carbon atom has a valency of four and the hydrogen atom a valency of one, the structural formulas of these isomers form ternary rooted trees. For example, the following figures indicates that two isomers of propyl-alcohols:

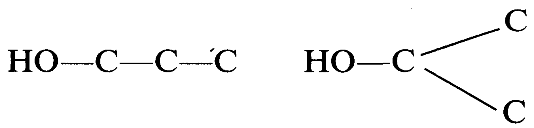

Figure 1. Propyl-alcohol

In this figure, we regard that the remaining valencies of carbon atoms are bonded with hydrogen atoms. Let $C(n)$ be the number of the isomers of the alcohols containing n carbon atoms. We define $C(0)=1$. Clearly $C(n)\geq 1$ $(n\geq 0)$ is nondecreasing. We define a power series $g(z)$ by

$$g(z)=\sum_{n\geq 0}C(n)z^{n}=1+z+z^{2}+2z^{3}+4z^{4}+8z^{5}+17z^{6}+\cdots, \tag{1}$$

which satisfies the functional equation:

$$g(z)=1+\frac{z}{6}(g(z)^{3}+3g(z)g(z^{2})+2g(z^{3})) \tag{2}$$

(cf. Temperley [5], Polya [4]). Regarding the alcohol enumeration problem considered in (2) Polya concluded that the number of isomeric hydrocarbons of

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formula C_nH_{2n+2} is asymptotically equal to

$$An^{-5/2}l^{-n},$$

where $A > 0$ and l are constants with

$$0.35 < l < 0.36$$

(cf. Polya [4]). We can deduce from this result that

$$l = r(g), \tag{3}$$

where $r(h)$ is the convergence radius of power series $h(z)$.

We replace the carbon atoms by boron atoms which have a valency of three. Then these structural formulas form binary rooted trees. In this paper, we study these rooted trees as purely mathematical model and do not care about the real existence of such chemical substances.

Let $B(n)$ be the number of isomers containing n boron atoms. We put $B(0) = 1$, as above. Clearly $B(n) \geq 1$ $(n \geq 0)$ is nondecreasing. We define a power series $f(z)$ by

$$f(z) = \sum_{n\geq 0} B(n)z^n = 1 + z + z^2 + 2z^3 + 3z^4 + 6z^5 + 11z^6 + \cdots. \tag{4}$$

We can see that $f(z)$ satisfies the functional equation (cf. Temperley [5]):

$$f(z) = 1 + \frac{z}{2}(f(z)^2 + f(z^2)). \tag{5}$$

The purpose of this paper is to estimate the convergence radius of $f(z)$ and $g(z)$. Also we prove the transcendency of the function $f(z)$ over $\boldsymbol{C}(z)$ and its values at algebraic points. Our theorem for $g(z)$ improves the result of Polya mentioned above.

THEOREM 1. *Let $f(z)$ be the power series defined by* (4). *Then*

$$0.402696 < r(f) < 0.402699,$$

and hence $B(n) < 0.402696^{-n}$ for all large n and $B(n) > 0.402699^{-n}$ for infinitely many n.

THEOREM 2. *Let $g(z)$ be the power series defined by* (1). *Then*

$$0.355179 < r(g) < 0.355183,$$

and hence $C(n) < 0.355179^{-n}$ for all large n and $C(n) > 0.355183^{-n}$ for infinitely many n.

A power series satisfying the functional equation such as (5) is one of the so-called *Mahler functions*. Then the transcendency of values of $f(\alpha)$ for algebraic numbers α are deduced from the transcendency of $f(z)$ as a function (cf. Ku. Nishioka [2], [3]). We state our results.

THEOREM 3. *The function* $f(z)$ *is transcendental over* $C(z)$.

COROLLARY. *If* α *is an algebraic number with* $0 < |\alpha| < r(f)$, *then the value* $f(\alpha)$ *is transcendental.*

Neither the transcendency of the function $g(z)$ over $\boldsymbol{C}(z)$ nor the transcendency of the value $g(\alpha)$ at an algebraic point has been proved so far.

2. Proof of Theorem 1

Assume that $u(z)$ and $v(z)$ are majorant and minorant power series of $f(z)$, respectively. Then we have

$$r(u) \le r(f) \le r(v). \tag{6}$$

Our idea of the proof is to choose majorant and minorant power series of $f(z)$ among algebraic functions.

