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CERTAIN CHARACTERIZATIONS OF REAL HYPERSURFACES OF TYPE A IN A COMPLEX SPACE FORM

By

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§ 1. Introduction

A complex n-dimensional Kähler manifold of constant holomorphic sectional curvature c is called a *complex space form*, which is denoted by $M_n(c)$. A complete and simply connected complex space form consists of a complex projective space P_nC, a complex Euclidean space C^n or a complex hyperbolic space H_nC, according as $c > 0$, $c = 0$ or $c < 0$.

In this paper, we consider a real hypersurface M in $M_n(c)$. Typical examples of M in P_nC are the six model spaces of type A_1, A_2, B, C, D and E (cf. [10]), and the ones of M in H_nC are the four model spaces of type A_0, A_1, A_2 and B (cf. [1]), which are all given as orbits under certain Lie subgroups of the group consisting of all isometries of P_nC or H_nC. Denote by (ϕ, ξ, η, g) the *almost contact metric structure* of M induced from the almost complex structure of $M_n(c)$ and A the shape operator of M. Eigenvalues and einvectors of A are called *principal curvatures and principal vectors*, respectively.

Many differential geometers have studied M from various points of view. In particular, Berndt [1] and Takagi [10] investigated the homogeneity of M. According to Takagi's classification theorem and Berndt's one, the principal curvatures and their multiplicities of homogeneous real hypersurfaces in $M_n(c)$ are given. Moreover, it is very interesting to characterize homogeneous real hypersurfaces of $M_n(c)$. There are many characterizations of homogeneous ones of type A since these examples have a lot of beautiful geometric properties, where *type A* means type A_1 or A_2 in P_nC and type A_0, A_1 or A_2 in H_nC. Okumura [8] and Montiel-Romero [7] proved the fact in P_nC and H_nC, respectively that M

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satisfies $A\phi = \phi A$ if and only if M is locally congruent to type A. The following theorem is proved by Kimura and Maeda [4] and Ki, Kim and Lee [2] for M in $P_n C$ and $H_n C$, respectively.

THEOREM A. *Let M be a real hypersurface of $M_n(c)$, $c \neq 0$, $n \geq 2$. If it satisfies*

$$\nabla_\xi A = 0, \quad g(A\xi, \xi) \neq 0,$$

then M is of type A, where ∇ is the Riemannian connection on M.

In his previous paper [9], the third named auther proved the following

THEOREM B. *Let M be a real hypersurface of $M_n(c)$, $c \neq 0$, $n \geq 2$. If it satisfies*

$$\nabla_\xi A = a(A\phi - \phi A), \quad 2a \neq -g(A\xi, \xi)$$

for some non-zero constant a, then M is of type A.

Motivated by these results, in this article we will give a generalization of Theorems A and B and another characterizations of homogeneous real hypersurfaces of type A in $M_n(c)$. The purpose of this paper is to prove the following

THEOREM 1. *Let M be a real hypersurface of $M_n(c)$, $c \neq 0$, $n \geq 2$. If it satisfies*

$$\nabla_\xi A = f(A\phi - \phi A) - df(\xi)I, \quad 2f \neq -g(A\xi, \xi) \tag{1.2}$$

for a smooth function f, where I denotes the identity transformation, then M is of type A.

THEOREM 2. *Let M be a real hypersurface of $M_n(c), c \neq 0, n \geq 2$. If it satisfies*

$$\mathscr{L}_\xi (H + fg) = 0, \quad 2f \neq -g(A\xi, \xi) \tag{1.3}$$

for a smooth function f, then M is of type A, where $\mathscr{L}_\xi$ is the Lie derivative with respect to ξ and H is the second fundamental form of M in $M_n(c)$, namely $H(X, Y) = g(AX, Y)$ for any vecror fields X and Y.

