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ORBIT TYPES OF THE COMPACT LIE GROUP E_7 IN THE COMPLEX FREUDENTHAL VECTOR SPACE $\mathfrak{P}^C$

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1. Introduction

Let $\mathfrak{J}$ be the exceptional Jordan algebra over R and $\mathfrak{J}^C$ its complexification. Then the simply connected compact exceptional Lie group F_4 acts on $\mathfrak{J}$ and F_4 has three orbit types which are

$$F_4/F_4, \quad F_4/Spin(9), \quad F_4/Spin(8).$$

Similarly the simply connected compact exceptional Lie group E_6 acts on $\mathfrak{J}^C$ and E_6 has five orbit types which are

$$E_6/E_6, \quad E_6/F_4, \quad E_6/Spin(10), \quad E_6/Spin(9), \quad E_6/Spin(8)$$

([5]). In this paper, we determine the orbit types of the simply connected compact exceptional Lie group E_7 in the complex Freudenthal vector space $\mathfrak{P}^C$. As a result, E_7 has seven orbit types which are

$$E_7/E_7, \quad E_7/E_6, \quad E_7/F_4, \quad E_7/Spin(11), \quad E_7/Spin(10),$$
$$E_7/Spin(9), \quad E_7/Spin(8).$$

2. Preliminaries

Let $\mathfrak{C}$ be the division Cayley algebra and $\mathfrak{J} = \{X \in M(3, \mathfrak{C}) \mid X^* = X\}$ the exceptional Jordan algebra with the Jordan multiplication $X \circ Y$, the inner product (X, Y) and the Freudenthal multiplication $X \times Y$. Let $\mathfrak{J}^C$ be the complexification of $\mathfrak{J}$ with the Hermitian inner product $\langle X, Y\rangle$. (The definitions of $X \circ Y$, (X, Y), $X \times Y$ and $\langle X, Y\rangle$ are found in [2]). Moreover, let $\mathfrak{P}^C = \mathfrak{J}^C \oplus \mathfrak{J}^C \oplus C \oplus C$ be the Freudenthal C-vector space with the Hermitian inner product $\langle P, Q\rangle$. For $P, Q \in \mathfrak{P}^C$, we can define a C-linear mapping $P \times Q$ of $\mathfrak{P}^C$.

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(The definitions of $\langle P, Q\rangle$ and $P \times Q$ are found in [2]). The complex conjugation in the complexified spaces $\mathfrak{C}^C$, $\mathfrak{J}^C$ or $\mathfrak{P}^C$ is denoted by τ. Now, the simply connected compact exceptional Lie groups F_4, E_6 and E_7 are defined by

$$F_4 = \{\alpha \in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J}) \mid \alpha(X \circ Y) = \alpha X \circ \alpha Y\},$$

$$E_6 = \{\alpha \in \mathrm{Iso}_C(\mathfrak{J}^C) \mid \tau\alpha\tau(X \times Y) = \alpha X \times \alpha Y, \langle \alpha X, \alpha Y\rangle = \langle X, Y\rangle\},$$

$$E_7 = \{\alpha \in \mathrm{Iso}_C(\mathfrak{P}^C) \mid \alpha(P \times Q)\alpha^{-1} = \alpha P \times \alpha Q, \langle \alpha P, \alpha Q\rangle = \langle P, Q\rangle\}$$

$$= \{\alpha \in \mathrm{Iso}_C(\mathfrak{P}^C) \mid \alpha(P \times Q)\alpha^{-1} = \alpha P \times \alpha Q, \alpha(\tau\lambda) = (\tau\lambda)\alpha\}$$