By the functional equation (5), we get the following recursive formulas:

$$B(n) = \frac{1}{2}\sum_{h=0}^{n-1} B(h)B(n-1-h) + \begin{cases} B((n-1)/2)/2 & (n\text{: odd}), \\ 0 & (n\text{: even}), \end{cases} \quad (n \ge 1). \tag{7}$$

Let $a > 0$ be a parameter and $\Phi_a(z)$ be an algebraic function defined by

$$\Phi_a(z) = 1 + \frac{z}{2}\left(1 + z^2 + z^4 + 2z^6 + \frac{3z^8}{1-az^2} + \Phi_a(z)^2\right), \quad \Phi_a(0) = 1,$$

so that we have

$$\Phi_a(z) = \frac{1 - \sqrt{1 - z(2 + z + z^3 + z^5 + 2z^7 + 3z^9/(1-az^2))}}{z}. \tag{8}$$

We denote the Taylor expansion of $\Phi_a(z)$ by

$$\Phi_a(z) = \sum_{n \ge 0} b_a(n) z^n.$$

Then we get the following recursive formulas:

$$b_a(n) = \frac{1}{2}\sum_{h=0}^{n-1} b_a(h)b_a(n-1-h) + \begin{cases} B((n-1)/2)/2 & (n = 1,3,5,7,9), \\ 3a^{(n-1)/2-4}/2 & (n-1: \text{ even, } n \geq 11), \\ 0 & (n-1: \text{ odd}), \end{cases} \quad (n \geq 1). \tag{9}$$

We note that $b_a(n) > 0$ $(n \geq 0)$. We have the following lemmas.

LEMMA 1. *$\Phi_2(z)$ is a majorant series of $f(z)$.*

LEMMA 2. *$\Phi_{1.94}(z)$ is a minorant series of $f(z)$.*

We deduce Theorem 1 from Lemma 1 and 2. By Lemma 1 and 2, we see that

$$r(\Phi_2) \leq r(f) \leq r(\Phi_{1.94}). \tag{10}$$

First we calculate the convergence radius of $\Phi_2(z)$. The roots of the following equation can be the singular points of $\Phi_2(z)$ defined by (8) with $a = 2$:

$$(1-2z^2)^2\left(1 - z\left(2 + z + z^3 + z^5 + 2z^7 + \frac{3z^9}{1-2z^2}\right)\right) = 0. \tag{11}$$

Let ζ_1 be a root of (11) having the minimum absolute value. Using a calculator, it is proved that ζ_1 is a real single root and $\zeta_1 = 0.402696\cdots$. Then $z = \zeta_1$ is a branch point of $\Phi_2(z)$ and

$$r(\Phi_2) = \zeta_1.$$

Therefore we have by (10)

$$0.402696 < r(\Phi_2) \leq r(f).$$

Next we calculate the convergence radius of $\Phi_{1.94}(z)$. We see that the roots of the following equation can be the singular points of $\Phi_{1.94}(z)$:

$$(1-1.94z^2)^2\left(1 - z\left(2 + z + z^3 + z^5 + 2z^7 + \frac{3z^9}{1-1.94z^2}\right)\right) = 0. \tag{12}$$

Let ζ_2 be a root of (12) having the minimum absolute value. Using a calculator, we see that ζ_2 is a real single root and $\zeta_2 = 0.402698\cdots$. Then $z = \zeta_2$ is a branch point of $\Phi_{1.94}(z)$ and

$$r(\Phi_{1.94}) = \zeta_2.$$

Therefore we have by (10)

$$r(f) \leq r(\Phi_{1.94}) < 0.402699.$$

Hence Theorem 1 follows.

It remains to prove Lemma 1 and 2.

PROOF OF LEMMA 1. We prove the following inequality by induction:

$$b_2(n) \geq B(n) \quad (n \geq 0).$$

This holds for $0 \leq n \leq 26$ by Table 1. Assume that $n \geq 26$ and

$$b_2(h) \geq B(h) \quad (0 \leq h \leq n).$$

Then we have by (7) and (9)

$$b_2(n+1) - B(n+1) = \sum_{h=0}^{n}(b_2(h)b_2(n-h) - B(h)B(n-h)) + \begin{cases} (3 \cdot 2^{n/2-4} - B(n/2))/2 & (n\text{: even}), \\ 0 & (n\text{: odd}). \end{cases}$$