§ 2. Preliminaries

First of all, we recall fundamental properties about real hypersurfaces of a complex space form. Let M be a real hypersurface of a complex n-dimensional complex space form $M_n(c)$ of constant holomorphic sectional curvature c, and let C be a unit normal vector field on a neighborhood in M. We denote by J the almost complex structure of $M_n(c)$. For a local vector field X on the neighborhood in M, the images of X and C under the linear transformation J can be represented as

$$JX = \phi X + \eta(X)C, \quad JC = -\xi,$$

where ϕ defines a skew-symmetric transformation on the tangent bundle TM of M, while η and ξ denote a 1-form and a vector field on the neighborhood in M, respectively. Then it is seen that $g(\xi, X) = \eta(X)$, where g denotes the Riemannian metric tensor on M induced from the metric tensor on $M_n(c)$. The set of tensors (ϕ, ξ, η, g) is called an *almost contact metric structure* on M. They satisfy the following properties:

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta(\xi) = 1,$$

where I denotes the identity transformation. Furthermore the covariant derivatives of the structure tensors are given by

$$\nabla_X \xi = \phi AX, \quad \nabla_X \phi(Y) = \eta(Y)AX - g(AX, Y)\xi \tag{2.1}$$

for any vector fields X and Y on M, where ∇ is the Riemannian connection on M and A denotes the shape operator of M in the direction of C.

Since the ambient space is of constant holomorphic sectional curvature c, the equations of Gauss and Codazzi are respectively given as follows:

$$\begin{aligned} R(X, Y)Z = \frac{c}{4}\{&g(Y, Z)X - g(X, Z)Y \\ &+ g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\} \\ &+ g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \tag{2.2}$$

$$\nabla_X A(Y) - \nabla_Y A(X) = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\}, \tag{2.3}$$

where R denotes the Riemannian curvature tensor of M and $\nabla_X A$ denotes the covariant derivative of the shape operator A with respect to X.

Next, we suppose that the structure vector field ξ is principal with the corresponding principal curvature α, namely $A\xi = \alpha\xi$. Then it is seen in [3] and

[6] that α is constant on M and it satisfies

$$2A\phi A = \frac{c}{2}\phi + \alpha(A\phi + \phi A). \tag{2.4}$$

§3. Proof of Theorems

Let M be a real hypersurface of $M_n(c)$, $c \neq 0$, $n \geq 2$. First of all, we shall give a sufficient condition for the structure vector field ξ to be principal. We suppose that ξ is principal, i.e., $A\xi = \alpha\xi$, where α is constant. Then, by (2.1) and (2.4), we get

$$\nabla_X A(\xi) = -\frac{c}{4}\phi X - \frac{1}{2}\alpha(A\phi - \phi A)X,$$

from which together with (2.3) it follows that

$$\nabla_\xi A = -\frac{1}{2}\alpha(A\phi - \phi A). \tag{3.1}$$

Taking account of this property and the already known some theorems, in order to prove our theorems, we shall assert the following

Proposition 3.1. *Let M be a real hypersurface of $M_n(c)$, $c \neq 0$, $n \geq 2$. If it satisfies*

$$\nabla_\xi A = f(A\phi - \phi A) - df(\xi)I \tag{3.2}$$

for a smooth function f, then ξ is principal, and hence $df(\xi) = 0$.

By the assumption (3.2) and (2.3), it turns out to be

$$\nabla_Y A(\xi) = f(A\phi - \phi A)Y - df(\xi)Y - \frac{c}{4}\phi Y. \tag{3.3}$$

Differentiating this equation with respect to X covariantly and using (2.1), we get

$$\begin{aligned} \nabla_X\nabla_Y A(\xi) = {} & f\{\nabla_X A(\phi Y) + g(Y,\xi)A^2X - g(AX, Y)A\xi \\ & - g(AY,\xi)AX + g(AX,AY)\xi - \phi\nabla_X A(Y)\} \\ & - \frac{c}{4}\{g(Y,\xi)AX - g(AX, Y)\xi\} - \nabla_Y A(\phi AX) \\ & + df(X)(A\phi - \phi A)Y \end{aligned} \tag{3.4}$$

