TSUKUBA J. MATH.
Vol. 23 No. 2 (1999), 383-416

KLEIN'S SURFACE OF GENUS THREE AND ASSOCIATED THETA CONSTANTS

By

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Introduction

Klein's surface of genus three is the closed Riemann surface R defined by the equation

$$y^7 = x(x-1)^2.$$

This surface is famous because the conformal automorphism group $Aut(R)$ has order $168 = 84(3-1)$, the maximum possible.

In 1970, Rauch and Lewittes wrote the beautiful paper [4] in which they found the period matrix of Klein's surface. It is difficult to determine the period matrix of a non-hyperelliptic Riemann surface of genus $g \geq 3$ explicitly.

Klein originally obtained his surface R in the form of the upper half-plane identified under the principal congruence subgroup of level seven, $\Gamma(7)$, of the modular group Γ. Rauch and Lewittes represented the surface as the unit circle uniformization of R. They found a canonical homology basis expertly and they had the matrix representations of the generators of the automorphism group $Aut(R)$. In order to find the period matrix with respect to this canonical homology basis they used these matrix representations.

The aim of our article is to decide the proportionalities of associated theta constants of Klein's surface.

In our article we represent R as a covering surface over P^1. Let T be a conformal automorphim of order 7 and let $\langle T \rangle$ be the cyclic group generated by T and $R/\langle T \rangle$ the surface obtained by identifying the equivalent points on R under $\langle T \rangle$. Then $R/\langle T \rangle$ becomes P^1 conformally and so R is considered as a 7-sheet covering surface of $R/\langle T \rangle \cong P^1$.

In the first section we calculate the period matrix of R by the different method from that of Rauch and Lewittes. First we choose a basis of the space

Received July 15, 1998.
Revised December 3, 1998.

$H^1(R)$ of abelian differentials and we construct a canonical basis of the first homology group $H_1(R) = H_1(R, Z)$ concretely. Second we consider the action of T on $H_1(R)$ and we obtain the representation of T on $H_1(R)$. This representation becomes a symplex modular matrix. Using these results, we compute the period matrix of R. Our matrix is different from that which was found by Rauch and Lewittes.

In the second section we start a brief review of the theta function and give some important lemmas. Particularly the transformation formula of the theta functions plays important role in the third section. Next we derive some identities which are necessary to determine the proportions between theta constants.

In the third section we decide the proportionalities of the theta constants. First we show that 36 theta constants are classified in 6 orbits by the action of the representation of T in the second section. Next using the theta identities, we determine the proportionalities of the theta constants. Lastly we have to determine the roots of unity in the proportionalities. This can only be decided by reference to the Fourier expansions of theta constants and we use numerical computations.

The proportionalities that we obtain means the homogeneous coordinate of the projective image of the Jacobian variety associated to Klein's surface by means of theta constants. This example will be important to study moduli problem.

1. Period Matrix

We consider Klein's surface R of genus three defined by the equation

$$y^7 = x(x-1)^2$$

This surface has an automorphism $T \in Aut(R)$ of order 7 defined by

$$T : (x, y) \mapsto (x, \rho y) \tag{1}$$

where $\rho = \exp(2\pi i/7)$.

Then $R/\langle T \rangle$ is conformal equivalent to P^1 and R is considered as a 7-sheets covering surface of $R/\langle T \rangle \cong P^1$ with three branch points over $0, 1, \infty$.

Since we can easily examine the divisors of differentials, we obtain a basis of the vector space of holomorphic differentials given by

$$\omega_1 = \frac{dx}{y^3}, \quad \omega_2 = \frac{(x-1)\,dx}{y^5}, \quad \omega_3 = \frac{(x-1)\,dx}{y^6}. \tag{2}$$

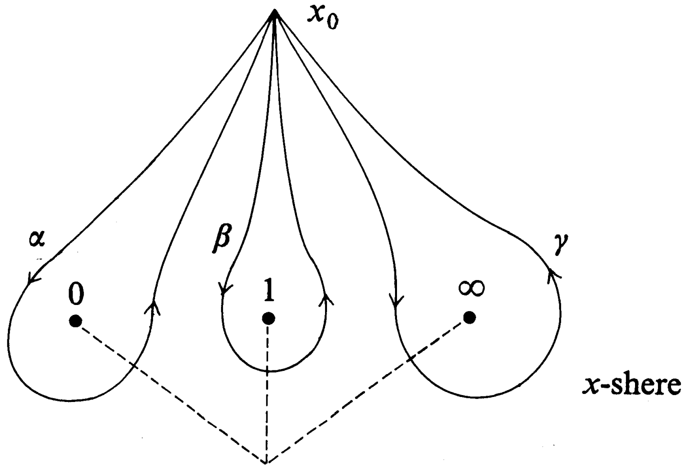


Figure 1

Then we define

$$T(\omega_j) = \omega_j \circ T^{-1},$$

so we have

$$T(\omega_1) = \rho^3\omega_1, \quad T(\omega_2) = \rho^5\omega_2, \quad T(\omega_3) = \rho^6\omega_3. \tag{3}$$

Here we construct a canonical homology basis on R. We consider the following three closed curve α, β, γ and three broken segments on x-sphere as illustrated in Fig. 1. Klein's surface R is a seven sheet covering of the x-shere. Its layers join along the twenty-one segments each of which lies over one of three broken segments.

The function y has seven branches $y_1, y_j = \rho^{j-1}y_1$ $(j = 2, \dots, 7)$. Let α_j be the lifting of α which is a path from y_j-branch to y_{j+1}-branch $(j = 1, \dots, 6)$ and α_7 the lifting of α from y_7-branch to y_1-branch. Let β_j be the lifting of β which is a path from y_j-branch to y_{j+2}-branch $(j = 1, \dots, 5)$ and β_k the lifting of β from y_k-branch to y_{k-5}-branch $(k = 6, 7)$. Let γ_j be the lifting of γ which is a path from y_j-branch to y_{j+4}-branch $(j = 1, 2, 3)$ and γ_k the lifting of γ from y_k-branch to y_{k-3}-branch $(k = 4, 5, 6, 7)$.

Then we get the following relations.

$$(4) \quad \begin{cases} \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7 \sim 0, \quad \beta_1+\beta_3+\beta_5+\beta_7+\beta_2+\beta_4+\beta_6 \sim 0 \\ \gamma_1+\gamma_5+\gamma_2+\gamma_6+\gamma_3+\gamma_7+\gamma_4 \sim 0, \\ \qquad \alpha_1+\beta_2+\gamma_4 \sim 0, \quad \alpha_2+\beta_3+\gamma_5 \sim 0, \quad \alpha_3+\beta_4+\gamma_6 \sim 0, \\ \qquad \alpha_4+\beta_5+\gamma_7 \sim 0, \quad \alpha_5+\beta_6+\gamma_1 \sim 0, \quad \alpha_6+\beta_7+\gamma_2 \sim 0, \\ \qquad \alpha_7+\beta_1+\gamma_3 \sim 0. \end{cases}$$

where $\alpha+\beta$ denotes the composition of α and β by joining the final point of α and the initial point of β, and $\sim$ means homotopic equivalence.

Moreover, we have

$$T(\alpha_j)=\alpha_k,\quad T(\beta_j)=\beta_k,\quad T(\gamma_j)=\gamma_k\ \ (k\equiv j+1(mod\,7)).$$

Then from (4) we have a canonical homology basis represented as follows:

$$(5)\quad \left\{\begin{array}{lll} A_1=\alpha_1+\alpha_2-\beta_1, & A_2=-\alpha_1+\gamma_1+\gamma_5, & A_3=\alpha_3-\beta_2-\beta_7-\gamma_3 \\ B_1=-\beta_3+\alpha_3+\alpha_4, & B_2=-\alpha_5+\gamma_5-\gamma_2, & B_3=-\gamma_7-\gamma_3-\gamma_6-\beta_4. \end{array}\right\}$$

These basis are obtained by drawing closed curves on 7 sheets carefully. See Appendix.

Now we consider the action of T on $H_1(R)=H_1(R,Z)$, the (first) homology group (with integral coefficients).

In general, let $\{A_1,A_2,A_3,B_1,B_2,B_3\}$ be a canonocal homology basis of R, then $\{T(A_1),T(A_2),T(A_3),T(B_1),T(B_2),T(B_3)\}$ become again a canonocal homology basis of R. We immediately see that there exists a 6×6 matrix M with integer entries whose inverse is also an integer matrix (thus $det\,M=1$). We now write $M=\begin{pmatrix} D & C \\ B & A\end{pmatrix}$ in 3×3 blocks, then we have

$$(6)\qquad \begin{pmatrix} T(A_1)\\ T(A_2)\\ T(A_3)\\ T(B_1)\\ T(B_2)\\ T(B_3)\end{pmatrix}=M\begin{pmatrix} A_1\\ A_2\\ A_3\\ B_1\\ B_2\\ B_3\end{pmatrix},\quad \begin{array}{l} {}^tAD-{}^tCB=I_3\\ {}^tAC={}^tCA\\ {}^tBD={}^tDB\end{array},$$

that is, $M\in Sp(3,Z)$, the homogeneous symplectic modular group.

In our case, we take $\{A_1,A_2,A_3,B_1,B_2,B_3\}$ in (5) as a canonical homology basis. Then, using (4), we get

$$(7)\quad \left\{\begin{array}{l} T(A_1)\approx A_1+A_3+B_1-B_2,\quad T(A_2)\approx -A_1-A_2-A_3-B_1+B_2+B_3,\\ T(A_3)\approx -A_3+B_2,\quad T(B_1)\approx -A_1-A_2-A_3+B_3,\\ T(B_2)\approx -A_2-A_3+B_3,\quad T(B_3)\approx -A_1-A_2-A_3+B_2, \end{array}\right.$$

where $\approx$ means homological equivalence. That is,

$$(8)\quad A=\begin{pmatrix}0&0&1\\0&0&1\\0&1&0\end{pmatrix},\quad B=\begin{pmatrix}-1&-1&-1\\0&-1&-1\\-1&-1&-1\end{pmatrix},$$
$$C=\begin{pmatrix}1&-1&0\\-1&1&1\\0&1&0\end{pmatrix},\quad D=\begin{pmatrix}1&0&1\\-1&-1&-1\\0&0&-1\end{pmatrix}.$$

We show the first equation in (7) as follows;

$A_1+A_3+B_1-B_2\approx\alpha_1+\alpha_2-\beta_1+\alpha_3-\beta_2-\beta_7-\gamma_3+\alpha_3+\alpha_4-\beta_3+\alpha_5-$ $\gamma_2-\gamma_5\approx\alpha_1+\alpha_2+2\alpha_3+\alpha_4+\alpha_5-\gamma_2-\gamma_3-\gamma_5-\beta_1-\beta_2-\beta_3-\beta_7\approx$ $\alpha_3-(\alpha_6+\beta_7+\gamma_2)-(\alpha_7+\beta_1+\gamma_3)-(\beta_3+\gamma_5)-\beta_2\approx\alpha_2+\alpha_3-\beta_2\approx T(A_1)$.