(where λ is the C-linear transformation of $\mathfrak{P}^C$ defined by $\lambda(X, Y, \xi, \eta) = (Y, -X, \eta, -\xi)$), respectively. Then we have a natural inclusion $F_4 \subset E_6 \subset E_7$, that is,

$$E_6 = \{\alpha \in E_7 \mid \alpha(0,0,1,0) = (0,0,1,0)\} \subset E_7,$$

$$F_4 = \{\alpha \in E_6 \mid \alpha E = E\} \subset E_6 \subset E_7,$$

where E is the 3×3 unit matrix. The groups F_4, E_6 and E_7 have the following subgroups

$$Spin(8) = \{\alpha \in F_4 \mid \alpha E_k = E_k, k = 1,2,3\} \subset F_4 \subset E_6 \subset E_7,$$

$$Spin(9) = \{\alpha \in F_4 \mid \alpha E_1 = E_1\} \subset F_4 \subset E_6 \subset E_7,$$

$$Spin(10) = \{\alpha \in E_6 \mid \alpha E_1 = E_1\} \subset E_6 \subset E_7,$$

$$Spin(11) = \{\alpha \in E_7 \mid \alpha(E_1,0,1,0) = (E_1,0,1,0)\} \subset E_7,$$

where E_k is the usual notation in $\mathfrak{J}^C$, e.g. $E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ ([2]).

3. Orbit Types of F_4 in $\mathfrak{J}$ and E_6 in $\mathfrak{J}^C$

We shall review the results of orbit types of F_4 in $\mathfrak{J}$ and E_6 in $\mathfrak{J}^C$.

LEMMA 1 ([1]). *Any element $X \in \mathfrak{J}$ can be transformed to a diagonal form by some $\alpha \in F_4$:*

$$\alpha X = \begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & 0 \\ 0 & 0 & \xi_3 \end{pmatrix}, \quad \xi_k \in \boldsymbol{R}, \quad (\text{which is briefly written by } (\xi_1, \xi_2, \xi_3)).$$

The order of ξ_1, ξ_2, ξ_3 can be arbitrarily exchanged under the action of F_4.

THEOREM 2 ([5]). *The orbit types of the group F_4 in $\mathfrak{J}$ are as follows.*
(1) *The orbit through (ξ,ξ,ξ) is F_4/F_4.*
(2) *The orbit through (ξ_1,ξ,ξ) (where $\xi_1 \neq \xi$) is $F_4/Spin(9)$.*
(3) *The orbit through (ξ_1,ξ_2,ξ_3) (where ξ_1,ξ_2,ξ_3 are distinct) is $F_4/Spin(8)$.*

LEMMA 3 ([3]). *Any element $X \in \mathfrak{J}^C$ can be transformed to a diagonal form by some $\alpha \in E_6$:*

$$\alpha X = \begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & 0 \\ 0 & 0 & \xi_3 \end{pmatrix}, \quad \xi_k \in C, \quad (\text{which is briefly written by } (\xi_1,\xi_2,\xi_3)).$$

The order of ξ_1,ξ_2,ξ_3 can be arbitrarily exchanged under the action of E_6.

THEOREM 4 ([5]). *The orbit types of the group E_6 in $\mathfrak{J}^C$ are as follows.*
(1) *The orbit through (0,0,0) is E_6/E_6.*
(2) *The orbit through (ξ_1,ξ_2,ξ_3) (where $|\xi_1| = |\xi_2| = |\xi_3| \neq 0$) is E_6/F_4.*
(3) *The orbit through $(\xi,0,0)$ (where $\xi \neq 0$) is $E_6/Spin(10)$.*
(4) *The orbit through (ξ_1,ξ_2,ξ_3) (where $|\xi_1| \neq |\xi_2| = |\xi_3| \neq 0$) is $E_6/Spin(9)$.*
(5) *The orbit through (ξ_1,ξ_2,ξ_3) (where $|\xi_1|,|\xi_2|,|\xi_3|$ are distinct) is $E_6/Spin(8)$.*

4. Orbit Types of E_7 in $\mathfrak{P}^C$

LEMMA 5 ([2]). *Any element $P \in \mathfrak{P}^C$ can be transformed to the following diagonal form by some $\alpha \in E_7$:*

$$\alpha P = \left(\begin{pmatrix} ar_1 & 0 & 0 \\ 0 & ar_2 & 0 \\ 0 & 0 & ar_3 \end{pmatrix}, \begin{pmatrix} br_1 & 0 & 0 \\ 0 & br_2 & 0 \\ 0 & 0 & br_3 \end{pmatrix}, ar, br \right), \quad \begin{matrix} r_k, r \in \boldsymbol{R}, 0 \leq r_k, 0 \leq r, \\ a, b \in C, |a|^2 + |b|^2 = 1. \end{matrix}$$