for any vector fields X and Y. Since the Ricci formula for the shape operator A is given by

$$\nabla_X\nabla_Y A(Z) - \nabla_Y\nabla_X A(Z) = R(X, Y)(AZ) - A(R(X, Y)Z),$$

it follows from (2.2), (2.3) and (3.4) that

$$\begin{aligned}(3.5)\quad & \nabla_X A(\phi AY) - \nabla_Y A(\phi AX) + f\{\nabla_X A(\phi Y) - \nabla_Y A(\phi X)\} \\ &= -\{fg(Y,\xi) + g(AY,\xi)\}A^2X + \{fg(X,\xi) + g(AX,\xi)\}A^2Y \\ &+ \{fg(AY,\xi) + g(A^2Y,\xi)\}AX - \{fg(AX,\xi) + g(A^2X,\xi)\}AY \\ &+ \frac{c}{4}\{fg(Y,\xi) + g(AY,\xi)\}X - \frac{c}{4}\{fg(X,\xi) + g(AX,\xi)\}Y \\ &+ \frac{c}{4}\{g(A\phi Y,\xi)\phi X - g(A\phi X,\xi)\phi Y\} - \frac{c}{2}g(\phi X, Y)\phi A\xi \\ &+ df(Y)(A\phi - \phi A)X - df(X)(A\phi - \phi A)Y\end{aligned}$$

for any vector fields X and Y.

Now, in order to prove Proposition 3.1, we shall express (3.5) in the simpler form. The inner product of (3.5) and ξ, combining with (2.3) and (3.2), implies

$$\begin{aligned}(3.6)\quad & fg((A\phi A\phi - \phi A\phi A)X, Y) \\ &+ f^2\{g(X,\xi)g(AY,\xi) - g(Y,\xi)g(AX,\xi)\} \\ &- df(\xi)\{g((A\phi + \phi A)X, Y) + 2fg(\phi X, Y)\} \\ &+ f\{g(X,\xi)g(A^2Y,\xi) - g(Y,\xi)g(A^2X,\xi)\} \\ &+ 2\{g(AX,\xi)g(A^2Y,\xi) - g(AY,\xi)g(A^2X,\xi)\} \\ &- df(X)g(A\phi Y,\xi) + df(Y)g(A\phi X,\xi) = 0\end{aligned}$$

for any vector fields X and Y. Since Y is arbitrary, we get

$$\begin{aligned}&\{f(A\phi A\phi - \phi A\phi A) - df(\xi)(A\phi + \phi A)\}X - 2fdf(\xi)\phi X \\ &+ \{fg(X,\xi) + 2g(AX,\xi)\}A^2\xi + \{f^2g(X,\xi) \\ &- 2g(A^2X,\xi)\}A\xi - f\{fg(AX,\xi) + g(A^2X,\xi)\}\xi \\ &+ df(X)\phi A\xi + g(A\phi X,\xi)\nabla f = 0\end{aligned}$$

for any vector field X, where we denote by ∇f the gradient of the function f. On the other hand, taking account of (2.1) and the skew-symmetry of the transformation ϕ, we have

$$g((A\phi A\phi - \phi A\phi A)X, \phi X) = g(X, \xi)g(A\phi AX, \xi). \tag{3.7}$$

Putting $Y = \phi X$ in (3.6) and applying the above property, we get

$$\begin{aligned} fg(X,\xi)\{g(A\phi AX,\xi) + fg(A\phi X,\xi) + g(A^2\phi X,\xi)\} \\ + 2\{g(AX,\xi)g(A^2\phi X,\xi) - g(A\phi X,\xi)g(A^2X,\xi)\} \\ - df(\xi)\{g((A\phi + \phi A)X,\phi X) + 2fg(\phi X,\phi X)\} \\ - df(X)g(A\phi^2X,\xi) + df(\phi X)g(A\phi X,\xi) = 0. \end{aligned} \tag{3.8}$$

Let T_0 be the distribution defined by the subspace $T_0(x) = \{u \in T_xM : g(u, \xi(x)) = 0\}$ of the tangent space T_xM of M at any point x, called *a holomorphic distribution*.

