TSUKUBA J. MATH.
Vol. 23 No. 2 (1999), 235–243

REAL HYPERSURFACES OF A COMPLEX PROJECTIVE SPACE

By

Syuji NAKAJIMA

1. Introduction

Let $P_n(C)$ be an n-dimensional complex projective space with Fubini-Study metric of constant holomorphic sectional curvature 4. Typical examples of real hypersurface in $P_n(C)$ are homogeneous ones. R. Takagi ([8]) showed that all homogeneous real hypersurfaces in $P_n(C)$ are realized as the tubes of constant radius over compact Hermitian symmetric spaces of rank 1 or 2. Namely, he proved the following

THEOREM 1.1. *Let M be a homogeneous real hypersurface of $P_n(C)$. Then M is locally congruent to one of the following*:

(A_1) *a geodesic hypersphere* (*that is, a tube over a hyperplane* $P_{n-1}(C)$),
(A_2) *a tube over a totally geodesic* $P_k(C)$ $(1 \leq k \leq n-2)$,
(B) *a tube over a complex quadric* Q_{n-1},
(C) *a tube over* $P_1(C) \times P_{(n-1)/2}(C)$ *and* $n(\geq 5)$ *is odd*,
(D) *a tube over a complex Grassmann* $G_{2,5}(C)$ *and* $n = 9$,
(E) *a tube over a Hermitian symmetric space* $SO(10)/U(5)$ *and* $n = 15$.

On the other hand, many differential geometers have studied real hypersurfaces in $P_n(C)$ by making use of the almost contact metric structure induced from the complex structure J of $P_n(C)$ (see, §2). It is well-known that there does not exist a real hypersurface with parallel second fundamental tensor of $P_n(C)$. Moreover Hamada ([2]) showed that there does not exist a real hypersurface with recurrent second fundamental tensor A of $P_n(C)$, i.e., there exists a 1-form α such that $\nabla A = A \otimes \alpha$. In this paper we consider the weaker condition.

The second fundamental tensor A is called a birecurrent second fundamental tensor if there exists a covariant tensor field α of order 2 such that $\nabla^2 A = A \otimes \alpha$.

Received April 14, 1998.
Revised December 21, 1998.

We may regard the parallel condition and the recurrent condition as a special case. First, we show the nonexistence of real hypersurfaces with birecurrent second fundamental tensor of $P_n(\boldsymbol{C})$.

Next, we characterize a homogeneous real hypersurface of type (A_1) and (A_2) under the condition that the structure vector is principal (see, §2).

The author would like to express his sincere gratitude to Professor Y. Matsuyama for his valuable suggestions during the preparation of this paper.

2. Preliminaries

Let M be an orientable real hypersurface of $P_n(\boldsymbol{C})$ and let N be a unit normal vector field on M. The Riemannian connections $\tilde{\nabla}$ in $P_n(\boldsymbol{C})$ and ∇ in M are related by the following formulas for any vector fields X and Y on M:

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \tag{1}$$

$$\tilde{\nabla}_X N = -AX, \tag{2}$$

where g denote the Riemannian metric of M induced from the Fubini-Study metric G of $P_n(\boldsymbol{C})$ and A is the second fundamental tensor of M in $P_n(\boldsymbol{C})$. An eigenvector of A is called a *principal curvature vector*. Also an eigenvalue of A is called a *principal curvature*. It is known that M has an almost contact metric structure induced from the complex structure J on $P_n(\boldsymbol{C})$, that is, we define a tensor field ϕ of type (1,1), a vector field ξ and a 1-form η on M by $g(\phi X, Y) = G(JX, Y)$ and $g(\xi, X) = \eta(X) = G(JX, N)$. Then we have

$$\phi^2 X = -X + \eta(X)\xi, \quad g(\xi, \xi) = 1, \quad \phi\xi = 0. \tag{3}$$