Moreover, any element $P \in \mathfrak{P}^C$ can be transformed to the following diagonal form by some $\varphi(A)\alpha \in \varphi(SU(2))E_7$:

$$\varphi(A)\alpha P = \left(\begin{pmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, r, 0 \right), \quad r_k, r \in R, 0 \leq r_k, 0 \leq r,$$

(which is briefly written by $(r_1,r_2,r_3;r)$), where $\varphi(A) \in \varphi(SU(2)) \subset E_8$ and commutes with any element $\alpha \in E_7$. The order of r_1,r_2,r_3,r can be arbitrarily exchanged under the action of E_7. (As for the definitions of the groups E_8 and $\varphi(SU(2))$, see

[2]). *The action of* $\varphi(A), A \in SU(2)$, *on* $\mathfrak{P}^C$ *is given by*

$$\varphi(A)P = \varphi\left(\begin{pmatrix} a & -\tau b \\ b & \tau a \end{pmatrix}\right)(X, Y, \xi, \eta)$$

$$= (aX + \tau(bY), aY - \tau(bX), a\xi + \tau(b\eta), a\eta - \tau(b\xi)).$$

THEOREM 6. *The group* E_7 *has the following seven orbit types in* $\mathfrak{P}^C$:

$$E_7/E_7, \quad E_7/E_6, \quad E_7/F_4, \quad E_7/Spin(11), \quad E_7/Spin(10),$$

$$E_7/Spin(9), \quad E_7/Spin(8).$$

More details,

(1) *The orbit through* $(0,0,0;0)$ *is* E_7/E_7.

(2) *The orbit through* $(0,0,0;1)$ *or* $(1,1,1;1)$ *is* E_7/E_6.

(3) *The orbit through* $(1,1,1;0)$ *or* $(1,1,1;r)$ (*where* $0 < r, 1 \neq r$) *is* E_7/F_4.

(4) *The orbit through* $(1,0,0;1)$ *or* $(1,r,r;1)$ (*where* $0 < r, 1 \neq r$) *is* $E_7/Spin(11)$.

(5) *The orbit through* $(1,0,0;r)$ (*where* $0 < r, 1 \neq r$) *is* $E_7/Spin(10)$.

(6) *The orbit through* $(1,1,r;0)$ *or* $(1,1,r;s)$ (*where* $0 < r, 0 < s$ *and* $1,r,s$ *are distinct*) *is* $E_7/Spin(9)$.

(7) *The orbit through* $(1,r,s;0)$ *or* $(1,r,s;t)$ (*where* r,s,t *are positive and* $1,r,s,t$ *are distinct*) *is* $E_7/Spin(8)$.

PROOF. From Lemma 5, the representatives of orbit types (up to a constant) can be given by the following.

$$\begin{array}{llll} (0,0,0;0), & (0,0,0;1), & (0,0,1;1), & (0,0,1;r), \\ (0,1,1;1), & (0,1,1;r), & (0,1,r;s), & (1,1,1;1), \\ (1,1,1;r), & (1,1,r;r), & (1,1,r;s), & (1,r,s;t) \end{array}$$

where r,s,t are positive, $0,1,r,s,t$ are distinct and the order of $0,1,r,s,t$ can be arbitrarily exchanged.

(1) The isotropy subgroup $(E_7)_{(0,0,0;0)}$ is obviously E_7. Therefore the orbit through $(0,0,0;0)$ is E_7/E_7.

(2) The isotropy subgroup $(E_7)_{(0,0,0;1)}$ is E_6. Therefore the orbit through $(0,0,0;1)$ is E_7/E_6.