Now, suppose that the structure vector field ξ is not principal. Then we can put $A\xi = \alpha\xi + \beta U$, where U is a unit vector field in the holomorphic distribution T_0, and α and β are smooth functions on M. So we may consider the case that the function β does not vanish identically on M. Let M_0 be the non-empty open subset of M consisting of points x at which $\beta(x) \neq 0$. And we put $AU = \beta\xi + \gamma U + \delta V$, where U and V are orthonormal vector fields in T_0, and γ and δ are smooth functions on M_0. And let $L(\xi, U)$ be a distribution spanned by ξ and U.

For any vector field X belonging to the holomorphic distribution T_0, (3.8) can be simplified as

$$\begin{aligned} 2\{g(AX,\xi)g(A^2\phi X,\xi) - g(A\phi X,\xi)g(A^2X,\xi)\} \\ - df(\xi)\{g((A\phi + \phi A)X,\phi X) + 2fg(\phi X,\phi X)\} \\ + \beta\{df(X)g(X,U) + df(\phi X)g(\phi X,U)\} = 0. \end{aligned}$$

Furthermore, we can see that this equation holds for any vector field X. By the polarization of the above equation, we have

$$\begin{aligned} 2\{g(AX,\xi)g(A^2\phi Y,\xi) - g(A\phi X,\xi)g(A^2Y,\xi) \\ + g(AY,\xi)g(A^2\phi X,\xi) - g(A\phi Y,\xi)g(A^2X,\xi)\} \\ - df(\xi)\{g((A\phi + \phi A)X,\phi Y) + g((A\phi + \phi A)Y,\phi X) \\ + 4fg(\phi X,\phi Y)\} + \beta\{df(X)g(Y,U) + df(\phi X)g(\phi Y,U) \\ + df(Y)g(X,U) + df(\phi Y)g(\phi X,U)\} = 0 \end{aligned}$$

for any vector fields X and Y. Hence we have

$$\begin{aligned}(3.9)\qquad & df(\xi)\{\phi(A\phi+\phi A)X+(A\phi+\phi A)\phi X+4f\phi^2X\} \\ & -2\{g(AX,\xi)\phi A^2\xi+g(A\phi X,\xi)A^2\xi-g(A^2\phi X,\xi)A\xi \\ & -g(A^2X,\xi)\phi A\xi\}+\beta\{df(X)U-df(\phi X)\phi U \\ & +g(X,U)\nabla f+g(\phi X,U)df(\phi I)\}=0.\end{aligned}$$

First, in order to prove Proposition 3.1, we shall assert the following

LEMMA 3.2. *The distribution $L(\xi, U)$ is A-invariant on M_0, namely we have*

$$AU=\beta\xi+\gamma U \tag{3.10}$$

on M_0.

PROOF. On the open subset M_0, by the forms $A\xi=\alpha\xi+\beta U$ and $AU=\beta\xi+\gamma U+\delta V$, it turns out to be

$$A^2\xi=(\alpha^2+\beta^2)\xi+\beta(\alpha+\gamma)U+\beta\delta V.$$

Thus we can rewrite (3.9) as

$$\begin{aligned}(3.11)\qquad & df(\xi)\{\phi(A\phi+\phi A)X+(A\phi+\phi A)\phi X+4f\phi^2X\} \\ & +2\{\alpha g(A^2\phi X,\xi)-(\alpha^2+\beta^2)g(A\phi X,\xi)\}\xi \\ & +2\beta\{g(A^2\phi X,\xi)-(\alpha+\gamma)g(A\phi X,\xi)\}U-2\beta\delta g(A\phi X,\xi)V \\ & +2\beta\{g(A^2X,\xi)-(\alpha+\gamma)g(AX,\xi)\}\phi U-2\beta\delta g(AX,\xi)\phi V \\ & +\beta\{df(X)U-df(\phi X)\phi U+g(X,U)\nabla f+g(\phi X,U)\,df(\phi I)\} \\ & =0\end{aligned}$$