It follows from (1) that

$$(\nabla_X \phi)Y = \eta(X)AX - g(AX, Y)\xi, \tag{4}$$

$$\nabla_X \xi = \phi AX. \tag{5}$$

Let $\tilde{R}$ and R be the curvature tensors of $P_n(\boldsymbol{C})$ and M, respectively. Then we have the following Gauss and Codazzi equations:

$$\begin{aligned} R(X, Y)Z = {} & g(Y,Z)X - g(X,Z)Y + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y \\ & - 2g(\phi X, Y)\phi Z + g(AY,Z)AX - g(AX,Z)AY, \end{aligned} \tag{6}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi. \tag{7}$$

In the following, we use the same terminology and notations as above unless otherwise stated. Now we prepare without proof the following results:

THEOREM 2.1 ([5]). *Let M be a real hypersurface of $P_n(\boldsymbol{C})$. Then the following are equivalent*:

(i) *M is locally congruent to one of homogeneous ones of type A_1 and A_2.*

(ii) $(\nabla_X A)Y = -\eta(Y)\phi X - g(\phi X, Y)\xi$ *for any* $X, Y \in TM$.

THEOREM 2.2 ([6]). *Let M be a real hypersurface of $P_n(\boldsymbol{C})$. Then the following are equivalent*:

(i) $\phi A = A\phi$.

(ii) *M is locally congruent to one of homogeneous real hypersurfaces of type A_1 and A_2.*

THEOREM 2.3 ([4]). *There are no real hypersurfaces M with $(R(X, Y)A)Z = 0$ for $X, Y, Z \in TM$ in $P_n(\boldsymbol{C})$, $n \geq 2$.*

THEOREM 2.4 ([3]). *Let M be a real hypersurface of $P_n(\boldsymbol{C})$. Then M has constant principal curvatures and ξ is a principal curvature vector if and only if M is locally congruent to a homogeneous real hypersurface.*

THEOREM 2.5 ([1]). *Let M be a real hypersurface of $P_n(\boldsymbol{C})$, $n \geq 3$. If the second fundamental tensor A satisfies*

$$(R(X, Y)A)Z = 0$$

for all tangent vectors X, Y, Z perpendicular to ξ, then M is locally congruent to a geodesic hypersphere.

PROPOSITION 2.6 ([5]). *If ξ is a principal curvature vector, then the corresponding principal curvature α is locally constant.*

PROPOSITION 2.7 ([5]). *Assume that ξ is a principal curvatur vector and corresponding principal curvature is α. If $AX = rX$ for $X \perp \xi$, then we have $A\phi X = ((\alpha r + 2)/(2r - \alpha))\phi X$.*

3. Main Theorem

First we consider a real hypersurface with birecurrent second fundamental tensor. Hamada ([2]) showed that there are no real hypersurfaces with recurrent second fundamental tensor of $P_n(\boldsymbol{C})$. We may regard the recurrent condition as special case of the birecurrent condition. We will prove the following:

THEOREM 1. *There exists no real hypersurfaces with birecurrent second fundamental tensor of* $P_n(\boldsymbol{C})$.

PROOF. The following equation holds for any $Y \in TM$.

$$\nabla_Y A^2 = (\nabla_Y A)A + A(\nabla_Y A).$$

Differentiating the above equation by $X \in TM$, we have

$$\nabla^2_{X,Y} A^2 = (\nabla^2_{X,Y} A)A + A(\nabla^2_{X,Y} A) + (\nabla_X A)(\nabla_Y A) + (\nabla_Y A)(\nabla_X A).$$

Here we suppose that the second fundamental tensor A is birecurrent, i.e., there exists a covariant tensor field α of order 2 satisfying $\nabla^2_{X,Y} A = \alpha(X, Y)A$. Hence we have from the above equation

$$\nabla^2_{X,Y} A^2 = 2\alpha(X, Y)A^2 + (\nabla_X A)(\nabla_Y A) + (\nabla_Y A)(\nabla_X A).$$

From this equation and commutativity of the trace and the derivation we obtain

$$\nabla^2_{X,Y}(tr\, A^2) = 2\alpha(X, Y)(tr\, A^2) + 2tr((\nabla_X A)(\nabla_Y A)).$$