(2′) The isotropy subgroup $(E_7)_{(1,1,1;1)}$ is conjugate to E_6 in E_7. In fact, we know that the following realization of the homogeneous space E_7/E_6 : $E_7/E_6 = \{P \in \mathfrak{P}^C \mid P \times P = 0, \langle P, P\rangle = 1\} = \mathfrak{M}$ ([4]). Since $1/2\sqrt{2}(E,E,1,1)$ and $(0,0,1,0) \in \mathfrak{M}$, there exists $\delta \in E_7$ such that

$$\delta\left(\frac{1}{2\sqrt{2}}(E,E,1,1)\right) = (0,0,1,0).$$

Hence the isotropy subgroup $(E_7)_{(E,E,1,1)}$ is conjugate to the isotropy subgroup $(E_7)_{(0,0,1,0)}$ is $E_7 : (E_7)_{(E,E,1,1)} \sim (E_7)_{(0,0,1,0)}$. On the other hand, since

$$\varphi\left(\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\right)(E,0,1,0) = \frac{1}{\sqrt{2}}(E,E,1,1),$$

we have $(E_7)_{(E,0,1,0)} = (E_7)_{(E,E,1,1)} \sim (E_7)_{(0,0,1,0)} = E_6$. Therefore the orbit through $(1,1,1;1)$ is E_7/E_6.

(3) The isotropy subgroup $(E_7)_{(1,1,1;0)}$ is F_4. In fact, for $\alpha \in E_7$ and $P \in \mathfrak{P}^C$, we have $\alpha(\tau\lambda((P \times P)P)) = \tau\lambda(\alpha((P \times P)P)) = \tau\lambda(\alpha(P \times P)\alpha^{-1}\alpha P) = \tau\lambda((\alpha P \times \alpha P)\alpha P)$. Now, let $P = (1,1,1;0)$. Since $\tau\lambda((P \times P)P) = 3/2(0,0,0;1)$, if $\alpha \in E_7$ satisfies $\alpha P = P$, then α also satisfies $\alpha(0,0,0;1) = (0,0,0;1)$. Hence $\alpha \in E_6$, so together with $\alpha E = E$, we have $\alpha \in F_4$. Therefore the orbit through $(1,1,1;0)$ is E_7/F_4.

(3′) The isotropy subgroup $(E_7)_{(1,1,1;r)}$ is F_4. In fact, let $P = (1,1,1;r)$. Since $\tau\lambda((P \times P)P) = 3/2(r,r,r;1)$, if $\alpha \in E_7$ satisfies $\alpha P = P \cdots$ (i), then α also satisfies $\alpha(r,r,r;1) = (r,r,r;1) \cdots$ (ii). Take (i)–(ii), then we have $\alpha(1-r,1-r,1-r;r-1) = (1-r,1-r,1-r;r-1)$. Since $1-r \neq 0$, we have $\alpha(1,1,1;-1) = (1,1,1;-1)$. Together with $\alpha P = P$, we have $\alpha(0,0,0;1) = (0,0,0;1)$ and $\alpha(1,1,1;0) = (1,1,1;0)$. Hence $\alpha \in E_6$ and hence $\alpha \in F_4$. Therefore the orbit through $(1,1,1;r)$ is E_7/F_4.

(4) The isotropy subgroup $(E_7)_{(1,0,0;1)}$ is $Spin(11)$. Therefore the orbit through $(1,0,0;1)$ is $E_7/Spin(11)$.

(4′) The isotropy subgroup $(E_7)_{(1,r,r;1)}$ is $Spin(11)$. In fact, let $P = (1,r,r;1)$. Since $\tau\lambda((P \times P)P) = 3/2(r^2,r,r;r^2)$, if $\alpha \in E_7$ satisfies $\alpha P = P \cdots$ (i), then α also satisfies $\alpha(r^2,r,r;r^2) = (r^2,r,r;r^2) \cdots$ (ii). Take (i)–(ii), then we have $\alpha(1-r^2,0,0;1-r^2) = (1-r^2,0,0;1-r^2)$. Since $1-r^2 \neq 0$, we have $\alpha(1,0,0;1) = (1,0,0;1)$. Hence $\alpha \in Spin(11)$. Therefore the orbit through $(1,r,r;1)$ is $E_7/Spin(11)$.