Putting $\delta(h) = b_2(h) - B(h) \geq 0$ $(0 \leq h \leq n)$, we get

$$\frac{1}{2}\sum_{h=0}^{n}(b_2(h)b_2(n-h) - B(h)B(n-h)) \geq \sum_{h=0}^{n} B(h)\delta(n-h) \geq B([n/2]) \sum_{h=0}^{[(n+1)/2]} \delta(h),$$

and so

$$b_2(n+1) - B(n+1) \geq B([n/2]) \sum_{h=0}^{[(n+1)/2]} \delta(h) + \begin{cases} (3 \cdot 2^{n/2-4} - B(n/2))/2 & (n\text{: even}), \\ 0 & (n\text{: odd}). \end{cases}$$

Thus, if n is odd, $b_2(n+1) \geq B(n+1)$. If n is even, we have

$$b_2(n+1) - B(n+1) \geq B(m) \sum_{h=0}^{m} \delta(h) + 3 \cdot 2^{m-5} - \frac{1}{2} B(m),$$

where $n = 2m$. Since $m \geq 13$ and $\delta(13) = 0.5$, we obtain $b_2(n+1) \geq B(n+1)$. This completes the proof.

PROOF OF LEMMA 2. First we explain why the value $a = 1.94$ is chosen for $\Phi_a(z)$ to be a minorant series of $f(z)$. By (7) and (9), we have

$$B(n) = b_a(n) \quad (1 \leq n \leq 10)$$

and

$$B(11) - b_a(11) = \frac{1}{2} B(5) - \frac{3}{2} a = 3\left(1 - \frac{1}{2} a\right).$$

Putting

$$a = 2 - \frac{2}{3} \varepsilon \quad (\varepsilon > 0),$$

we get

$$B(11) - b_a(11) = 3\left(1 - 1 + \frac{1}{3} \varepsilon\right) = \varepsilon.$$

Hence we obtain

$$\begin{aligned} B(12) - b_a(12) &= \frac{1}{2} \sum_{h=0}^{11} (B(h)B(11-h) - b_a(h) b_a(11-h)) \\ &= \frac{1}{2} \sum_{h=0}^{11} B(h)B(11-h) - \frac{1}{2}\left(2b_a(0)b_a(11) + \sum_{h=1}^{10} b_a(h) b_a(11-h)\right) \\ &= \frac{1}{2} \sum_{h=0}^{11} B(h)B(11-h) - \frac{1}{2}\left(2B(0)(B(11)-\varepsilon) + \sum_{h=1}^{10} B(h)B(11-h)\right) \\ &= \frac{1}{2} \sum_{h=0}^{11} B(h)B(11-h) - \frac{1}{2} \sum_{h=0}^{11} B(h)B(11-h) + \varepsilon = \varepsilon, \end{aligned}$$

so that we have

$$b_a(13) = \frac{3}{2}a^2 + \frac{1}{2}\sum_{h=0}^{12} b_a(h)b_a(12-h)$$

$$= \frac{3}{2}a^2 + \frac{1}{2}\left(2b_a(0)b_a(12) + 2b_a(1)b_a(11) + \sum_{h=2}^{10} b_a(h)b_a(12-h)\right)$$

$$= \frac{3}{2}a^2 + \frac{1}{2}\left(2(B(12)-\varepsilon) + 2(B(11)-\varepsilon) + \sum_{h=2}^{10} B(h)B(12-h)\right)$$

$$= \frac{3}{2}a^2 - 2\varepsilon + \frac{1}{2}\sum_{h=0}^{12} B(h)B(12-h).$$

Thus we get

$$B(13) - b_a(13) = \frac{1}{2}B(6) - \frac{3}{2}a^2 + 2\varepsilon$$

$$= \frac{11}{2} - \frac{3}{2}\left(2 - \frac{2}{3}\varepsilon\right)^2 + 2\varepsilon = -\frac{2}{3}\varepsilon^2 + 6\varepsilon - \frac{1}{2},$$

and so $B(13) \geq b_a(13)$ if and only if

$$-\frac{2}{3}\varepsilon^2 + 6\varepsilon - \frac{1}{2} \geq 0,$$

which implies

$$\frac{9-\sqrt{78}}{2} \leq \varepsilon \leq \frac{9+\sqrt{78}}{2}.$$

Since

$$a = 2 - \frac{2}{3}\varepsilon \leq 2 - \frac{9-\sqrt{78}}{3} = \frac{\sqrt{78}-3}{3} = 1.9439\cdots,$$

we may choose $a = 1.94$.