Replacing X by Y, and subtracting from the above equation, we have

$$(\alpha(X, Y) - \alpha(Y, X))(tr\, A^2) = 0.$$

Since there exists no real hypersurfaces with $A = 0$, we have $\alpha(X, Y) = \alpha(Y, X)$. Then we obtain $(R(X, Y)A)Z = 0$ for $X, Y, Z \in TM$. This shows the assertion from Theorem 2.3. □

We denote by $\xi^\perp$ the subbundle of TM consisting of vectors perpendicular to ξ. In what follows $e_1, \dots, e_{2n-2}$ stands for an orthonormal basis of $\xi^\perp$ at a point in M. Next, we will prove the following:

THEOREM 2. *There exist no real hypersurfaces in* $P_n(\boldsymbol{C})$, $n \geq 3$, *satisfying the following condition*:

(8) $$g((\nabla^2_{X,Y} A)Z, W) = \alpha(X, Y)g(AZ, W) \quad \textit{for } X, Y, Z, W \in \xi^\perp,$$

where $\alpha(X, Y)$ *is a covariant tensor field of order* 2. *And the structure vector* ξ *is principal.*

PROOF.

$$\begin{aligned}\nabla^2_{X,Y}(tr\,A^2) &= \sum_{j=1}^{2n-2} g((\nabla^2_{X,Y}A^2)e_j, e_j) + g((\nabla^2_{X,Y}A^2)\xi, \xi)\\ &= \sum_{j=1}^{2n-2} g((\nabla^2_{X,Y}A)Ae_j, e_j) + \sum_{j=1}^{2n-2} g(A(\nabla^2_{X,Y}A)e_j, e_j)\\ &\quad + \sum_{j=1}^{2n-2} g((\nabla_X A)(\nabla_Y A)e_j, e_j) + \sum_{j=1}^{2n-2} g((\nabla_Y A)(\nabla_X A)e_j, e_j)\\ &\quad + g((\nabla^2_{X,Y}A)A\xi, \xi) + g(A(\nabla^2_{X,Y}A)\xi, \xi)\\ &\quad + g((\nabla_X A)(\nabla_Y A)\xi, \xi) + g((\nabla_Y A)(\nabla_X A)\xi, \xi).\end{aligned}$$

Since the structure vector ξ is principal, $Ae_j \in \xi^\perp$. By using the assumption (8), we have

$$\begin{aligned}\nabla^2_{X,Y}(tr\,A^2) &= 2\alpha(X, Y) \sum_{j=1}^{2n-2} g(A^2e_j, e_j) + 2\sum_{j=1}^{2n-2} g((\nabla_X A)e_j, (\nabla_Y A)e_j)\\ &\quad + g((\nabla^2_{X,Y}A)A\xi, \xi) + g(A(\nabla^2_{X,Y}A)\xi, \xi) + 2g((\nabla_X A)\xi, (\nabla_Y A)\xi).\end{aligned}$$

Replacing X by Y, and subtracting from the above equation, we have

$$0 = (\alpha(X, Y) - \alpha(Y, X)) \sum_{j=1}^{2n-2} g(A^2e_j, e_j), \tag{9}$$

because the structure vector ξ is principal.

Here we assume that $\sum_{j=1}^{2n-2} g(A^2e_j, e_j) = 0$. Let Z be a vector field orthogonal to ξ such that $AZ = \lambda Z$. Then it is known (see, Prop. 2.7) that

$$A\phi Z = \frac{\alpha\lambda + 2}{2\lambda - \alpha}\phi Z.$$

Hence we have from the assumption

$$\lambda = \frac{\alpha\lambda + 2}{2\lambda - \alpha} = 0.$$

This is a contradiction. So (9) implies that $\alpha(X, Y)$ is symmetric tensor. Therefore we have from (8)

$$g((R(X, Y)A)Z, W) = 0 \quad \text{for } X, Y, Z \text{ and } W \in \xi^\perp. \tag{10}$$