Similarly we can show other equations.

To compute the period matrix of Klein's surface, we will represent A_2, A_3, B_1, B_2, B_3 by A_1 and $T^j(A_1)$ $(j=1,\dots,6)$.

We have

$$(9)\quad \begin{cases} A_2\approx T^2(A_1)+T^5(A_1), & A_3\approx -A_1+T(A_1)+T^6(A_1)\\ B_1\approx +T^2(A_1), \quad B_2\approx +T^2(A_1)+T^6(A_1), & B_3\approx -A_1-T^4(A_1)\end{cases}$$

Considering the relation $A_1+T(A_1)+\cdots+T^6(A_1)\approx 0$, we know that $\{T(A_1),\dots,T^6(A_1)\}$ form a basis of the first homology group.

We put

$$\int_{A_j}\omega_i=a_{ij},\quad \int_{B_j}\omega_i=b_{ij}.$$

We note

$$(10)\quad \begin{cases} T(\omega_1)=\rho^3\omega_1, & T(\omega_2)=\rho^5\omega, & T(\omega_3)=\rho^6\omega_3,\\ T^2(\omega_1)=\rho^6\omega_1, & T^2(\omega_2)=\rho^3\omega, & T^2(\omega_3)=\rho^5\omega_3,\\ T^3(\omega_1)=\rho^2\omega_1, & T^3(\omega_2)=\rho\omega_2, & T^3(\omega_3)=\rho^4\omega_3,\\ T^4(\omega_1)=\rho^5\omega_1, & T^4(\omega_2)=\rho^6\omega_2, & T^4(\omega_3)=\rho^3\omega_3,\\ T^5(\omega_1)=\rho\omega_1, & T^5(\omega_2)=\rho^4\omega_2, & T^5(\omega_3)=\rho^2\omega_3,\\ T^6(\omega_1)=\rho^4\omega_1, & T^6(\omega_2)=\rho^2\omega_2, & T^6(\omega_3)=\rho\omega_3.\end{cases}$$

Using (9) and (10), we carry out integrations and we obtain the following equations.

$$
(11)\quad \begin{cases}
a_{12} = \int_{A_2} \omega_1 = \int_{T^2(A_1)} \omega_1 + \int_{T^5(A_1)} \omega_1 = (\rho + \rho^6) a_{11}, \\
a_{13} = \int_{A_3} \omega_1 = -\int_{A_1} \omega_1 + \int_{T(A_1)} \omega_1 + \int_{T^6(A_1)} \omega_1 = (-1 + \rho^3 + \rho^4) a_{11}, \\
a_{22} = \int_{A_2} \omega_2 = \int_{T^2(A_1)} \omega_2 + \int_{T^5(A_1)} \omega_2 = (\rho^3 + \rho^4) a_{21}, \\
a_{23} = \int_{A_3} \omega_2 = -\int_{A_1} \omega_2 + \int_{T(A_1)} \omega_2 + \int_{T^6(A_1)} \omega_2 = (-1 + \rho^2 + \rho^5) a_{21}, \\
a_{32} = \int_{A_2} \omega_3 = \int_{T^2(A_1)} \omega_3 + \int_{T^5(A_1)} \omega_3 = (\rho^2 + \rho^5) a_{31}, \\
a_{33} = \int_{A_3} \omega_3 = -\int_{A_1} \omega_3 + \int_{T(A_1)} \omega_3 + \int_{T^6(A_1)} \omega_3 = (-1 + \rho + \rho^6) a_{31}, \\
b_{11} = \int_{B_1} \omega_1 = \int_{T^2(A_1)} \omega_1 = \rho a_{11}, \\
b_{12} = \int_{B_2} \omega_1 = \int_{T^2(A_1)} \omega_1 + \int_{T^6(A_1)} \omega_1 = (\rho + \rho^3) a_{11}, \\
b_{13} = \int_{B_3} \omega_1 = -\int_{A_1} \omega_1 - \int_{T^4(A_1)} \omega_1 = -(1 + \rho^2) a_{11}, \\
b_{21} = \int_{B_1} \omega_2 = \int_{T^2(A_1)} \omega_2 = \rho^4 a_{21}, \\
b_{22} = \int_{B_2} \omega_2 = \int_{T^2(A_1)} \omega_2 + \int_{T^6(A_1)} \omega_2 = (\rho^4 + \rho^5) a_{21}, \\
b_{23} = \int_{B_3} \omega_2 = -\int_{A_1} \omega_2 - \int_{T^4(A_1)} \omega_2 = -(1 + \rho) a_{21}, \\
b_{31} = \int_{B_1} \omega_3 = \int_{T^2(A_1)} \omega_3 = \rho^2 a_{31}, \\
b_{32} = \int_{B_2} \omega_3 = \int_{T^2(A_1)} \omega_3 + \int_{T^6(A_1)} \omega_3 = (\rho^2 + \rho^6) a_{31}, \\
b_{33} = \int_{B_3} \omega_3 = -\int_{A_1} \omega_3 - \int_{T^4(A_1)} \omega_3 = -(1 + \rho^4) a_{31}.
\end{cases}
$$

Here we put

$$a_{11} = a, \quad a_{21} = b, \quad a_{31} = c.$$

Then, from (11), we have

$$(12)\quad (a_{ij}\ b_{ij}) = \begin{pmatrix} a & (\rho+\rho^6)a & (-1+\rho^3+\rho^4)a & \rho a & (\rho+\rho^3)a & -(1+\rho^2)a \\ b & (\rho^3+\rho^4)b & (-1+\rho^2+\rho^5)b & \rho^4 b & (\rho^4+\rho^5)b & -(1+\rho)b \\ c & (\rho^5+\rho^2)c & (-1+\rho+\rho^6)c & \rho^2 c & (\rho^2+\rho^6)c & -(1+\rho^4)c \end{pmatrix}.$$

Moreover we normalize this period matrix by multiplying $(a_{ij})^{-1}$ on the left side of $(a_{ij}\ b_{ij})$, that is

$$(a_{ij})^{-1}(a_{ij}\ b_{ij}) = (I_3\ \tau)$$

$$\tau = \frac{1}{7}\begin{pmatrix} -3-6(\rho^3+\rho^5+\rho^6) & +2-3(\rho^3+\rho^5+\rho^6) & -1-2(\rho^3+\rho^5+\rho^6) \\ +2-3(\rho^3+\rho^5+\rho^6) & +1-5(\rho^3+\rho^5+\rho^6) & +3-(\rho^3+\rho^5+\rho^6) \\ -1-2(\rho^3+\rho^5+\rho^6) & +3-(\rho^3+\rho^5+\rho^6) & +2-3(\rho^3+\rho^5+\rho^6) \end{pmatrix}.$$

Here, $\rho^3+\rho^5+\rho^6 = (-1+\sqrt{7}i)/2$, so we have finally

THEOREM A. *The normalized period matrix $(I_3\ \tau)$ of klein's surface is given by*

$$(13)\qquad \tau = \begin{pmatrix} \frac{3\sqrt{7}i}{7} & +\frac{1}{2}+\frac{3\sqrt{7}i}{14} & \frac{\sqrt{7}i}{7} \\ +\frac{1}{2}+\frac{3\sqrt{7}i}{14} & +\frac{1}{2}+\frac{5\sqrt{7}i}{14} & +\frac{1}{2}+\frac{\sqrt{7}i}{14} \\ \frac{\sqrt{7}i}{7} & +\frac{1}{2}+\frac{\sqrt{7}i}{14} & +\frac{1}{2}+\frac{3\sqrt{7}i}{14} \end{pmatrix}.$$

2. Theta Functions and Theta Identities

We shall give a brief account of the theta function together with an explanation of notations.

A g-characteristic is a $2\times g$ matrix of integers, written

$$\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} = \begin{bmatrix} \varepsilon_1 & \varepsilon_2 & \cdots & \varepsilon_g \\ \varepsilon_1' & \varepsilon_2' & \cdots & \varepsilon_g' \end{bmatrix}.$$

$\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}$ is even or odd according as $\sum_j \varepsilon_j\varepsilon_j' \equiv 0$ or $1 (mod 2)$. A reduced characteristic is a g-characteristic each of whose entries is zero or one.

There are 2^{2g} reduced characteristic: $2^{g-1}(2^g+1)$ are even; $2^{g-1}(2^g-1)$ are odd. Given a g-characteristic $\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}$, the g-theta function with characteristic is defined by

$$\theta\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}(z,\tau) = \sum_{n\in Z^g} \exp \pi i\left\{{}^t\left(n+\frac{\varepsilon}{2}\right)\tau\left(n+\frac{\varepsilon}{2}\right) + 2^t\left(n+\frac{\varepsilon}{2}\right)\left(z+\frac{\varepsilon'}{2}\right)\right\}$$

where $z \in C^g$ and $\tau \in H_g$ the space of $g\times g$ symmetric matrices with positive definite imaginary part. $\varepsilon, \varepsilon', z$ and n are treated as column vectors in matrix calculations.

The following lemma is an obvious consequence of definitions.

LEMMA 1. *If $\varepsilon = \hat{\varepsilon} + 2\nu$ and $\varepsilon' = \hat{\varepsilon}' + 2\nu'$, where ν and ν' are integral g-vectors, then*

$$\theta\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}(z,\tau) = (-1)^{\sum \varepsilon_j \nu_j'}\theta\begin{bmatrix} \hat{\varepsilon} \\ \hat{\varepsilon}' \end{bmatrix}(z,\tau).$$

This formula will be called, the reduction formula.

