

NOTE ON MACAULAY SEMIGROUPS

By

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Almost all of ideal theory of a commutative ring R concerns properties of ideals of R with respect to the multiplication “ \times ” on R . Abandoning the addition “ $+$ ” on R we extract the multiplication on R . Then we have the idea of the algebraic system S of a semigroup. We denote the operation on S by addition. S is called a grading monoid. Concretely, a submonoid S of a torsion-free abelian (additive) group is called a grading monoid (or a g -monoid). Many terms in commutative ring theory are defined analogously for S . For example, a non-empty subset I of S is called an ideal of S if $S + I \subset I$. Let I be an ideal of S with $I \subsetneq S$. If $s_1 + s_2 \in I$ (for $s_1, s_2 \in S$) implies $s_1 \in I$ or $s_2 \in I$, then I is called a prime ideal of S . If there exists an element $s \in S$ such that $I = S + s$, then I is called a principal ideal of S . The group $q(S) = \{s_1 - s_2 \mid s_1, s_2 \in S\}$ is called the quotient group of S . A subsemigroup of $q(S)$ containing S is called an oversemigroup of S . Let Γ be a totally ordered abelian (additive) group. A mapping v of a torsion-free abelian group G onto Γ is called a valuation on G if $v(x + y) = v(x) + v(y)$ for all $x, y \in G$. Then v is called a Γ -valued valuation on G . The subsemigroup $\{x \in G \mid v(x) \geq 0\}$ of G is called the valuation semigroup of G associated with v . A \mathbf{Z} -valued valuation is called a discrete valuation of rank 1. The valuation semigroup associated with a discrete valuation of rank 1 is called a discrete valuation semigroup of rank 1. An element x of an extension semigroup T of S is called integral over S if $nx \in S$ for some $n \in \mathbf{N}$. Let \bar{S} be the set of all integral elements of $q(S)$ over S . Then \bar{S} is an oversemigroup of S , and is called the integral closure of S . If $\bar{S} = S$, then S is called an integrally closed semigroup (or a normal semigroup). An ideal I of S is called a cancellation ideal of S if $I + J_1 = I + J_2$ (for ideals J_1, J_2 of S) implies $J_1 = J_2$. The maximum number n so that there exists a chain $P_1 \subsetneq P_2 \subsetneq \cdots \subsetneq P_n (\subsetneq S)$ of prime ideals of S is called the dimension of S . Many propositions for commutative rings are known to hold for S . The author conjectures that almost all propositions of multi-

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plicative ideal theory of R hold for S . It is usually expected that ideal theory of S is simpler than that of R . Therefore investigating the ideal theory of S may be an auxiliary means for investigating that of R . Of course we may say that the ideal theory of S has its proper significance (cf. [M2])). For an example, Anderson-Anderson [AA] posed a question: Is every cancellation ideal of a quasi-local domain principal? The answer to this question is open. However, every cancellation ideal of S is principal ([M1]). For another example, let $\Sigma'(D)$ (resp. $\Sigma'(S)$) be the set of all semistar-operations on D (resp. S). Assume that S is integrally closed and of dimension n . Then S is a valuation semigroup if and only if $n + 1 \leq |\Sigma'(S)| \leq 2n + 1$ ([OMS]). If a similar property holds for D is open. The aim of this paper is to show that almost all the propositions in Chapter 3 of Kaplansky's Commutative Rings [K] hold for g -monoids.

Since this paper is a semigroup version of commutative ring theory, a g -monoid is denoted by R . Let A be a non-empty set. Assume that, for every $r \in R$ and $a \in A$, there is defined $r + a \in A$ such that, for every $r_1, r_2 \in R$ and $a \in A$, we have $(r_1 + r_2) + a = r_1 + (r_2 + a)$ and $0 + a = a$. Then A is called an R -module. Let A be an R -module and $r \in R$. If $r + a_1 = r + a_2$ (for $a_1, a_2 \in A$) implies $a_1 = a_2$, then r is called a non-zerodivisor on A . If r is not a non-zerodivisor, then r is called a zerodivisor on A . The set of zerodivisors on A is denoted by $Z(A)$. Let B be a submodule of an R -module A , and $r \in R$. If $r + a \in B$ (for $a \in A$) implies $a \in B$, then r is called a non-zerodivisor on A modulo B . A non-zerodivisor on A modulo B is also called a non-zerodivisor on A/B . If r is not a non-zerodivisor on A/B , then r is called a zerodivisor. The set of zerodivisors on A/B is denoted by $Z(A/B)$. If $\{x_1, \dots, x_n\}$ is a finite subset of R , then the ideal $\bigcup_{i=1}^n (R + x_i)$ of R is denoted by (x_1, \dots, x_n) . The ordered sequence of elements x_1, \dots, x_n of R is called a regular sequence on A , if $(x_1, \dots, x_n) + A \subsetneq A$ and if $x_1 \notin Z(A)$, $x_2 \notin Z(A/((x_1) + A))$, \dots , $x_n \notin Z(A/((x_1, \dots, x_{n-1}) + A))$.