Since the structure vector ξ is principal, from Gauss equation (6) we have the following

$$g((R(X, Y)A)Z, \xi) = 0 \quad \text{for } X, Y, Z \in \xi^{\perp}.$$

Hence we have

$$(R(X, Y)A)Z = 0 \quad \text{for } X, Y, Z \in \xi^{\perp}.$$

Therefore, in the case of $n \geq 3$, M is locally congruent to a real hypersurface of type (A_1) (cf. Theorem 2.5). So, we shall check the equation (8) for a real hypersurface of type (A_1). Then the second fundamental tensor A of type (A_1) is expressed as (cf. [8]):

$$AX = tX \quad \text{for } X \in \xi^{\perp},$$

where t is constant and $t > 0$. Making use of Theorem 2.1 and (5), and substituting the above equation into (8), we get

$$-g(\phi Y, Z)g(\phi X, W) - g(\phi X, Z)g(\phi Y, W) = \alpha(X, Y)g(Z, W).$$

Putting $Y = Z = \phi X$ and $W = X$, we have $\|X\|^2 = 0$ for any $X \in \xi^{\perp}$. This is a contradiction. □

Next, our aim is to prove the following.

THEOREM 3. *Let M be a real hypersurface of $P_n(C)$, $n \geq 2$. If the second fundamental tensor A satisfies*

$$g((\nabla^2_{X,Y}A)Z, W) = -g(\phi AX, W)g(\phi Y, Z) - g(\phi AX, Z)g(\phi Y, W)$$

for any $X, Y, Z, W \in \xi^{\perp}$, and the structure vector ξ is principal. Then M is locally congruent to one of homogeneous real hypersurfaces of type (A_1) and (A_2).

PROOF. We assume that the second fundamental tensor A satisfies

$$\begin{aligned} &g((\nabla_X(\nabla_Y A))Z - (\nabla_{\nabla_X Y}A)Z, W) \\ &\quad = -g(\phi AX, W)g(\phi Y, Z) - g(\phi AX, Z)g(\phi Y, W). \end{aligned}$$

Exchanging X and Y in the above equation, we have the following

$$\begin{aligned} (11) \qquad &g((R(X, Y)A)Z, W) \\ &\quad = -g(\phi AX, W)g(\phi Y, Z) - g(\phi AX, Z)g(\phi Y, W) \\ &\qquad + g(\phi AY, W)g(\phi X, Z) + g(\phi AY, Z)g(\phi X, W). \end{aligned}$$

From Gauss equation (6) the left hand side of (11) is

$$\begin{aligned} g(R(X,Y)AZ - AR(X,Y)Z, W) \\ &= g(Y,AZ)g(X,W) - g(X,AZ)g(Y,W) \\ &\quad + g(\phi Y,AZ)g(\phi X,W) - g(\phi X,AZ)g(\phi Y,W) \\ &\quad - 2g(\phi X,Y)g(\phi AZ,W) + g(AY,AZ)g(AX,W) \\ &\quad - g(AX,AZ)g(AY,W) - g(Y,Z)g(X,AW) \\ &\quad + g(X,Z)g(Y,AW) - g(\phi Y,Z)g(\phi X,AW) \\ &\quad + g(\phi X,Z)g(\phi Y,AW) + 2g(\phi X,Y)g(\phi Z,AW) \\ &\quad - g(AY,Z)g(AX,AW) + g(AX,Z)g(AY,AW). \end{aligned}$$

And hence the equation (11) asserts that

$$\begin{aligned} (12)\qquad & g(HX,W)g(\phi Y,Z) + g(HX,Z)g(\phi Y,W) \\ & - g(HY,W)g(\phi X,Z) - g(HY,Z)g(\phi X,W) - 2g(HZ,W)g(\phi X,Y) \\ & - g(AY,Z)g(X,W) + g(Y,Z)g(AX,W) \\ & + g(AX,Z)g(Y,W) - g(X,Z)g(AY,W) \\ & - g(A^2Y,Z)g(AX,W) + g(A^2X,W)g(AY,Z) \\ & + g(A^2X,Z)g(AY,W) - g(A^2Y,W)g(AX,Z) = 0, \end{aligned}$$

where H is tensor field of type (1,1) which is defined by

$$HX = (A\phi - \phi A)X.$$

Here let $e_1,\dots,e_{2n-2}$ be an orthonormal basis of $\xi^\perp$. And the index i runs from 1 to $2n-2$. Putting $Y = \phi e_i$, $Z = e_i$ in (12), and taking summation on i.

$$(2n+1)g(HX,W) + g(AHAX,W) = 0$$

for any $X, W \in \xi^\perp$.