We see that the theta constants with odd characteristic vanish identically so that "theta constants" henceforce means "with even characteristic". Then we have following lemma.

LEMMA 2.

$$\text{(I)}\quad \theta\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}(0,\tau)\theta\begin{bmatrix} \delta \\ \delta' \end{bmatrix}(0,\tau) = \sum_{P\in(Z/2Z)^g} \theta\begin{bmatrix} ((\varepsilon+\delta)/2)-P \\ \varepsilon'+\delta' \end{bmatrix}(0,2\tau) \times \theta\begin{bmatrix} ((\varepsilon-\delta)/2)+P \\ \varepsilon'-\delta' \end{bmatrix}(0,2\tau)$$

$$\text{(II)}\quad \theta^2\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}(0,\tau) = \sum_{P\in(Z/2Z)^g} (-1)^{(\varepsilon-P)\varepsilon'}\theta\begin{bmatrix} \varepsilon-P \\ 0 \end{bmatrix}(0,2\tau)\theta\begin{bmatrix} P \\ 0 \end{bmatrix}(0,2\tau)$$

PROOF. See [3 pp. 63].

Moreover, we have following important lemma.

LEMMA 3. *Let $\tau \in H_g$ and $M = \begin{pmatrix} D & C \\ B & A \end{pmatrix} \in Sp(g,Z)$. Then*

$$det|C\tau + D| \neq 0$$

and we have

$$\hat{\tau} = M \circ \tau = (A\tau + B)(C\tau + D)^{-1}$$

and then

$$\hat{\tau} \in H_g.$$

Define, further,

$$\begin{bmatrix} \tilde{\varepsilon} \\ \tilde{\varepsilon}' \end{bmatrix} = M \circ \begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}$$

with

$$\tilde{\varepsilon} = D\varepsilon - C\varepsilon' + dv(C^tD)$$

$$\tilde{\varepsilon}' = -B\varepsilon + A\varepsilon' + dv(A^tB)$$

where $dv(N)$ is the vector consisting of diagonal elements of N. Then

$$\theta\begin{bmatrix} \tilde{\varepsilon} \\ \tilde{\varepsilon}' \end{bmatrix}(0, \hat{\tau}) = \kappa(M) \exp \pi i \phi\left(M, \begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}\right) \sqrt{det|C\tau + D|}\theta\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}(0, \tau)$$

where

$$\phi\left(M, \begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}\right) = \frac{1}{4}\{-{}^t\varepsilon^t DB\varepsilon + 2^t\varepsilon''CB\varepsilon - {}^t\varepsilon''CA\varepsilon' + 2^t(D\varepsilon - C\varepsilon')\, dv(A^tB)\}$$

and $\kappa(M)$ is an eighth root of unity.

PROOF. See [3 pp. 86–105].

REMARK. The really difficult part of this lemma is the determination of $\kappa(M)$. It is extremely important to observe that lemma 3 implies that $M \circ \begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}$ has the same character, i.e., is even or odd, as $\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}$.

LEMMA 4. *If $M_1, M_2 \in Sp(g, Z)$, then*

$$M_2M_1 \circ \begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} \equiv M_2 \circ M_1 \circ \begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} \ (mod\, 2)$$

Now we derive theta identities among theta constants which are necessary in §3. First, we use the formula (I) of lemma 2 to obtain

(14)
$$\theta\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}(0,\tau)$$
$$=\theta^2\begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}(0,2\tau)+\theta^2\begin{bmatrix}0&1&0\\1&0&0\end{bmatrix}(0,2\tau)$$
$$+\theta^2\begin{bmatrix}0&0&1\\1&0&0\end{bmatrix}(0,2\tau)+\theta^2\begin{bmatrix}0&1&1\\1&0&0\end{bmatrix}(0,2\tau)$$

(15)
$$\theta\begin{bmatrix}0&0&0\\0&1&1\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0&0&0\\1&1&1\end{bmatrix}(0,\tau)$$
$$=\theta^2\begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}(0,2\tau)-\theta^2\begin{bmatrix}0&1&0\\1&0&0\end{bmatrix}(0,2\tau)$$
$$-\theta^2\begin{bmatrix}0&0&1\\1&0&0\end{bmatrix}(0,2\tau)+\theta^2\begin{bmatrix}0&1&1\\1&0&0\end{bmatrix}(0,2\tau)$$

(16)
$$\theta\begin{bmatrix}0&0&0\\0&1&0\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0&0&0\\1&1&0\end{bmatrix}(0,\tau)$$
$$=\theta^2\begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}(0,2\tau)-\theta^2\begin{bmatrix}0&1&0\\1&0&0\end{bmatrix}(0,2\tau)$$
$$+\theta^2\begin{bmatrix}0&0&1\\1&0&0\end{bmatrix}(0,2\tau)-\theta^2\begin{bmatrix}0&1&1\\1&0&0\end{bmatrix}(0,2\tau)$$

(17)
$$\theta\begin{bmatrix}0&0&0\\0&0&1\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0&0&0\\1&0&1\end{bmatrix}(0,\tau)$$
$$=\theta^2\begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}(0,2\tau)+\theta^2\begin{bmatrix}0&1&0\\1&0&0\end{bmatrix}(0,2\tau)$$
$$-\theta^2\begin{bmatrix}0&0&1\\1&0&0\end{bmatrix}(0,2\tau)-\theta^2\begin{bmatrix}0&1&1\\1&0&0\end{bmatrix}(0,2\tau)$$

(18)
$$\theta\begin{bmatrix}0&1&0\\0&0&0\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0&1&0\\1&0&0\end{bmatrix}(0,\tau)$$
$$=2\theta\begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}0&1&0\\1&0&0\end{bmatrix}(0,2\tau)$$
$$+2\theta\begin{bmatrix}0&0&1\\1&0&0\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}0&1&1\\1&0&0\end{bmatrix}(0,2\tau)$$

(19)
$$\theta\begin{bmatrix}0&1&0\\0&0&1\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0&1&0\\1&0&1\end{bmatrix}(0,\tau)$$
$$=2\theta\begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}0&1&0\\1&0&0\end{bmatrix}(0,2\tau)$$
$$-2\theta\begin{bmatrix}0&0&1\\1&0&0\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}0&1&1\\1&0&0\end{bmatrix}(0,2\tau)$$

(20)
$$\theta\begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0&0&1\\1&0&0\end{bmatrix}(0,\tau)$$
$$=2\theta\begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}0&0&1\\1&0&0\end{bmatrix}(0,2\tau)$$
$$+2\theta\begin{bmatrix}0&1&0\\1&0&0\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}0&1&1\\1&0&0\end{bmatrix}(0,2\tau)$$

(21)
$$\theta\begin{bmatrix}0&0&1\\0&1&0\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0&0&1\\1&1&0\end{bmatrix}(0,\tau)$$
$$=2\theta\begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}0&0&1\\1&0&0\end{bmatrix}(0,2\tau)$$
$$-2\theta\begin{bmatrix}0&1&0\\1&0&0\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}0&1&1\\1&0&0\end{bmatrix}(0,2\tau)$$

(22)
$$\theta\begin{bmatrix}0&1&1\\0&0&0\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0&1&1\\1&0&0\end{bmatrix}(0,\tau)$$
$$=2\theta\begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}0&1&1\\1&0&0\end{bmatrix}(0,2\tau)$$
$$+2\theta\begin{bmatrix}0&1&0\\1&0&0\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}0&0&1\\1&0&0\end{bmatrix}(0,2\tau)$$

(23)
$$\theta\begin{bmatrix}0&1&1\\0&1&1\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0&1&1\\1&1&1\end{bmatrix}(0,\tau)$$
$$=2\theta\begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}0&1&1\\1&0&0\end{bmatrix}(0,2\tau)$$
$$-2\theta\begin{bmatrix}0&1&0\\1&0&0\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}0&0&1\\1&0&0\end{bmatrix}(0,2\tau)$$

From (14) × (17)–(15) × (16) = (18) × (19), we obtain

($\mathscr{I}$-1)
$$\theta\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0&0&0\\0&0&1\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0&0&0\\1&0&1\end{bmatrix}(0,\tau)$$
$$=\theta\begin{bmatrix}0&0&0\\0&1&0\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0&0&0\\1&1&0\end{bmatrix}(0,\tau)$$
$$\times\theta\begin{bmatrix}0&0&0\\0&1&1\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0&0&0\\1&1&1\end{bmatrix}(0,\tau)$$
$$+\theta\begin{bmatrix}0&1&0\\0&0&0\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0&1&0\\1&0&0\end{bmatrix}(0,\tau)$$
$$\times\theta\begin{bmatrix}0&1&0\\0&0&1\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0&1&0\\1&0&1\end{bmatrix}(0,\tau)$$

From (14) × (16)–(15) × (17) = (20) × (21), we obtain

$$(\mathscr{I}\text{-}2)\quad \theta\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0&0&0\\0&1&0\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0&0&0\\1&1&0\end{bmatrix}(0,\tau)$$

$$=\theta\begin{bmatrix}0&0&0\\0&0&1\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0&0&0\\1&0&1\end{bmatrix}(0,\tau)$$

$$\times\theta\begin{bmatrix}0&0&0\\0&1&1\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0&0&0\\1&1&1\end{bmatrix}(0,\tau)$$

$$+\theta\begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0&0&1\\1&0&0\end{bmatrix}(0,\tau)$$

$$\times\theta\begin{bmatrix}0&0&1\\0&1&0\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0&0&1\\1&1&0\end{bmatrix}(0,\tau)$$

From (14) × (15)–(16) × (17) = (22) × (23), we obtain

$$(\mathscr{I}\text{-}3)\quad \theta\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0&0&0\\0&1&1\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0&0&0\\1&1&1\end{bmatrix}(0,\tau)$$

$$=\theta\begin{bmatrix}0&0&0\\0&1&0\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0&0&0\\1&1&0\end{bmatrix}(0,\tau)$$

$$\times\theta\begin{bmatrix}0&0&0\\0&0&1\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0&0&0\\1&0&1\end{bmatrix}(0,\tau)$$

$$+\theta\begin{bmatrix}0&1&1\\0&0&0\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0&1&1\\1&0&0\end{bmatrix}(0,\tau)$$

$$\times\theta\begin{bmatrix}0&1&1\\0&1&1\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0&1&1\\1&1&1\end{bmatrix}(0,\tau)$$