THEOREM 1. *Let A be an R -module, and let x, y be a regular sequence on A . Then $x \notin Z(A/(y + A))$.*

PROOF. Assume that $x + a = y + a_1$ (for $a, a_1 \in A$). Since $y \notin Z(A/(x + A))$, we have $a_1 \in x + A$. Since $x \notin Z(A)$, we have $a \in y + A$, and hence $x \notin Z(A/(y + A))$.

Let A be an R -module. If $Z(A) = \emptyset$, then A is called torsion-free. Theorem 1 implies the following,

THEOREM 2. *Let A be a torsion-free R -module, and x_1, \dots, x_n a regular sequence on A . Then the sequence obtained by interchanging x_i and x_{i+1} is a regular sequence on A if and only if $x_{i+1} \notin Z(A/((x_1, \dots, x_{i-1}) + A))$.*

Let A be an R -module. If $r_1 + a = r_2 + a$ (for $r_1, r_2 \in R$ and $a \in A$) implies $r_1 = r_2$, then A is called cancellative. If every ideal of a g -monoid R is finitely generated, then R is called a Noetherian semigroup.

LEMMA 3 ([M3, Proposition 1]). *Let R be a Noetherian semigroup, and A a finitely generated R -module. Then A satisfies the ascending chain condition on submodules.*

THEOREM 4. *Let R be a Noetherian semigroup, and A a finitely generated torsion-free cancellative R -module. Let x_1, \dots, x_n be a regular sequence on A . Then any permutation of the x 's is a regular sequence on A .*

PROOF. Set $S = ((x_1, \dots, x_{n-2}) + A : x_n)_A$. By Theorem 2, it suffices to show that $S \subset (x_1, \dots, x_{n-2}) + A$. Suppose the contrary. Take s in S with $s \notin (x_1, \dots, x_{n-2}) + A$. Since $x_n \notin Z(A/((x_1, \dots, x_{n-1}) + A))$, we have $s \in (x_1, \dots, x_{n-1}) + A$, and hence $s = x_{n-1} + a$ for some $a \in A$. It follows that $x_n + a \in (x_1, \dots, x_{n-2}) + A$, and hence $a \in S$. Then we have $S = x_{n-1} + S$; a contradiction to Lemma 3.

THEOREM 5. *Let A be an R -module, and x_1, \dots, x_n a regular sequence on A . Then $(x_1), (x_1, x_2), \dots, (x_1, \dots, x_n)$ form a properly ascending chain.*

Let A be an R -module, and I an ideal of R . Let x_1, \dots, x_n be a regular sequence in I on A . If x_1, \dots, x_n, x is not a regular sequence on A for each $x \in I$, then x_1, \dots, x_n is called a maximal regular sequence in I on A .

REMARK. Let R be a Noetherian semigroup. Then two maximal regular sequences on R need not have the same length.

For example, let \mathbf{Z}_0 be the monoid of non-negative integers and let $R = \mathbf{Z}_0 \oplus \mathbf{Z}_0$. Set $p = (1, 0), q = (0, 1)$ and $x = (1, 1)$. Then p, q is a maximal regular sequence on R . Also, x is a maximal regular sequence on R .

Let A be an R -module, and I an ideal of R . Then the maximum of lengths of all regular sequences in I on A is called the grade of I on A , and is denoted by $G(I, A)$.

THEOREM 6. *Let A be an R -module, and I an ideal of R with $I + A \subsetneq A$. Let x_1, \dots, x_n be a maximal regular sequence in I on A . Then there exists a prime ideal P such that x_1, \dots, x_n is a maximal regular sequence in P on A .*

PROOF. Set $P = Z(A/((x_1, \dots, x_n) + A))$. Then P is a prime ideal containing I , and x_1, \dots, x_n is a maximal regular sequence in P on A .

LEMMA 7. *Let R be a Noetherian semigroup. Then there exists only a finite number of prime ideals of R .*

PROOF. Let x_1, \dots, x_n be the set of all irreducible elements of R any two of which are not associated. Let P be a prime ideal of R . Then P is generated by a subset of $\{x_1, \dots, x_n\}$.