Now we prove the inequality

$$B(n) \geq 3 \cdot 1.94^{n-4} \quad (n \geq 0), \tag{13}$$

by induction on n. We see by Table 1 that (13) holds for $0 \leq n \leq 7$. Assume that $n \geq 7$ and

$$B(h) \geq 3 \cdot 1.94^{h-4} \quad (0 \leq h \leq n).$$

Then we have by (7)

$$B(n+1) \geq \frac{1}{2}\sum_{h=0}^{n} B(h)B(n-h)$$

$$\geq \frac{1}{2}(2B(0)B(n) + 2B(1)B(n-1) + 2B(2)B(n-2) + 2B(3)B(n-3))$$

$$= B(n) + B(n-1) + B(n-2) + 2B(n-3),$$

and so

$$B(n+1) \geq 3 \cdot 1.94^{n-4} + 3 \cdot 1.94^{n-5} + 3 \cdot 1.94^{n-6} + 6 \cdot 1.94^{n-7}$$

$$\geq 3 \cdot 1.94^{n-3}.$$

Finally we prove

$$B(n) \geq b_{1.94}(n) \quad (n \geq 0), \tag{14}$$

by induction on n. We see by Table 1 that (14) holds for $0 \leq n \leq 10$. Assume that $n \geq 10$ and

$$B(h) \geq b_{1.94}(h) \quad (0 \leq h \leq n).$$

By (7) and (9), we have

$$B(n+1) - b_{1.94}(n+1) = \sum_{h=0}^{n}(B(h)B(n-h) - b_{1.94}(h)b_{1.94}(n-h)) + \begin{cases} (B(n/2) - 3 \cdot 1.94^{n/2-4})/2 & (n\text{: even}), \\ 0 & (n\text{: odd}). \end{cases}$$

This and (13) imply $B(n+1) \geq b_{1.94}(n+1)$, and the proof is completed.

3. Proof of Theorem 2

As in the proof of the proceeding theorem, we shall choose a suitable majorant and minorant of $g(z)$ among algebraic functions.

By the functional equation (2), we get the following recursive formulas:

$$C(n) = \frac{1}{6}\sum_{h=0}^{n-1}\sum_{i=0}^{n-1-h} C(h)C(i)C(n-1-h-i) + \frac{1}{2}\sum_{h=0}^{[(n-1)/2]} C(h)C(n-1-2h) + \begin{cases} C((n-1)/3)/3 & (3 \text{ divides } n-1), \\ 0 & (3 \text{ does not divide } n-1), \end{cases} \quad (n \geq 1). \tag{15}$$

Table 1: $b_2(n)$, $B(n)$, $b_{1.94}(n)$, $\delta(n)$ $(1 \le n \le 26)$.

n	$b_2(n)$	$B(n)$	$b_{1.94}(n)$	$\delta(n)$
1	1	1	1	0
2	1	1	1	0
3	2	2	2	0
4	3	3	3	0
5	6	6	6	0
6	11	11	11	0
7	23	23	23	0
8	46	46	46	0
9	98	98	98	0
10	207	207	207	0
11	451	451	450.···	0
12	983	983	982.···	0
13	2179.5	2179	2178.···	0.5
14	4850.5	4850	4849.···	0.5
15	10906.5	10905	10903.···	1.5
16	24633.5	24631	24629.···	2.5
17	56017.5	56011	56005.···	6.5
18	127925	127912	127901.···	13
19	293575.5	293547	293515.···	28.5
20	676223	676157	676094.···	66
21	1563518.5	1563372	1563204.···	146.5
22	3626501.5	3626149	3625792.···	352.5
23	8437179.5	8436379	8435475.···	800.5
24	19682222.5	19680277	19678254.···	1945.5
25	46031154.5	46026618	46021651.···	4536.5
26	107901616	107890609	107879110.···	11007

For $a > 0$ and any positive integer l, we put

$$h_a(z) = C(0) + C(1)z + \frac{C(2)z^2}{1 - az} = 1 + z + \frac{z^2}{1 - az},$$

$$k_a(z) = C(0) + C(1)z + C(2)z^2 + \frac{C(3)z^3}{1 - az} = 1 + z + z^2 + \frac{2z^3}{1 - az}.$$