Second, we use the formula (II) of lemma 2 to obtain

$$(24)\quad \theta^2\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}(0,\tau)=\theta^2\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}(0,2\tau)+\theta^2\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}(0,2\tau)$$

$$+\theta^2\begin{bmatrix}0&1&0\\0&0&0\end{bmatrix}(0,2\tau)+\theta^2\begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}(0,2\tau)$$

$$+\theta^2\begin{bmatrix}1&1&0\\0&0&0\end{bmatrix}(0,2\tau)+\theta^2\begin{bmatrix}1&0&1\\0&0&0\end{bmatrix}(0,2\tau)$$

$$+\theta^2\begin{bmatrix}0&1&1\\0&0&0\end{bmatrix}(0,2\tau)+\theta^2\begin{bmatrix}1&1&1\\0&0&0\end{bmatrix}(0,2\tau)$$

$$(25)\qquad \theta^2\begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}(0,\tau)=\theta^2\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}(0,2\tau)-\theta^2\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}(0,2\tau)$$

$$+\theta^2\begin{bmatrix}0&1&0\\0&0&0\end{bmatrix}(0,2\tau)+\theta^2\begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}(0,2\tau)$$

$$-\theta^2\begin{bmatrix}1&1&0\\0&0&0\end{bmatrix}(0,2\tau)-\theta^2\begin{bmatrix}1&0&1\\0&0&0\end{bmatrix}(0,2\tau)$$

$$+\theta^2\begin{bmatrix}0&1&1\\0&0&0\end{bmatrix}(0,2\tau)-\theta^2\begin{bmatrix}1&1&1\\0&0&0\end{bmatrix}(0,2\tau)$$

$$(26)\qquad \theta^2\begin{bmatrix}0&0&0\\0&1&0\end{bmatrix}(0,\tau)=\theta^2\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}(0,2\tau)+\theta^2\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}(0,2\tau)$$

$$-\theta^2\begin{bmatrix}0&1&0\\0&0&0\end{bmatrix}(0,2\tau)+\theta^2\begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}(0,2\tau)$$

$$-\theta^2\begin{bmatrix}1&1&0\\0&0&0\end{bmatrix}(0,2\tau)+\theta^2\begin{bmatrix}1&0&1\\0&0&0\end{bmatrix}(0,2\tau)$$

$$-\theta^2\begin{bmatrix}0&1&1\\0&0&0\end{bmatrix}(0,2\tau)-\theta^2\begin{bmatrix}1&1&1\\0&0&0\end{bmatrix}(0,2\tau)$$

$$(27)\qquad \theta^2\begin{bmatrix}0&0&0\\0&0&1\end{bmatrix}(0,\tau)=\theta^2\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}(0,2\tau)+\theta^2\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}(0,2\tau)$$

$$+\theta^2\begin{bmatrix}0&1&0\\0&0&0\end{bmatrix}(0,2\tau)-\theta^2\begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}(0,2\tau)$$

$$+\theta^2\begin{bmatrix}1&1&0\\0&0&0\end{bmatrix}(0,2\tau)-\theta^2\begin{bmatrix}1&0&1\\0&0&0\end{bmatrix}(0,2\tau)$$

$$-\theta^2\begin{bmatrix}0&1&1\\0&0&0\end{bmatrix}(0,2\tau)-\theta^2\begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}(0,2\tau)$$

(28)
$$\theta^2\begin{bmatrix}0&0&0\\1&1&0\end{bmatrix}(0,\tau)=\theta^2\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}(0,2\tau)-\theta^2\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}(0,2\tau)$$
$$-\theta^2\begin{bmatrix}0&1&0\\0&0&0\end{bmatrix}(0,2\tau)+\theta^2\begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}(0,2\tau)$$
$$+\theta^2\begin{bmatrix}1&1&0\\0&0&0\end{bmatrix}(0,2\tau)-\theta^2\begin{bmatrix}1&0&1\\0&0&0\end{bmatrix}(0,2\tau)$$
$$-\theta^2\begin{bmatrix}0&1&1\\0&0&0\end{bmatrix}(0,2\tau)+\theta^2\begin{bmatrix}1&1&1\\0&0&0\end{bmatrix}(0,2\tau)$$

(29)
$$\theta^2\begin{bmatrix}0&0&0\\1&0&1\end{bmatrix}(0,\tau)=\theta^2\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}(0,2\tau)-\theta^2\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}(0,2\tau)$$
$$+\theta^2\begin{bmatrix}0&1&0\\0&0&0\end{bmatrix}(0,2\tau)-\theta^2\begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}(0,2\tau)$$
$$-\theta^2\begin{bmatrix}1&1&0\\0&0&0\end{bmatrix}(0,2\tau)+\theta^2\begin{bmatrix}1&0&1\\0&0&0\end{bmatrix}(0,2\tau)$$
$$-\theta^2\begin{bmatrix}0&1&1\\0&0&0\end{bmatrix}(0,2\tau)+\theta^2\begin{bmatrix}1&1&1\\0&0&0\end{bmatrix}(0,2\tau)$$

(30)
$$\theta^2\begin{bmatrix}0&0&0\\0&1&1\end{bmatrix}(0,\tau)=\theta^2\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}(0,2\tau)+\theta^2\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}(0,2\tau)$$
$$-\theta^2\begin{bmatrix}0&1&0\\0&0&0\end{bmatrix}(0,2\tau)-\theta^2\begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}(0,2\tau)$$
$$-\theta^2\begin{bmatrix}1&1&0\\0&0&0\end{bmatrix}(0,2\tau)-\theta^2\begin{bmatrix}1&0&1\\0&0&0\end{bmatrix}(0,2\tau)$$
$$+\theta^2\begin{bmatrix}0&1&1\\0&0&0\end{bmatrix}(0,2\tau)+\theta^2\begin{bmatrix}1&1&1\\0&0&0\end{bmatrix}(0,2\tau)$$

$$(31)\quad \theta^2\begin{bmatrix}0&0&0\\1&1&1\end{bmatrix}(0,\tau)=\theta^2\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}(0,2\tau)-\theta^2\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}(0,2\tau)$$
$$-\theta^2\begin{bmatrix}0&1&0\\0&0&0\end{bmatrix}(0,2\tau)-\theta^2\begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}(0,2\tau)$$
$$+\theta^2\begin{bmatrix}1&1&0\\0&0&0\end{bmatrix}(0,2\tau)+\theta^2\begin{bmatrix}1&0&1\\0&0&0\end{bmatrix}(0,2\tau)$$
$$+\theta^2\begin{bmatrix}0&1&1\\0&0&0\end{bmatrix}(0,2\tau)-\theta^2\begin{bmatrix}1&1&1\\0&0&0\end{bmatrix}(0,2\tau)$$

$$(32)\quad \theta^2\begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}(0,\tau)=2\theta\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}(0,2\tau)$$
$$+2\theta\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}1&0&1\\0&0&0\end{bmatrix}(0,2\tau)$$
$$+2\theta\begin{bmatrix}0&1&0\\0&0&0\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}0&1&1\\0&0&0\end{bmatrix}(0,2\tau)$$
$$+2\theta\begin{bmatrix}1&1&0\\0&0&0\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}1&1&1\\0&0&0\end{bmatrix}(0,2\tau)$$

$$(33)\quad \theta^2\begin{bmatrix}0&0&1\\0&1&0\end{bmatrix}(0,\tau)=2\theta\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}(0,2\tau)$$
$$+2\theta\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}1&0&1\\0&0&0\end{bmatrix}(0,2\tau)$$
$$-2\theta\begin{bmatrix}0&1&0\\0&0&0\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}0&1&1\\0&0&0\end{bmatrix}(0,2\tau)$$
$$-2\theta\begin{bmatrix}1&1&0\\0&0&0\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}1&1&1\\0&0&0\end{bmatrix}(0,2\tau)$$

(34)
$$\theta^2\begin{bmatrix}0&0&1\\1&0&0\end{bmatrix}(0,\tau)=2\theta\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}(0,2\tau)$$
$$-2\theta\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}1&0&1\\0&0&0\end{bmatrix}(0,2\tau)$$
$$+2\theta\begin{bmatrix}0&1&0\\0&0&0\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}0&1&1\\0&0&0\end{bmatrix}(0,2\tau)$$
$$-2\theta\begin{bmatrix}1&1&0\\0&0&0\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}1&1&1\\0&0&0\end{bmatrix}(0,2\tau)$$

(35)
$$\theta^2\begin{bmatrix}0&0&1\\1&1&0\end{bmatrix}(0,\tau)=2\theta\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}(0,2\tau)$$
$$-2\theta\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}1&0&1\\0&0&0\end{bmatrix}(0,2\tau)$$
$$-2\theta\begin{bmatrix}0&1&0\\0&0&0\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}0&1&1\\0&0&0\end{bmatrix}(0,2\tau)$$
$$+2\theta\begin{bmatrix}1&1&0\\0&0&0\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}1&1&1\\0&0&0\end{bmatrix}(0,2\tau)$$

(36)
$$\theta^2\begin{bmatrix}0&1&0\\0&0&0\end{bmatrix}(0,\tau)=2\theta\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}0&1&0\\0&0&0\end{bmatrix}(0,2\tau)$$
$$+2\theta\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}1&1&0\\0&0&0\end{bmatrix}(0,2\tau)$$
$$+2\theta\begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}0&1&1\\0&0&0\end{bmatrix}(0,2\tau)$$
$$+2\theta\begin{bmatrix}1&0&1\\0&0&0\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}1&1&1\\0&0&0\end{bmatrix}(0,2\tau)$$

$$(37)\quad \theta^2\begin{bmatrix}0&1&0\\1&0&0\end{bmatrix}(0,\tau)=2\theta\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}0&1&0\\0&0&0\end{bmatrix}(0,2\tau)$$
$$-2\theta\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}1&1&0\\0&0&0\end{bmatrix}(0,2\tau)$$
$$+2\theta\begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}0&1&1\\0&0&0\end{bmatrix}(0,2\tau)$$
$$-2\theta\begin{bmatrix}1&0&1\\0&0&0\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}1&1&1\\0&0&0\end{bmatrix}(0,2\tau)$$

$$(38)\quad \theta^2\begin{bmatrix}0&1&0\\0&0&1\end{bmatrix}(0,\tau)=2\theta\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}0&1&0\\0&0&0\end{bmatrix}(0,2\tau)$$
$$+2\theta\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}1&1&0\\0&0&0\end{bmatrix}(0,2\tau)$$
$$-2\theta\begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}0&1&1\\0&0&0\end{bmatrix}(0,2\tau)$$
$$-2\theta\begin{bmatrix}1&0&1\\0&0&0\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}1&1&1\\0&0&0\end{bmatrix}(0,2\tau)$$

$$(39)\quad \theta^2\begin{bmatrix}0&1&0\\1&0&1\end{bmatrix}(0,\tau)=2\theta\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}0&1&0\\0&0&0\end{bmatrix}(0,2\tau)$$
$$-2\theta\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}1&1&0\\0&0&0\end{bmatrix}(0,2\tau)$$
$$-2\theta\begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}0&1&1\\0&0&0\end{bmatrix}(0,2\tau)$$
$$+2\theta\begin{bmatrix}1&0&1\\0&0&0\end{bmatrix}(0,2\tau)\theta\begin{bmatrix}1&1&1\\0&0&0\end{bmatrix}(0,2\tau)$$