(5) The isotropy subgroup $(E_7)_{(1,0,0;r)}$ is $Spin(10)$. In fact, for $\alpha \in E_7$ and $P \in \mathfrak{P}^C$, we have $\alpha((P \times P)\tau\lambda P) = (\alpha(P \times P)\alpha^{-1})\alpha(\tau\lambda P) = (\alpha P \times \alpha P)\tau\lambda(\alpha P)$. Now, let $P = (1,0,0;r)$. Since $(P \times P)\tau\lambda P = -1/2(r^2,0,0;r)$, if $\alpha \in E_7$ satisfies $\alpha P = P \cdots$ (i), then α also satisfies $\alpha(r^2,0,0;r) = (r^2,0,0;r) \cdots$ (ii). Take (i)–(ii), then we have $\alpha(1-r^2,0,0;0) = (1-r^2,0,0;0)$. Since $1-r^2 \neq 0$, we have $\alpha(1,0,0;0) = (1,0,0;0) \cdots$ (iii). Take (i)–(iii), then $\alpha(0,0,0;r) = (0,0,0;r)$, that is, $\alpha(0,0,0;1) = (0,0,0;1)$. Hence $\alpha \in E_6$ and $\alpha E_1 = E_1$. Thus $\alpha \in Spin(10)$. Therefore the orbit through $(1,0,0;r)$ is $E_7/Spin(10)$.

(6) The isotropy subgroup $(E_7)_{(1,1,r;0)}$ is $Spin(9)$. In fact, let $P = (1,1,r;0)$. Since $\tau\lambda((P \times P)P) = 3/2(0,0,0;r)$, if $\alpha \in E_7$ satisfies $\alpha P = P$, then α also satisfies $\alpha(0,0,0;1) = (0,0,0;1)$. Hence $\alpha \in E_6$, so together with $\alpha P = P$, we have $\alpha \in Spin(9)$ (Theorem 4.(4)). Therefore the orbit through $(1,1,r;0)$ is $E_7/Spin(9)$.

(6′) The isotropy subgroup $(E_7)_{(1,1,r;s)}$ is $Spin(9)$. In fact, let $P = (1,1,r;s)$. Since $\tau\lambda((P \times P)P) = 3/2(rs,rs,s;r)$, if $\alpha \in E_7$ satisfies $\alpha P = P \cdots$ (i), then α also satisfies $\alpha(rs,rs,s;r) = (rs,rs,s;r) \cdots$ (ii). Take (i) $\times r -$ (ii)$\times s$, then we have $\alpha(r(1-s^2), r(1-s^2), r^2-s^2; 0) = (r(1-s^2), r(1-s^2), r^2-s^2; 0)$. Since $r(1-s^2)$, r^2-s^2 are non-zero and $r(1-s^2) \neq r^2-s^2$, from (6) we have $\alpha \in Spin(9)$. Therefore the orbit through $(1,1,r;s)$ is $E_7/Spin(9)$.

(7) The isotropy subgroup $(E_7)_{(1,r,s;0)}$ is $Spin(8)$. In fact, let $P = (1,r,s;0)$. Since $\tau\lambda((P \times P)P) = 3/2(0,0,0;rs)$, if $\alpha \in E_7$ satisfies $\alpha P = P$, then α also satisfies $\alpha(0,0,0;1) = (0,0,0;1)$. Hence $\alpha \in E_6$, so together with $\alpha P = P$, we have $\alpha \in Spin(8)$ (Theorem 4.(5)). Therefore the orbit through $(1,r,s;0)$ is $E_7/Spin(8)$.

(7′) The isotropy subgroup $(E_7)_{(1,r,s;t)}$ is $Spin(8)$. In fact, let $P = (1,r,s;t)$. Since $\tau\lambda((P \times P)P) = 3/2(rst,st,rt;rs)$, if $\alpha \in E_7$ satisfies $\alpha P = P \cdots$ (i), then α also satisfies $\alpha(rst,st,rt;rs) = (rst,st,rt;rs) \cdots$ (ii). Take (i) $\times rs -$ (ii) $\times t$, then we have $\alpha(rs(1-t^2), s(r^2-t^2), r(s^2-t^2); 0) = (rs(1-t^2), s(r^2-t^2), r(s^2-t^2); 0)$. Since $rs(1-t^2), s(r^2-t^2)$ and $r(s^2-t^2)$ are non-zero and distinct, from (7) we have $\alpha \in Spin(8)$. Therefore the orbit through $(1,r,s;t)$ is $E_7/Spin(8)$.

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