Let $\Psi_a(z)$ be an algebraic function defined by $\Psi(0) = 1$ and

$$\Psi_a(z) = 1 + \frac{z}{6}\Psi_a(z)^3 + \frac{z}{2}h_a(z^2)\Psi_a(z) + \frac{z}{3}k_a(z^3). \tag{16}$$

We put

$$\Psi_a(z) = \sum_{n \ge 0} d_a(n) z^n.$$

Then it follows from (16) that

$$\begin{aligned} d_a(n) = &\frac{1}{6}\sum_{h=0}^{n-1}\sum_{i=0}^{n-1-h} d_a(h)d_a(i)d_a(n-1-h-i) \\ &+ \frac{1}{2}\sum_{h=0}^{2} d_a(h)d_a(n-1-2h) + \frac{1}{2}\sum_{h=3}^{[(n-1)/2]} a^{h-2}d_a(n-1-2h) \\ &+ \begin{cases} C((n-1)/3)/3 & (n = 1, 4, 7), \\ 2 \cdot a^{(n-1)/3-3}/3 & (3 \text{ divides } n,\ n-1 \ge 10), \\ 0 & (3 \text{ does not divide } n-1), \end{cases} \quad (n \ge 1). \end{aligned} \tag{17}$$

We have the following lemmas.

LEMMA 3. *$\Psi_{2.01}(z)$ is a majorant series of $g(z)$.*

LEMMA 4. *$\Psi_2(z)$ is a minorant series of $g(z)$.*

We deduce Theorem 2 from Lemma 3 and 4. By Lemma 3 and 4, we have

$$r(\Psi_{2.01}) \le r(g) \le r(\Psi_2).$$

Let $D_a(z)$ denote the discriminant of equation

$$X = 1 + \frac{z}{6}X^3 + \frac{z}{2}h_a(z^2)X + \frac{z}{3}k_a(z^3)$$

corresponding to (16). Then we have

$$D_a(z) = (zh_a(z^2) - 2)^3 + z(zk_a(z^3) + 3)^2.$$

We note that

$$r(\Psi_a) \le \zeta(a),$$

where $\zeta(a)$ is a root of $D_a(z) = 0$ having the minimal absolute value. If $\zeta(a)$ is a single root,

$$r(\Psi_a) = \zeta(a).$$

Using a calculator, we see that

$$\zeta(2) = 0.355182\ldots, \quad \zeta(2.01) = 0.355179\ldots,$$

and that $\zeta(2)$ and $\zeta(2.01)$ are single roots. Hence

$$0.355179 < r(g) < 0.355183.$$

Proof of Lemma 3. First we show that

$$d_{2.01}(n) \ge C(n) \quad (n \ge 0) \tag{18}$$

by induction on n. This holds for $0 \le n \le 24$ by Table 2. Suppose that $n \ge 24$ and

$$d_{2.01}(h) \ge C(h) \quad (0 \le h \le n).$$

We have to show that $d_{2.01}(n+1) \ge C(n+1)$. Put $d(n) = d_{2.01}(n)$ for brevity and

$$\delta(h) = d(h) - C(h) \ge 0 \quad (0 \le h \le n).$$

Since

$$\sum_{h=0}^{n}\sum_{i=0}^{n-h}(d(h)d(i)d(n-h-i) - C(h)C(i)C(n-h-i))$$

$$\ge 3\sum_{h=0}^{n}\sum_{i=0}^{n-h} C(h)\delta(i)C(n-h-i) \ge 3\sum_{h=0}^{[n/2]}\sum_{k=0}^{2h} C(k)\delta(2h-k)C(n-2h),$$

we get

$$d(n+1) - C(n+1) \geq \frac{1}{2}\sum_{h=0}^{[n/2]}\sum_{k=0}^{2h} C(k)\delta(2h-k)C(n-2h)$$

$$+\frac{1}{2}\sum_{h=0}^{2}(d(h)d(n-2h) - C(h)C(n-2h))$$

$$+\frac{1}{2}\sum_{h=3}^{[n/2]}(2.01^{h-2}d(n-2h) - C(h)C(n-2h))$$

$$+\begin{cases} (2\cdot 2.01^{(n/3)-3} - C(n/3))/3 & (3 \text{ divides } n), \\ 0 & (3 \text{ does not divide } n), \end{cases}$$