From $(32)\times(33)+(34)\times(35)=(24)\times(26)+(25)\times(28)-(27)\times(30)-(29)\times(31)$, we obtain,

$$(\mathscr{I}\text{-4})\qquad \theta^2\begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}(0,\tau)\theta^2\begin{bmatrix}0&0&1\\0&1&0\end{bmatrix}(0,\tau)$$

$$+\,\theta^2\begin{bmatrix}0&0&1\\1&0&0\end{bmatrix}(0,\tau)\theta^2\begin{bmatrix}0&0&1\\1&1&0\end{bmatrix}(0,\tau)$$

$$=\theta^2\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}(0,\tau)\theta^2\begin{bmatrix}0&0&0\\0&1&0\end{bmatrix}(0,\tau)$$

$$+\,\theta^2\begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}(0,\tau)\theta^2\begin{bmatrix}0&0&0\\1&1&0\end{bmatrix}(0,\tau)$$

$$-\,\theta^2\begin{bmatrix}0&0&0\\0&0&1\end{bmatrix}(0,\tau)\theta^2\begin{bmatrix}0&0&0\\0&1&1\end{bmatrix}(0,\tau)$$

$$-\,\theta^2\begin{bmatrix}0&0&0\\1&0&1\end{bmatrix}(0,\tau)\theta^2\begin{bmatrix}0&0&0\\1&1&1\end{bmatrix}(0,\tau)$$

From $(32)^2+(33)^2+(34)^2+(35)^2=(24)^2+(25)^2+(26)^2+(28)^2-(27)^2-(29)^2-(30)^2-(31)^2$, we obtain,

$$(\mathscr{I}\text{-5})\qquad \theta^4\begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}(0,\tau)+\theta^4\begin{bmatrix}0&0&1\\0&1&0\end{bmatrix}(0,\tau)$$

$$+\,\theta^4\begin{bmatrix}0&0&1\\1&0&0\end{bmatrix}(0,\tau)+\theta^4\begin{bmatrix}0&0&1\\1&1&0\end{bmatrix}(0,\tau)$$

$$=\theta^4\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}(0,\tau)+\theta^4\begin{bmatrix}0&0&0\\0&1&0\end{bmatrix}(0,\tau)$$

$$+\,\theta^4\begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}(0,\tau)+\theta^4\begin{bmatrix}0&0&0\\1&1&0\end{bmatrix}(0,\tau)$$

$$-\,\theta^4\begin{bmatrix}0&0&0\\0&0&1\end{bmatrix}(0,\tau)-\theta^4\begin{bmatrix}0&0&0\\0&1&1\end{bmatrix}(0,\tau)$$

$$-\,\theta^4\begin{bmatrix}0&0&0\\1&0&1\end{bmatrix}(0,\tau)-\theta^4\begin{bmatrix}0&0&0\\1&1&1\end{bmatrix}(0,\tau)$$

From $(32)\times(34)+(33)\times(35)=(24)\times(25)+(26)\times(28)-(27)\times(29)-(30)\times$

(31), we obtain

$$(\mathscr{I}\text{-}6)\qquad \theta^2\begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}(0,\tau)\theta^2\begin{bmatrix}0&0&1\\1&0&0\end{bmatrix}(0,\tau)$$
$$+\theta^2\begin{bmatrix}0&0&1\\0&1&0\end{bmatrix}(0,\tau)\theta^2\begin{bmatrix}0&0&1\\1&1&0\end{bmatrix}(0,\tau)$$
$$=\theta^2\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}(0,\tau)\theta^2\begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}(0,\tau)$$
$$+\theta^2\begin{bmatrix}0&0&0\\0&1&0\end{bmatrix}(0,\tau)\theta^2\begin{bmatrix}0&0&0\\1&1&0\end{bmatrix}(0,\tau)$$
$$-\theta^2\begin{bmatrix}0&0&0\\0&0&1\end{bmatrix}(0,\tau)\theta^2\begin{bmatrix}0&0&0\\1&0&1\end{bmatrix}(0,\tau)$$
$$-\theta^2\begin{bmatrix}0&0&0\\0&1&1\end{bmatrix}(0,\tau)\theta^2\begin{bmatrix}0&0&0\\1&1&1\end{bmatrix}(0,\tau)$$

From $(36)\times(37)+(38)\times(39)=(24)\times(25)+(27)\times(29)-(26)\times(28)-(30)\times$ (31), we obtain

$$(\mathscr{I}\text{-}7)\qquad \theta^2\begin{bmatrix}0&1&0\\0&0&0\end{bmatrix}(0,\tau)\theta^2\begin{bmatrix}0&1&0\\1&0&0\end{bmatrix}(0,\tau)$$
$$+\theta^2\begin{bmatrix}0&1&0\\0&0&1\end{bmatrix}(0,\tau)\theta^2\begin{bmatrix}0&1&0\\1&0&1\end{bmatrix}(0,\tau)$$
$$=\theta^2\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}(0,\tau)\theta^2\begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}(0,\tau)$$
$$+\theta^2\begin{bmatrix}0&0&0\\0&0&1\end{bmatrix}(0,\tau)\theta^2\begin{bmatrix}0&0&0\\1&0&1\end{bmatrix}(0,\tau)$$
$$-\theta^2\begin{bmatrix}0&0&0\\0&1&0\end{bmatrix}(0,\tau)\theta^2\begin{bmatrix}0&0&0\\1&1&0\end{bmatrix}(0,\tau)$$
$$-\theta^2\begin{bmatrix}0&0&0\\0&1&1\end{bmatrix}(0,\tau)\theta^2\begin{bmatrix}0&0&0\\1&1&1\end{bmatrix}(0,\tau)$$

From $(36)^2+(37)^2+(38)^2+(39)^2=(24)^2+(25)^2+(27)^2+(29)^2-(26)^2-(28)^2-(30)^2-(31)^2$, we obtain

$$(\mathscr{I}\text{-}8)\qquad \begin{aligned}&\theta^4\begin{bmatrix}0&1&0\\0&0&0\end{bmatrix}(0,\tau)+\theta^4\begin{bmatrix}0&1&0\\1&0&0\end{bmatrix}(0,\tau)\\&+\theta^4\begin{bmatrix}0&1&0\\0&0&1\end{bmatrix}(0,\tau)+\theta^4\begin{bmatrix}0&1&0\\1&0&1\end{bmatrix}(0,\tau)\\&=\theta^4\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}(0,\tau)+\theta^4\begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}(0,\tau)\\&+\theta^4\begin{bmatrix}0&0&0\\0&0&1\end{bmatrix}(0,\tau)+\theta^4\begin{bmatrix}0&0&0\\1&0&1\end{bmatrix}(0,\tau)\\&-\theta^4\begin{bmatrix}0&0&0\\0&1&0\end{bmatrix}(0,\tau)-\theta^4\begin{bmatrix}0&0&0\\1&1&0\end{bmatrix}(0,\tau)\\&-\theta^4\begin{bmatrix}0&0&0\\0&1&1\end{bmatrix}(0,\tau)-\theta^4\begin{bmatrix}0&0&0\\1&1&1\end{bmatrix}(0,\tau)\end{aligned}$$

From $(36)\times(38)+(37)\times(39)=(24)\times(27)+(25)\times(29)-(26)\times(30)-(28)\times(31)$, we obtain

$$(\mathscr{I}\text{-}9)\qquad \begin{aligned}&\theta^2\begin{bmatrix}0&1&0\\0&0&0\end{bmatrix}(0,\tau)\theta^2\begin{bmatrix}0&1&0\\0&0&1\end{bmatrix}(0,\tau)\\&+\theta^2\begin{bmatrix}0&1&0\\1&0&0\end{bmatrix}(0,\tau)\theta^2\begin{bmatrix}0&1&0\\1&0&1\end{bmatrix}(0,\tau)\\&=\theta^2\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}(0,\tau)\theta^2\begin{bmatrix}0&0&0\\0&0&1\end{bmatrix}(0,\tau)\\&+\theta^2\begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}(0,\tau)\theta^2\begin{bmatrix}0&0&0\\1&0&1\end{bmatrix}(0,\tau)\\&-\theta^2\begin{bmatrix}0&0&0\\0&1&0\end{bmatrix}(0,\tau)\theta^2\begin{bmatrix}0&0&0\\0&1&1\end{bmatrix}(0,\tau)\\&-\theta^2\begin{bmatrix}0&0&0\\1&1&0\end{bmatrix}(0,\tau)\theta^2\begin{bmatrix}0&0&0\\1&1&1\end{bmatrix}(0,\tau)\end{aligned}$$

We use these 9 theta identities $(\mathscr{I}\text{-}1),\ldots,(\mathscr{I}\text{-}9)$ in §3.