for any vector field X. The inner product of (3.11) and ξ implies that

$$\alpha g(\phi X,A^2\xi)-(\alpha^2+\beta^2)g(\phi X,A\xi)=0$$

for any vector field X. This gives us

$$\alpha A^2\xi-(\alpha^2+\beta^2)A\xi=0$$

on M_0 and hence we have

$$\beta\{(\alpha\gamma-\beta^2)U+\alpha\delta V\}=0.$$

Consequently, we have

$$\beta^2 = \alpha\gamma, \quad \delta = 0 \tag{3.12}$$

on M_0. So it completes the proof. ■

Furthermore, by (3.12), we also get

$$A^2\xi = (\alpha + \gamma)A\xi \tag{3.13}$$

on M_0.

Let M' be a closed subset in M containing all points x where $f(x) = 0$. Suppose that $M_0 - M'$ is not empty. Then we have the following

LEMMA 3.3. *If* (3.2) *is satisfied, then we have*

$$A\phi U = -\lambda\phi U, \quad \lambda = f + \alpha + \gamma \tag{3.14}$$

on $M_0 - M'$.

PROOF. By using the polarization of (3.8) together with (3.13), we have

$$\begin{aligned}
&fg(X,\xi)\{g(A\phi AY,\xi) + fg(A\phi Y,\xi) + g(A^2\phi Y,\xi)\} \\
&\quad + fg(Y,\xi)\{g(A\phi AX,\xi) + fg(A\phi X,\xi) + g(A^2\phi X,\xi)\} \\
&\quad - df(\xi)\{g((A\phi + \phi A)X,\phi Y) + 4fg(\phi X,\phi Y) + g((A\phi + \phi A)Y,\phi X)\} \\
&\quad - df(X)g(A\phi^2 Y,\xi) + df(\phi X)g(A\phi Y,\xi) \\
&\quad - df(Y)g(A\phi^2 X,\xi) + df(\phi Y)g(A\phi X,\xi) = 0
\end{aligned}$$

for any vector fields X and Y. Putting $Y = \xi$, we have

$$f\{g(A\phi AX,\xi) + fg(A\phi X,\xi) + g(A^2\phi X,\xi)\} = 0$$

because $A\phi A\xi$ is orthogonal to ξ. Since f has no zero points on $M_0 - M'$, we have

$$A\phi A\xi + f\phi A\xi + \phi A^2\xi = 0.$$

This equation, by (3.13), completes the proof. ■

Next, we give the following

LEMMA 3.4. *Assume that* $A^2\xi + hA\xi = 0$, *where* h *is a smooth function on* $M_0 - M'$. *Then it satisfies*

$$f\lambda^2 + \left(4f\gamma - 2h\gamma + \frac{c}{4}\right)\lambda - f^2\gamma - \frac{c}{4}(2h + 2\alpha + \gamma) - \beta dh(\phi U) = 0 \tag{3.15}$$

on $M_0 - M'$.

PROOF. Differentiating our assumption $A^2\xi + hA\xi = 0$ with respect to X and taking account of (2.1), (2.3) and (3.3), we get

$$\begin{aligned}&\nabla_X A(A\xi) + fA(A\phi - \phi A)X + fh(A\phi - \phi A)X + A^2\phi AX \\ &\quad + hA\phi AX - df(\xi)(AX + hX) - \frac{c}{4}A\phi X - \frac{c}{4}h\phi X + dh(X)A\xi = 0\end{aligned}$$

for any vector field X. The inner product of this equation with any vector field Y implies

$$\begin{aligned}&g(\nabla_X A(Y), A\xi) + fg(A(A\phi - \phi A)X, Y) + fhg((A\phi - \phi A)X, Y) \\ &\quad + g(A^2\phi AX, Y) + hg(A\phi AX, Y) - df(\xi)g(AX + hX, Y) \\ &\quad - \frac{c}{4}g(A\phi X, Y) - \frac{c}{4}hg(\phi X, Y) + dh(X)g(A\xi, Y) = 0.\end{aligned}$$