using (15) and (17). Noting that $d(h)d(n-2h) \geq C(h)C(n-2h)$ if $h = 0, 1, 2$ and $2.01^{h-2} > C(h)$ if $h = 3, 4, 5$, we have

$$d(n+1) - C(n+1) \geq \frac{1}{2}\sum_{h=0}^{5}\sum_{k=0}^{2h} C(k)\delta(2h-k)C(n-2h)$$

$$+\frac{1}{2}\sum_{h=6}^{[n/2]}\left(\sum_{k=0}^{2h} C(k)\delta(2h-k) - C(h) + 2.01^{h-2}\right)C(n-2h)$$

$$+\begin{cases} (2\cdot 2.01^{(n/3)-3} - C(n/3))/3 & (3 \text{ divides } n), \\ 0 & (3 \text{ does not divide } n). \end{cases}$$

Using a calculator and Table 2, we see that

$$\sum_{k=0}^{2h} C(k)\delta(2h-k) - C(h) + 2.01^{h-2} > 0 \quad (h = 6, 7),$$

$$\sum_{k=0}^{2h} C(k)\delta(2h-k) - C(h) + 2.01^{h-2} \geq 1 \quad (8 \leq h \leq 11),$$

and

$$\sum_{k=0}^{2h} C(k)\delta(2h-k) \geq \sum_{k=h}^{2h} C(k)\delta(2h-k) \geq C(h)\sum_{k=0}^{12}\delta(k) > C(h) \quad (h \geq 12).$$

Hence we get

$$\left(\sum_{k=0}^{2h} C(k)\delta(2h-k) - C(h) + 2.01^{h-2}\right)C(n-2h) \geq C(n-2h) \quad (n \geq 8),$$

and so

$$d(n+1) - C(n+1) \geq \frac{1}{2}\sum_{h=8}^{[n/2]} C(n-2h)$$

$$+ \begin{cases} (2 \cdot 2.01^{n/3-3} - C(n/3))/3 & (3 \text{ divides } n), \\ 0 & (3 \text{ does not divide } n). \end{cases}$$

If n is not divided by 3, then $d(n+1) \geq C(n+1)$. Assume that n is divided by 3 and write $n = 3m$. Noting that $m \geq 8$, we have

$$d(n+1) - C(n+1) \geq \frac{1}{2}C(m) + 2.01^{m-3} - \frac{1}{3}C(m) > 0,$$

and (18) is proved.

PROOF OF LEMMA 4. First we prove

$$C(n) \geq 2^{n-2} \quad (n \geq 0) \tag{19}$$

by induction on n. Since $\{C(n)\}_{n \geq 0} = \{1, 1, 1, 2, 4, 8, \ldots\}$, (19) holds for $0 \leq n \leq 5$. Suppose that $n \geq 5$ and

$$C(h) \geq 2^{h-2} \quad (0 \leq h \leq n).$$

We show that $C(n+1) \geq 2^{n-1}$. By (15), we have

$$C(n+1) \geq \frac{1}{6}\sum_{h=0}^{n}\sum_{i=0}^{n-h} C(h)C(i)C(n-h-i) + \frac{1}{2}\sum_{h=0}^{[n/2]} C(h)C(n-2h).$$