3. The Proportionalities of Theta Constants.

We will decide the proportionalities of theta constants. We take our $M \in Sp(3, Z)$ in §2, that is $M = \begin{pmatrix} D & C \\ B & A \end{pmatrix}$, and

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & -1 & -1 \\ 0 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix},$$

$$C = \begin{pmatrix} +1 & -1 & 0 \\ -1 & +1 & +1 \\ 0 & +1 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix}.$$

Then we find

$$\hat{\tau} = M \circ \tau = (A\tau + B)(C\tau + D)^{-1} = \tau,$$

$$det|C\tau + D| = 1$$

and

$$M^7 = I_6.$$

We now apply lemma 3 and lemma 4 to $36 = 2^{g-1}(2^g + 1) = 2^2(2^3 + 1)$ 3-theta constants. We find

$$dv(C^tD) = \begin{pmatrix} +1 \\ -1 \\ 0 \end{pmatrix}, \quad dv(A^tB) = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

and thus that

$$\tilde{\varepsilon} = \begin{pmatrix} 1 & 0 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{pmatrix} - \begin{pmatrix} +1 & -1 & 0 \\ -1 & +1 & +1 \\ 0 & +1 & 0 \end{pmatrix} \begin{pmatrix} \varepsilon_1' \\ \varepsilon_2' \\ \varepsilon_3' \end{pmatrix} + \begin{pmatrix} +1 \\ -1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \varepsilon_1 + \varepsilon_3 - \varepsilon_1' + \varepsilon_2' + 1 \\ -\varepsilon_1 - \varepsilon_2 - \varepsilon_3 + \varepsilon_1' - \varepsilon_2' - \varepsilon_3' - 1 \\ -\varepsilon_3 - \varepsilon_2' \end{pmatrix}$$

$$\tilde{\varepsilon}' = -\begin{pmatrix} -1 & -1 & -1 \\ 0 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \varepsilon_1' \\ \varepsilon_2' \\ \varepsilon_3' \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} +\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_3' - 1 \\ +\varepsilon_2 + \varepsilon_3 + \varepsilon_3' - 1 \\ +\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_2' - 1 \end{pmatrix}.$$

We also find that

$$\phi\left(M,\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}\right)$$

$$=\frac{1}{4}\left\{-(\varepsilon_1\varepsilon_2\varepsilon_3)\begin{pmatrix}-1&0&0\\0&+1&+1\\0&+1&+1\end{pmatrix}\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\end{pmatrix}+2(\varepsilon'_1\varepsilon'_2\varepsilon'_3)\begin{pmatrix}-1&0&0\\0&-1&-1\\0&-1&-1\end{pmatrix}\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\end{pmatrix}\right.$$

$$-(\varepsilon'_1\varepsilon'_2\varepsilon'_3)\begin{pmatrix}0&0&0\\0&+1&0\\0&0&+1\end{pmatrix}\begin{pmatrix}\varepsilon'_1\\ \varepsilon'_2\\ \varepsilon'_3\end{pmatrix}+2\left((\varepsilon_1\varepsilon_2\varepsilon_3)\begin{pmatrix}1&-1&0\\0&-1&0\\1&-1&-1\end{pmatrix}\right.$$

$$\left.\left.-(\varepsilon'_1\varepsilon'_2\varepsilon'_3)\begin{pmatrix}+1&-1&0\\-1&+1&+1\\0&+1&0\end{pmatrix}\right)\begin{pmatrix}-1\\-1\\-1\end{pmatrix}\right\}$$

$$=\frac{1}{4}\{+\varepsilon_1^2-\varepsilon_2^2-2\varepsilon_2\varepsilon_3-\varepsilon_3^2-2(\varepsilon_1\varepsilon'_1+\varepsilon_2\varepsilon'_2+\varepsilon_2\varepsilon'_3+\varepsilon_3\varepsilon'_2+\varepsilon_3\varepsilon'_3)$$

$$-\varepsilon_2'^2-\varepsilon_3'^2+2(+\varepsilon_2+\varepsilon_3+\varepsilon'_2+\varepsilon'_3)\}.$$

Thus, applying lemma 3 to $\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(0,\pi)$ and its transforms while always using the reduced characteristic which occurs and lemma 1, we obtain following equations.

First, we have

$$M\circ\begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}=\begin{bmatrix}2&-2&-1\\0&0&0\end{bmatrix}\equiv\begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}\quad(mod\,2),$$

$$\phi\left(M,\begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}\right)=-\frac{1}{4}$$

and lemma 1 implies

$$\theta\begin{bmatrix}2&-2&-1\\0&0&0\end{bmatrix}(0,\tau)=\theta\begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}(0,\tau).$$

Thus

$$\theta\begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}(0,\tau)=\kappa(M)\exp\left(+\frac{\pi i}{4}\right)\theta\begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}(0,\tau).\tag{40}$$

Consequently we have

$$\kappa(M) = \exp\left(-\frac{\pi i}{4}\right).$$

Second, applying lemma 3 in succession to $\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(0,\tau)$ in the same way, we obtain,

(41) $$\theta\begin{bmatrix}1 & 1 & 0\\ 1 & 1 & 1\end{bmatrix}(0,\tau) = \exp\left(-\frac{\pi i}{4}\right)\theta\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}(0,\tau)$$

(42) $$\theta\begin{bmatrix}0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}(0,\tau) = \exp\left(+\frac{\pi i}{4}\right)\theta\begin{bmatrix}1 & 1 & 0\\ 1 & 1 & 1\end{bmatrix}(0,\tau) = \theta\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}(0,\tau)$$

(43) $$\theta\begin{bmatrix}1 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}(0,\tau) = \exp\left(-\frac{\pi i}{4}\right)\theta\begin{bmatrix}0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}(0,\tau)$$
$$= \exp\left(-\frac{\pi i}{4}\right)\theta\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}(0,\tau)$$

(44) $$\theta\begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 1\end{bmatrix}(0,\tau) = \theta\begin{bmatrix}1 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}(0,\tau) = \exp\left(-\frac{\pi i}{4}\right)\theta\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}(0,\tau)$$

(45) $$\theta\begin{bmatrix}0 & 1 & 1\\ 0 & 0 & 0\end{bmatrix}(0,\tau) = \exp\left(+\frac{\pi i}{4}\right)\theta\begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 1\end{bmatrix}(0,\tau) = \theta\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}(0,\tau)$$

(46) $$\theta\begin{bmatrix}0 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}(0,\tau) = \exp\left(-\frac{\pi i}{4}\right)\theta\begin{bmatrix}0 & 1 & 1\\ 0 & 0 & 0\end{bmatrix}(0,\tau)$$
$$= \exp\left(-\frac{\pi i}{4}\right)\theta\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}(0,\tau)$$

(47) $$\theta\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}(0,\tau) = \theta\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}(0,\tau)$$

(48) $$\theta\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}(0,\tau) = \exp\left(+\frac{\pi i}{4}\right)\theta\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}(0,\tau)$$
$$= \exp\left(+\frac{\pi i}{4}\right)\theta\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}(0,\tau)$$

(49) $$\theta\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}(0,\tau) = \theta\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}(0,\tau) = \exp\left(\frac{+\pi i}{4}\right)\theta\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}(0,\tau)$$

(50) $$\theta\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}(0,\tau)=\theta\begin{bmatrix}0&1&0\\0&0&0\end{bmatrix}(0,\tau)=\exp\left(\frac{+\pi i}{4}\right)\theta\begin{bmatrix}0&0&0\\0&0&1\end{bmatrix}(0,\tau)$$

(51) $$\theta\begin{bmatrix}0&0&0\\0&1&0\end{bmatrix}(0,\tau)=\theta\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}(0,\tau)=\exp\left(+\frac{\pi i}{4}\right)\theta\begin{bmatrix}0&0&0\\0&0&1\end{bmatrix}(0,\tau)$$

(52) $$\theta\begin{bmatrix}0&0&1\\1&1&0\end{bmatrix}(0,\tau)=\theta\begin{bmatrix}0&0&0\\0&1&0\end{bmatrix}(0,\tau)=\exp\left(+\frac{\pi i}{4}\right)\theta\begin{bmatrix}0&0&0\\0&0&1\end{bmatrix}(0,\tau)$$

(53) $$\theta\begin{bmatrix}0&0&0\\1&1&1\end{bmatrix}(0,\tau)=\exp\left(-\frac{\pi i}{4}\right)\theta\begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}(0,\tau)$$

(54) $$\theta\begin{bmatrix}1&0&1\\0&0&0\end{bmatrix}(0,\tau)=\exp\left(+\frac{\pi i}{4}\right)\theta\begin{bmatrix}0&0&0\\1&1&1\end{bmatrix}(0,\tau)=\theta\begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}(0,\tau)$$

(55) $$\theta\begin{bmatrix}1&1&1\\1&0&1\end{bmatrix}(0,\tau)=\exp\left(+\frac{\pi i}{4}\right)\theta\begin{bmatrix}1&0&1\\0&0&0\end{bmatrix}(0,\tau)$$
$$=\exp\left(+\frac{\pi i}{4}\right)\theta\begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}(0,\tau)$$

(56) $$\theta\begin{bmatrix}0&0&1\\1&0&0\end{bmatrix}(0,\tau)=\exp\left(-\frac{\pi i}{4}\right)\theta\begin{bmatrix}1&1&1\\1&0&1\end{bmatrix}(0,\tau)=\theta\begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}(0,\tau)$$

(57) $$\theta\begin{bmatrix}1&1&1\\0&0&0\end{bmatrix}(0,\tau)=\theta\begin{bmatrix}0&0&1\\1&0&0\end{bmatrix}(0,\tau)=\theta\begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}(0,\tau)$$

(58) $$\theta\begin{bmatrix}1&0&1\\0&1&0\end{bmatrix}(0,\tau)=\theta\begin{bmatrix}1&1&1\\0&0&0\end{bmatrix}(0,\tau)=\theta\begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}(0,\tau)$$

(59) $$\theta\begin{bmatrix}0&1&0\\0&0&1\end{bmatrix}(0,\tau)=\theta\begin{bmatrix}0&0&0\\1&0&1\end{bmatrix}(0,\tau)$$

(60) $$\theta\begin{bmatrix}1&1&0\\1&1&0\end{bmatrix}(0,\tau)=\exp\left(-\frac{\pi i}{4}\right)\theta\begin{bmatrix}0&1&0\\0&0&1\end{bmatrix}(0,\tau)$$
$$=\exp\left(-\frac{\pi i}{4}\right)\theta\begin{bmatrix}0&0&0\\1&0&1\end{bmatrix}(0,\tau)$$