Exchanging X and Y in the above equation and substituting the second one from the first one, we have

$$\begin{aligned}&g(\nabla_X A(Y) - \nabla_Y A(X), A\xi) + fg((A^2\phi - 2A\phi A + \phi A^2)X, Y) \\ &\quad + g((A^2\phi A + A\phi A^2)X, Y) + 2hg(A\phi AX, Y) \\ &\quad - \frac{c}{4}g((A\phi + \phi A)X, Y) - \frac{c}{2}hg(\phi X, Y) \\ &\quad + dh(X)g(A\xi, Y) - dh(Y)g(A\xi, X) = 0\end{aligned}$$

for any vector fields X and Y. Putting $X = U$ and $Y = \phi U$ in this equation and taking account of (2.3), (3.10), (3.12) and (3.14), we can easily see that the equation (3.15) holds. ■

Now, we are in position to prove Proposition 3.1, namely to prove the fact that under the condition (3.2), the structure vector ξ is principal. We suppose that the open set $M_0 - M'$ is not empty. Then, differentiating the form $A\xi = \alpha\xi + \beta U$

with respect to ξ covariantly on $M_0 - M'$, we have by (2.1)

$$\nabla_\xi A(\xi) = d\alpha(\xi)\xi + \alpha\beta\phi U + d\beta(\xi)U - \beta A\phi U + \beta\nabla_\xi U.$$

This, combining with the assumption (3.2) and (3.14), implies

$$d(f+\alpha)(\xi)\xi + d\beta(\xi)U + \beta(2f + 2\alpha + \gamma)\phi U + \beta\nabla_\xi U = 0.$$

From the inner product of ξ and U respectively, we get

$$\nabla_\xi U = -(2f + 2\alpha + \gamma)\phi U, \quad d(f+\alpha)(\xi) = 0, \quad d\beta(\xi) = 0 \tag{3.16}$$

on $M_0 - M'$, where we have used that $g(\nabla_\xi U, \xi) = 0$ and $g(\nabla_\xi U, U) = 0$. By making use of (3.2) and (3.10), $\gamma = g(AU, U)$ gives us $d\gamma(\xi) = -df(\xi)$. Therefore, from (3.14) and (3.16), we get $d\lambda(\xi) = -df(\xi)$. Differentiating (3.14) with respect to ξ covariantly, and taking account of (2.1) and the above property, we get

$$\nabla_\xi A(\phi U) - g(AU,\xi)A\xi - \lambda g(AU,\xi)\xi + (A\phi + \lambda\phi)\nabla_\xi U - df(\xi)\phi U = 0.$$

By (3.2), (3.10), (3.12), (3.14) and the first equation of (3.16), the above equation gives the following

$$(f + \alpha + \gamma)(f + 2\alpha + 2\gamma) = 0, \quad df(\xi) = 0 \tag{3.17}$$

on $M_0 - M'$. Since $f \neq 0$, we have $\alpha + \gamma \neq 0$ on $M_0 - M'$ by the above equation.

Now, we consider the first case $f + \alpha + \gamma = 0$. By (3.14) and (3.16), we get

$$A\phi U = 0, \quad \nabla_\xi U = \gamma\phi U. \tag{3.18}$$

Differentiating $A\xi = \alpha\xi + \beta U$ with respect to any vector field X covariantly, and taking account of (2.1), (3.3) and the second equation of (3.17), we get

$$f(A\phi - \phi A)X - \frac{c}{4}\phi X + A\phi AX - d\alpha(X)\xi - \alpha\phi AX - d\beta(X)U - \beta\nabla_X U = 0.$$