Since the structure vector ξ is principal, the above equation holds for all tangent vectors X, W. Hence we have

$$(13)\qquad (2n+1)H + AHA = 0$$

at a point in M. From Proposition 2.7, we can take orthonormal vectors e_j, ϕe_j, ξ $(j = 1,\dots,n-1)$ which are principal vectors. Let λ_j and α be principal curvatures of e_j and ξ, respectively. Then principal curvatures of ϕe_j are $(\alpha\lambda_j + 2)/(2\lambda_j - \alpha)$, say λ_j'. By using the orthonormal basis we have

$$He_j = (A\phi - \phi A)e_j = (\lambda_j' - \lambda_j)\phi e_j,$$

$$H\phi e_j = (A\phi - \phi A)\phi e_j = (\lambda_j' - \lambda_j)e_j,$$

$$H\xi = 0.$$

Hence we have the following expression of H.

$$H = \left(\begin{array}{ccc|ccc|c} & & & k_1 & & 0 & 0 \\ & \mathbf{0} & & & \ddots & & \vdots \\ & & & 0 & & k_{n-1} & 0 \\ \hline k_1 & & 0 & & & & 0 \\ & \ddots & & & \mathbf{0} & & \vdots \\ 0 & & k_{n-1} & & & & 0 \\ \hline 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \end{array}\right),$$

where $k_j = \lambda_j' - \lambda_j$ $(j = 1, \ldots, n-1)$.

By virture of the above expression of H, it follows from (13) that

$$(\lambda_j' - \lambda_j)(\lambda_j\lambda_j' + 2n + 1) = 0 \quad (j = 1, \ldots, n-1).$$

Then by Proposition 2.6, we see that all principal curvatures are locally constant. Hence our real hypersurface M is homogeneous one by Theorem 2.4.

Due to Takagi's work ([8]), we find that a principal curvature of homogeneous real hypersurfaces in $P_n(C)$ is one of the following:

$$r_1 = t, \quad r_2 = -\frac{1}{t}, \quad r_3 = \frac{1+t}{1-t}, \quad r_4 = \frac{t-1}{t+1}, \quad \alpha = t - \frac{1}{t},$$

where $t = \cot\theta$ $(0 < \theta < \pi/4)$.

Here we assume that there exist k $(1 \leq k \leq n-1)$ such that $\lambda_k\lambda_k' = -2n - 1$ (≤ -5). We note that a real hypersurface of type (A_1) has two distinct principal curvatures r_1 and α, type (A_2) has three distinct principal curvatures r_1, r_2 and α, type (B) has three distinct principal curvatures r_3, r_4, and α, a real hypersurface of type (C), (D) and (E) has five distinct principal curvatures. Now let $\lambda_k = r_i$ $(i = 1, 2)$. Then $\lambda_k' = r_i$ $(i = 1, 2)$ from Proposition 2.7. Hence we have $\lambda_k\lambda_k' = r_i^2$

$(i = 1, 2)$, which contradicts with the assumption. On the other hand, let $\lambda_k = r_3$. Then $\lambda_k = r_4$ from Proposition 2.7. Hence we have $\lambda_k \lambda_k' = -1$, which contradicts with the assumption. Hence we have $\lambda_j' - \lambda_j = 0$ $(j = 1, \ldots, n-1)$.

This means that $H = 0$ (i.e. $A\phi = \phi A$). By Theorem 2.2, M is one of homogeneous real hypersurfaces of type A_1 and A_2.

Conversely, a homogeneous real hypersurface of type (A) satisfies the assumption because of Theorem 2.1. This completes the proof of Theorem 3. □

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Department of Mathematics,
Chuo University
1-13-27 Kasuga, Bunkyo-ku
Tokyo 112-8551, Japan