(61) $$\theta\begin{bmatrix}0&1&1\\1&0&0\end{bmatrix}(0,\tau)=\exp\left(+\frac{\pi i}{2}\right)\theta\begin{bmatrix}1&1&0\\1&1&0\end{bmatrix}(0,\tau)$$
$$=\exp\left(+\frac{\pi i}{4}\right)\theta\begin{bmatrix}0&0&0\\1&0&1\end{bmatrix}(0,\tau)$$

(62) $$\theta\begin{bmatrix}1 & 0 & 1\\ 1 & 1 & 1\end{bmatrix}(0,\tau)=\exp\left(-\frac{\pi i}{4}\right)\theta\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & 0\end{bmatrix}(0,\tau)=\theta\begin{bmatrix}0 & 0 & 0\\ 1 & 0 & 1\end{bmatrix}(0,\tau)$$

(63) $$\theta\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}(0,\tau)=\exp\left(+\frac{\pi i}{4}\right)\theta\begin{bmatrix}1 & 0 & 1\\ 1 & 1 & 1\end{bmatrix}(0,\tau)$$
$$=\exp\left(+\frac{\pi i}{4}\right)\theta\begin{bmatrix}0 & 0 & 0\\ 1 & 0 & 1\end{bmatrix}(0,\tau)$$

(64) $$\theta\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 1\end{bmatrix}(0,\tau)=\exp\left(\frac{5\pi i}{4}\right)\theta\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}(0,\tau)$$
$$=\exp\left(-\frac{\pi i}{2}\right)\theta\begin{bmatrix}0 & 0 & 0\\ 1 & 0 & 1\end{bmatrix}(0,\tau)$$

(65) $$\theta\begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 0\end{bmatrix}(0,\tau)=\theta\begin{bmatrix}0 & 0 & 0\\ 1 & 1 & 0\end{bmatrix}(0,\tau)$$

(66) $$\theta\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 1\end{bmatrix}(0,\tau)=\exp\left(-\frac{\pi i}{4}\right)\theta\begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 0\end{bmatrix}(0,\tau)$$
$$=\exp\left(-\frac{\pi i}{4}\right)\theta\begin{bmatrix}0 & 0 & 0\\ 1 & 1 & 0\end{bmatrix}(0,\tau)$$

(67) $$\theta\begin{bmatrix}1 & 0 & 1\\ 1 & 0 & 1\end{bmatrix}(0,\tau)=\exp\left(+\frac{\pi i}{2}\right)\theta\begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 0\end{bmatrix}(0,\tau)$$
$$=\exp\left(+\frac{\pi i}{4}\right)\theta\begin{bmatrix}0 & 0 & 0\\ 1 & 1 & 0\end{bmatrix}(0,\tau)$$

(68) $$\theta\begin{bmatrix}0 & 1 & 1\\ 0 & 1 & 1\end{bmatrix}(0,\tau)=\exp\left(-\frac{\pi i}{2}\right)\theta\begin{bmatrix}1 & 0 & 1\\ 1 & 0 & 1\end{bmatrix}(0,\tau)$$
$$=\exp\left(-\frac{\pi i}{4}\right)\theta\begin{bmatrix}0 & 0 & 0\\ 1 & 1 & 0\end{bmatrix}(0,\tau)$$

(69) $$\theta\begin{bmatrix}1 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}(0,\tau)=\exp\left(+\frac{\pi i}{4}\right)\theta\begin{bmatrix}0 & 1 & 1\\ 0 & 1 & 1\end{bmatrix}(0,\tau)=\theta\begin{bmatrix}0 & 0 & 0\\ 1 & 1 & 0\end{bmatrix}(0,\tau)$$

(70) $$\theta\begin{bmatrix}0&1&0\\1&0&1\end{bmatrix}(0,\tau)=\exp\left(+\frac{\pi i}{4}\right)\theta\begin{bmatrix}1&1&0\\0&0&0\end{bmatrix}(0,\tau)$$
$$=\exp\left(+\frac{\pi i}{4}\right)\theta\begin{bmatrix}0&0&0\\1&1&0\end{bmatrix}(0,\tau).$$

By the action $M\circ$ of the cyclic subgroup of $Sp(3,Z)$ generated by M, 36 theta constants are classified in 6 orbits; $\theta\begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}(0,\pi)$ is M-invariant and other 5 orbits consist of 7 theta constants. Here we remark that no theta constant is zero. We can confirm this fact by numerical computations at the last part of this article.

Now we take

$$\theta\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}(0,\tau),\quad \theta\begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}(0,\tau),\quad \theta\begin{bmatrix}0&0&0\\0&0&1\end{bmatrix}(0,\tau)$$
$$\theta\begin{bmatrix}0&0&0\\1&0&1\end{bmatrix}(0,\tau),\quad \theta\begin{bmatrix}0&0&0\\1&1&0\end{bmatrix}(0,\tau),\quad \theta\begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}(0,\tau),$$

which are representatives of 6 orbits.

We put

$$\frac{\theta\begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}(0,\tau)}{\theta\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}(0,\tau)}=X,\quad \frac{\theta\begin{bmatrix}0&0&0\\0&0&1\end{bmatrix}(0,\tau)}{\theta\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}(0,\tau)}=Y,\quad \frac{\theta\begin{bmatrix}0&0&0\\1&0&1\end{bmatrix}(0,\tau)}{\theta\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}(0,\tau)}=Z$$

$$\frac{\theta\begin{bmatrix}0&0&0\\1&1&0\end{bmatrix}(0,\tau)}{\theta\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}(0,\tau)}=U,\quad \frac{\theta\begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}(0,\tau)}{\theta\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}(0,\tau)}=V$$

Substituting above representatives in theta identities ($\mathscr{I}$-1), …, ($\mathscr{I}$-9) in §2 and considering that no representative is zero, we obtain,

(71) $$Y^2=\frac{1-i}{\sqrt{2}}X^2ZU+\frac{1+i}{\sqrt{2}}Y^2ZU$$

(72) $$\frac{1+i}{\sqrt{2}}Z+U-V=0$$

(73) $$\frac{1+i}{\sqrt{2}}XZ - XU + YZU = 0$$

(74) $$V^2 + iX^2Y^2 = +2iY^2 + iX^2Z^2 + X^2U^2$$

(75) $$V^4 = -3Y^4 - Z^4 + U^4$$

(76) $$X^2V^2 + iY^2 = 2X^2 - Y^2Z^2 + iY^2U^2$$

(77) $$-Y^4 + iZ^2U^2 = 2X^2 + Y^2Z^2 - iY^2U^2$$

(78) $$1 + X^4 + 2Y^4 = 0$$

(79) $$(Y^2 + iX^2)(iZ^2 - U^2) = 0.$$

From (79), we obtain

$$Y^2 + iX^2 = 0 \quad \text{or} \quad iZ^2 - U^2 = 0$$

But if $iZ^2 - U^2 = 0$, then we have contradiction. In fact, if $U = ((1+i)/\sqrt{2})Z$, then from (73) we have $YZU = 0$ which contradicts because no representative is zero, if $U = -((1+i)/\sqrt{2})Z$, then from (72) we have $V = 0$ which contradicts, too. Therefore, we have

(80) $$Y^2 = -iX^2$$

and

(81) $$X^4 = 1.$$

From (71) and (80), we have

$$-iX^2 = \frac{1-i}{\sqrt{2}}X^2ZU + \frac{1-i}{\sqrt{2}}X^2ZU,$$

and since $X \neq 0$, we have

(82) $$ZU = \frac{1-i}{2\sqrt{2}}.$$

From (72) and (82),

$$V^2 = +iZ^2 + \sqrt{2}(1+i)ZU + U^2$$

or

(83) $$V^2 = 1 + iZ^2 + U^2$$

Thus

$$(V^2 - 1)^2 = (+iZ^2 + U^2)^2$$

or

(84) $$V^4 - 2V^2 + 1 = \frac{1}{2} - Z^4 + U^4.$$

From (75) and (80), we have

(85) $$V^4 = 3 - Z^4 + U^4.$$

Thus from (84) and (85), we obtain

(86) $$V^2 = \frac{7}{4}$$

(87) $$+iZ^2 + U^2 = \frac{3}{4}.$$

From (74), (80) and (81),

(88) $$V^2 + 1 = X^2(2 + iZ^2 + U^2).$$

From (86), (87) and (88), we obtain

(89) $$X^2 = 1$$

and from (80)

(90) $$Y^2 = -i.$$

Summing up these, we have

PROPOSITION. *Klein's surface has the following proportionalities of theta constants.*

$$X^2 = 1, \quad Y^2 = -i, \quad V^2 = \frac{7}{4}$$

$$ZU = \frac{1-i}{2\sqrt{2}}, \quad \frac{1+i}{\sqrt{2}}Z + U = V.$$

To decide the values of $X, \ldots, V$, there remains only the problem of determining which roots of unity or signs to take. This can only be decided by reference to the Fourier expansions of the theta constants, and we use numerical computations. For example, we will compute the value of X. We have

$$\theta\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}(0,\tau) = \sum_{(n_1,n_2,n_3)} \exp\pi i\left\{ + n_1n_2 + \frac{1}{2}(n_2+n_3)^2 + \frac{\sqrt{7}i}{14}(6n_1^2+5n_2^2+3n_3^2+6n_1n_2+4n_1n_3+2n_2n_3)\right\}$$

and

$$\theta\begin{bmatrix}0 & 0 & 0\\ 1 & 0 & 0\end{bmatrix}(0,\tau) = \sum_{(n_1,n_2,n_3)} \exp\pi i\left\{ + n_1 + n_1n_2 + \frac{1}{2}(n_2+n_3)^2 + \frac{\sqrt{7}i}{14}(6n_1^2+5n_2^2+3n_3^2+6n_1n_2+4n_1n_3+2n_2n_3)\right\}$$

where the explicit period τ is computed in §1.