By taking the inner product of this equation with ξ and U respectively, we get

$$d\alpha(X) = f\beta g(\phi X, U), \quad d\beta(X) = \left(f\gamma - \frac{c}{4}\right)g(\phi X, U), \tag{3.19}$$

where we have used (3.10) and the first equation of (3.18). Owing to $\beta^2 = \alpha\gamma$, it is easily seen that

$$2\beta d\beta(X) = \gamma d\alpha(X) + \alpha d\gamma(X),$$

from which together with (3.19), it turns out to be

$$\beta\left(f\alpha + f\gamma - \frac{c}{2}\right)g(\phi X, U) + \alpha df(X) = 0$$

for any vector field X, where we have used $f + \alpha + \gamma = 0$. This implies $\beta(f^2 + c/2) + \alpha df(\phi U) = 0$. Hence, by the first equation of (3.12) and (3.15), we get $\beta = 0$ on $M_0 - M'$, where we have used that $\lambda = 0$ and $h = f$. It is a contradiction.

Lastly, we suppose that $f + 2\alpha + 2\gamma = 0$ in the first equation of (3.17). Putting $X = \xi$ and $Y = U$ in (3.5) and from the inner product of ξ and U respectively, we obtain

$$\beta g(\phi\nabla_U U, U) = (f + \gamma)(f + \alpha + \gamma) + \gamma(f + \alpha) + \frac{c}{4}$$

and

$$\beta(f + \alpha + 2\gamma)g(\phi\nabla_U U, U) = f(f + 2\gamma)(f + \alpha + \gamma) + \gamma^2(f + \alpha) - \frac{c}{4}(f + \alpha),$$

where we have used (3.2), (3.10), (3.13), (3.14), (3.16) and $df(\xi) = d\gamma(\xi) = 0$. Combining the above two equations, we get

$$(f + \alpha + \gamma)\left(f\alpha + 2f\gamma + 2\alpha\gamma + 2\gamma^2 + \frac{c}{2}\right) = 0.$$

By the supposed condition $f + 2\alpha + 2\gamma = 0$, we have $f^2 = c$. Therefore, we obtain $\alpha = 0$, where we have used (3.15), $f + 2\alpha + 2\gamma = 0$ and $h = \lambda = f/2$. Hence $\beta = 0$ on $M_0 - M'$ by the first equation of (3.12). It is also a contradition.

Consequently, these two cases mean that the subset $M_0 - M'$ is empty and hence the subset M_0 is contained in the subset M'. Hence it satisfies

$$\nabla_\xi A = 0, \quad g(A\xi, \xi) \neq 0$$

on M_0. Since Theorem A is a local property, we see that the structure vector field ξ is principal on M_0. Then it is a contradiction. Therefore the subset M_0 of M is empty and hence ξ is principal on M. Thus, comparing (3.1) with (3.2), we get $df(\xi) = 0$. It completes the proof of Proposition 3.1. ■

The following is immediate from Proposition 3.1.

COROLLARY 3.5. *Let M be a real hypersurface of $M_n(c)$, $c \neq 0$, $n \geq 2$. If it satisfies $\nabla_\xi A = 0$, then ξ is principal.*

REMARK. Kimura and Maeda [4] proved Corollary 3.5 in the case where $c > 0$.

PROOF OF THEOREM 1. By Proposition 3.1, the structure vector ξ is principal and $df(\xi) = 0$. Combining (3.1) with the assumption (1.2) of Theorem 1, we have

$$(2f + \alpha)(A\phi - \phi A) = 0,$$

which implies that $A\phi - \phi A = 0$ by the assumption. Thus, the real hypersurface M is of type A. ■

PROOF OF THEOREM 2. Since $\mathscr{L}_\xi(H + fg)(X, Y) = g(\nabla_\xi A(X), Y) - fg((A\phi - \phi A)X, Y) + df(\xi)g(X, Y)$ for any vector fields X and Y, by the assumption (1.3) of Theorem 2, we have

$$\nabla_\xi A = f(A\phi - \phi A) - df(\xi)I.$$

Hence Theorem 2 is proved by Theorem 1. ■

REMARK. Theorem B which was introduced in § 1 can be obtained by Theorem 1.

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