Noting that $n \geq 5$, we get

$$C(n+1) \geq \frac{1}{6}(3C(0)^2C(n) + 6C(0)C(1)C(n-1)$$

$$+ 6C(0)C(2)C(n-2) + 3C(1)^2C(n-2))$$

$$+ \frac{1}{2}(C(0)C(n) + C(1)C(n-2)),$$

and so

$$C(n+1) \geq C(n) + C(n-1) + 2C(n-2)$$

$$\geq 2^{n-2} + 2^{n-3} + 2 \cdot 2^{n-4} = 2^{n-1}.$$

We have to prove the inequality

$$C(n) \geq d_2(n) \quad (n \geq 0). \tag{20}$$

This holds for $0 \leq n \leq 7$ by Table 2. Assume that $n \geq 7$ and

$$C(h) \geq d_2(h) \quad (0 \leq h \leq n). \tag{21}$$

It follows from (15) and (17) that

$$\begin{aligned} &C(n+1) - d_2(n+1) \\ &\quad \geq \frac{1}{6}\sum_{h=0}^{n}\sum_{i=0}^{n-h}(C(h)C(i)C(n-h-i) - d_2(h)d_2(i)d_2(n-h-i)) \\ &\quad + \frac{1}{2}\sum_{h=0}^{2}(C(h)C(n-2h) - d_2(h)d_2(n-2h)) \\ &\quad + \frac{1}{2}\sum_{h=3}^{[n/2]}(C(h)C(n-2h) - 2^{h-2}d_2(n-2h)) \\ &\quad + \begin{cases} (C(n/3) - 2\cdot 2^{(n/3)-3})/3 & (3 \text{ divides } n), \\ 0 & (3 \text{ does not divide } n). \end{cases} \end{aligned}$$

Hence we have using (19) and (21),

$$C(n+1) \geq d_2(n+1)$$

and the inequality (20) is proved.

4. Proof of Theorem 3

We use the following lemma.

LEMMA 5 (Ke. Nishioka [1], cf. Ku. Nishioka [3]). *Suppose that* $h(z) \in \boldsymbol{C}[[z]]$ *satisfies the following functional equation for an integer* $q > 1$:

$$h(z) = \varphi(z, h(z^q)),$$

where $\varphi(z,u)$ *is a rational function in* z, u *over* $\boldsymbol{C}$. *If* $h(z)$ *is algebraic over* $\boldsymbol{C}(z)$, *then* $h(z) \in \boldsymbol{C}(z)$.

PROOF OF THEOREM 3. Assume that $f(z)$ is algebraic over $\boldsymbol{C}(z)$. Then by Lemma 5, we may put

Table 2: $d_{2.01}(n)$, $C(n)$, $d_2(n)$, $\delta(n)$ $(1 \le n \le 24)$.

n	$d_{2.01}(n)$	$C(n)$	$d_2(n)$	$\delta(n)$
1	1	1	1	0
2	1	1	1	0
3	2	2	2	0
4	4	4	4	0
5	8	8	8	0
6	17	17	17	0
7	39.005	39	39	0.005
8	89.01	89	89	0.01
9	211.040···	211	211	0.040···
10	507.090···	507	507	0.090···
11	1238.265···	1238	1238	0.265···
12	3057.641···	3057	3057	0.641
13	7640.240···	7639	7638.5	1.240···
14	19244.···	19241	19240	3.···
15	48871.···	48865	48859.5	6.···
16	124924.···	124906	124894	18.···
17	321240.···	321198	321159	42.···
18	830343.···	830219	830127	124.···
19	2156328.···	2156010	2155747.···	318.···
20	5623020.···	5622109	5621458.···	911.···
21	14718271.···	14715813	14714048.···	4223.···
22	38656055.···	38649152	38644609.···	6903.···
23	101840877.···	101821927	101809755.···	18950.···
24	269063227.···	269010485	268978422.···	52742.···

$$f(z) = \frac{a(z)}{b(z)}, \quad a(z), b(z) \in \boldsymbol{C}[z], \quad (a(z), b(z)) = 1.$$

Then we have by the functional equation (5)

$$2a(z)b(z)b(z^2) = 2a(z)^2 b(z^2)b(z)^2 + za(z)^2 b(z^2) + za(z^2)b(z)^2. \tag{22}$$

Suppose that $b(z)$ is not a constant. Since $f(z) \in \boldsymbol{C}[\![z]\!]$ and $(a(z), b(z)) = 1$, we have $b(0) \neq 0$. Let ξ be a root of $b(z)$ having the minimum argument $\arg \xi \in (0, 2\pi]$ and let ξ_1 be one of $\sqrt{\xi}$ with $\arg \xi_1 = (\arg \xi)/2$. Noting that $\arg \xi > 0$, we get $\arg \xi_1 < \arg \xi$, and so $b(\xi_1) \neq 0$. Substituting $z = \xi_1$ in (22), we have

$$\xi_1 a(\xi) b(\xi_1)^2 = 0. \tag{23}$$

Since $(a(z), b(z)) = 1$ and $b(\xi) = 0$, we get $a(\xi) \neq 0$ and so by (23), $b(\xi_1) = 0$, which contradicts $b(\xi_1) \neq 0$. Therefore, $b(z)$ is a constant and hence $f(z)$ is a polynomial. This contradicts the fact that $B(n) \geq 1$ for all $n \geq 0$.

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