Experience has shown (see Rauch and Lebowits, chapter IV; pp. 269–270) that very good accuracy in computing 1-theta constants is achieved by using only the terms at the "origin" and in the "first layer" i.e., altogether the terms for which $-1 \le n \le 1$ where n is the single summention index. The analogue for the triple series of the 3-theta constants is the "origin and the first layer" i.e., the twenty seven terms for which $-1 \le n_1, n_2, n_3 \le 1$. Using the values

$$\pi = 3.14159, \quad \sqrt{7} = 2.64575$$

we find

$$\theta\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}(0,\tau) \doteq 1.0565 + 0.4405i \quad \theta\begin{bmatrix}0 & 0 & 0\\ 1 & 0 & 0\end{bmatrix}(0,\tau) \doteq 1.0589 + 0.4389$$

hence we find

$$X = \frac{\theta\begin{bmatrix}0 & 0 & 0\\ 1 & 0 & 0\end{bmatrix}(0,\tau)}{\theta\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}(0,\tau)} \doteq \frac{1.0589 + 0.4389i}{1.0565 + 0.4405i} \doteq 1.1443 - 0.0070i$$

In the same way,

$$\theta\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}(0,\tau) \doteq 1.0572 - 0.4406i \quad \theta\begin{bmatrix}0 & 0 & 0\\ 1 & 0 & 1\end{bmatrix}(0,\tau) \doteq 0.8112 - 0.0277i$$

$$\theta\begin{bmatrix}0 & 0 & 0\\ 1 & 1 & 0\end{bmatrix}(0,\tau) \doteq 0.8112 - 0.0028i \quad \theta\begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}(0,\tau) \doteq 1.3906 + 0.5760i$$

Hence we find

$$Y=\frac{\theta\begin{bmatrix}0&0&0\\0&0&1\end{bmatrix}(0,\tau)}{\theta\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}(0,\tau)}\doteq\frac{1.0572-0.4406i}{1.0565+0.4405i}\doteq 0.8056-0.8057i$$

$$Z=\frac{\theta\begin{bmatrix}0&0&0\\1&0&1\end{bmatrix}(0,\tau)}{\theta\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}(0,\tau)}\doteq\frac{0.8112-0.0277i}{1.0565+0.4405i}\doteq 0.7381-0.3378i$$

$$U=\frac{\theta\begin{bmatrix}0&0&0\\1&1&0\end{bmatrix}(0,\tau)}{\theta\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}(0,\tau)}\doteq\frac{0.8112-0.0028i}{1.0565+0.4405i}\doteq 0.8558-0.3603i$$

$$V=\frac{\theta\begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}(0,\tau)}{\theta\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}(0,\tau)}\doteq\frac{1.3906+0.5760i}{1.0565+0.4405i}\doteq 1.5051-0.0036i$$

We return to Proposition. From the numerical computations, we may conclude

$$X=1,\quad Y=\frac{1-i}{\sqrt{2}},\quad Z=\frac{1+\sqrt{7}+(1-\sqrt{7})i}{4\sqrt{2}},$$

$$U=\frac{\sqrt{7}-i}{4},\quad V=\frac{\sqrt{7}}{2}$$

We collect these results.

THEOREM B. *Klein's surface has the following proportionalities of theta constants;*

$$\frac{\theta\begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}(0,\tau)}{\theta\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}(0,\tau)}=X=1,\quad \frac{\theta\begin{bmatrix}0&0&0\\0&0&1\end{bmatrix}(0,\tau)}{\theta\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}(0,\tau)}=Y=\frac{1-i}{\sqrt{2}},$$

$$\frac{\theta\begin{bmatrix}0&0&0\\1&0&1\end{bmatrix}(0,\tau)}{\theta\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}(0,\tau)}=Z=\frac{1+\sqrt{7}+(1-\sqrt{7})i}{4\sqrt{2}},\quad \frac{\theta\begin{bmatrix}0&0&0\\1&1&0\end{bmatrix}(0,\tau)}{\theta\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}(0,\tau)}=U=\frac{\sqrt{7}-i}{4},$$

$$\frac{\theta\begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}(0,\tau)}{\theta\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}(0,\tau)}=V=\frac{\sqrt{7}}{2}$$

and other

$$\frac{\theta\begin{bmatrix}\varepsilon\\\varepsilon'\end{bmatrix}(0,\tau)}{\theta\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}(0,\tau)}$$

is decided by the relations (40), ..., (70).

REMARK 1. We see that

$$\frac{\theta\begin{bmatrix}\varepsilon\\\varepsilon'\end{bmatrix}(0,\tau)}{\theta\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}(0,\tau)}\in Q(\sqrt{2},\sqrt{7},i).$$

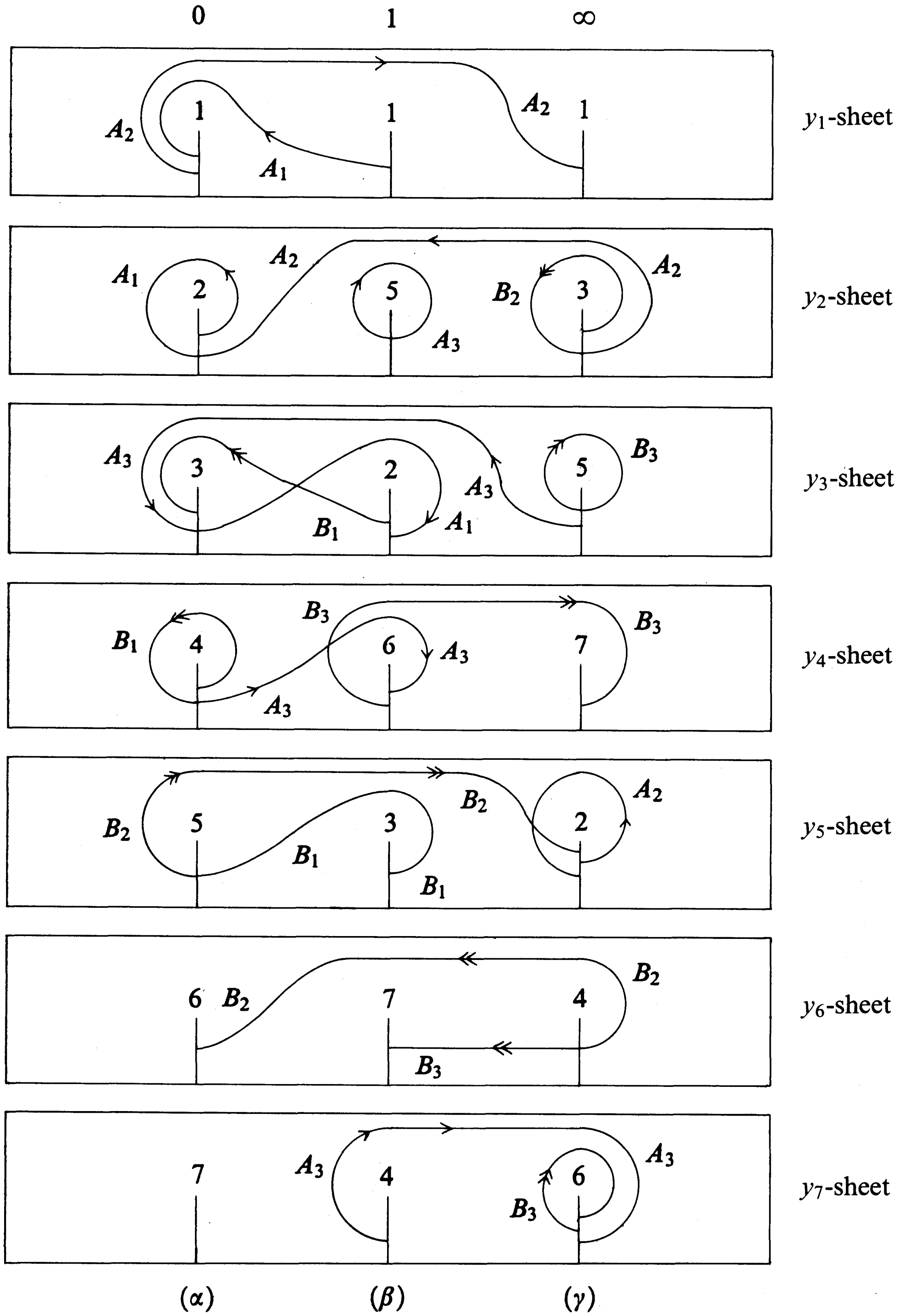
REMARK 2. Moreover we see that

$$X^2=1,\quad Y^2=-i,\quad Z^2=\frac{\sqrt{7}-3i}{8},\quad U^2=\frac{3-\sqrt{7}i}{8},\quad V^2=\frac{7}{4}$$

Thus

$$\frac{\theta^2\begin{bmatrix}\varepsilon\\\varepsilon'\end{bmatrix}(0,\tau)}{\theta^2\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}(0,\tau)}\in Q(\sqrt{7},i).$$

Appendix


0
1
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A_2
A_1
y_1-sheet
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B_2
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y_4-sheet
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A_2
y_5-sheet
B_2
B_3
y_6-sheet
A_3
B_3
y_7-sheet
(α)
(β)
(γ)

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CORRIGENDUM TO "A ZERO DENSITY ESTIMATE FOR DEDEKIND ZETA FUNCTIONS OF PURE EXTENSION FIELDS"

By

Koichi KAWADA

The introduction of the recent article [1] contains an erroneous account on the reducibility of the polynomial $x^k - n$, and the following description should be substituted for the last paragraph on p. 357 of [1]:

"Let $\mathscr{I}_k$ be the set of all the integers n such that $x^k - n$ is irreducible in $\boldsymbol{Q}[x]$. One may show that if $x^k - n$ is reducible then $|n|$ is a p-th power of an integer for some $p|k$, namely,

$$\mathscr{I}_k \supset \boldsymbol{Z} \Big\backslash \left(\bigcup_{p|k} \boldsymbol{Z}^p \right),$$

where $\boldsymbol{Z}$ is the integer ring, and $\boldsymbol{Z}^p$ denotes the set of all the integers of the form $\pm m^p$ with $m \in \boldsymbol{Z}$."

This change does not influence the central argument of [1] at all, and its body remains valid.

The author is deeply obliged to Professor Gerhard Turnwald for his indication on the above matter, and for the following information (see Ch. VI, §9, Theorem 9.1 of Lang [2]). When $4 \nmid k$, the polynomial $x^k - n$ is reducible if, and only if, n is a p-th power of an integer for some prime $p|k$. When $4|k$, however, it is reducible if, and only if, n is a p-th power of an integer for some $p|k$, or n is of the form $-4m^4$ with an integer m. Therefore the relation (1) in [1] is correct only when $4 \nmid k$, and fails to hold when $4|k$.

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