THEOREM 8. *Let R be a Noetherian semigroup, A a finitely generated torsion-free cancellative R -module, and J a k -generated ideal of R with $J + A \subsetneq A$. Then $G(J, A) \leq k$.*

PROOF. Let $J = (x_1, \dots, x_k)$. Suppose that there exists a regular sequence y_1, \dots, y_{k+1} in J on A . By Theorem 4, we may assume that $y_1 = x_1 + r_1$, $y_2 = x_1 + r_2$ for $r_1, r_2 \in R$. Choose $a \in A - (J + A)$. Then we have $r_1 + a \notin y_1 + A$ and $y_2 + (r_1 + a) \in y_1 + A$. Hence $y_2 \in Z(A/(y_1 + A))$; a contradiction.

Let A be an R -module. If any two maximal regular sequences in I on A have the same length for every ideal I with $I + A \subsetneq A$, then A is said to satisfy property (*). If A satisfies property (*), we say also that (R, A) satisfies property (*).

THEOREM 9. *In Theorem 8, let $J = (x_1, \dots, x_k)$, and assume that $G(J, A) = k$ and (R, A) satisfies property (*). Then x_1, \dots, x_k is a maximal regular sequence in J on A .*

PROOF. Assume that $x_{i_1} \notin Z(A)$, $x_{i_2} \notin Z(A/((x_{i_1}) + A))$, \dots , $x_{i_h} \notin Z(A/((x_{i_1}, \dots, x_{i_{h-1}}) + A))$ for $1 < h < k$. Then we have $J \not\subset Z(A/((x_{i_1}, \dots, x_{i_h}) + A))$. Hence there exists i_{h+1} such that $x_{i_{h+1}} \notin Z(A/((x_{i_1}, \dots, x_{i_h}) + A))$. Thus x_{i_1}, \dots, x_{i_k} is a regular sequence on A . Then Theorem 4 completes the proof.

THEOREM 10. *Let R be a Noetherian semigroup, and A a finitely generated torsion-free cancellative R -module which satisfies property (*). Let I be an ideal of R , and $x \in R$ with $J = (I, x) \not\subseteq R$. Then $G(J, A) \leq 1 + G(I, A)$.*

PROOF. Let x_1, \dots, x_m be a maximal regular sequence in I on A . If $J \subset Z(A/((x_1, \dots, x_m) + A))$, we have $G(J, A) \leq 1 + G(I, A)$. Assume that $J \not\subset Z(A/((x_1, \dots, x_m) + A))$. Then x_1, \dots, x_m, x is a regular sequence in J on A . Suppose that $J \not\subset Z(A/((x_1, \dots, x_m, x) + A))$. Then there exists $y \in I$ such that x_1, \dots, x_m, x, y is a regular sequence in J on A . Then x_1, \dots, x_m, y is a regular sequence in I on A by Theorem 4; a contradiction.

THEOREM 11. *Let R be a Noetherian semigroup, and A a finitely generated torsion-free cancellative R -module which satisfies property (*). Let I be an ideal of R contained in a maximal ideal M . Assume that $G(I, A) < G(M, A)$. Then there exists a prime ideal P of R such that $G(P, A) = 1 + G(I, A)$.*

PROOF. Let x_1, \dots, x_k be a maximal regular sequence in I on A , and set $I_0 = (x_1, \dots, x_k)$. We may take $x \in M - Z(A/(I_0 + A))$. By Theorem 10, we have $G((I, x), A) = k + 1$. Then $P = Z(A/((I, x) + A))$ is a desired prime ideal.

THEOREM 12. *Let R be a Noetherian semigroup, and A a finitely generated torsion-free cancellative R -module with property (*). Let $I = (x_1, \dots, x_n)$ be a proper ideal of R . Then $G(I, A) = n$ if and only if x_1, \dots, x_n is a regular sequence on A .*

PROOF. The necessity: Let $J = (x_1, \dots, x_{n-1})$, and assume that the assertion holds for x_1, \dots, x_{n-1} . We have $G(J, A) = n - 1$ by Theorem 10, and hence x_1, \dots, x_{n-1} is a regular sequence on A . Since $G(I, A) = n$, we have $I \not\subset Z(A/((x_1, \dots, x_{n-1}) + A))$. It follows that x_1, \dots, x_n is a regular sequence on A .

Let P be a prime ideal of R . Then the maximum number n so that there exists a chain $P_1 \subsetneq P_2 \subsetneq \dots \subsetneq P_n = P$ of prime ideals of R is called the height of P , and is denoted by $\text{ht}(P)$. For an ideal I of R , the minimum of $\text{ht}(P)$, P ranging over the prime ideals containing I is called the height of I , and is denoted by $\text{ht}(I)$.

THEOREM 13. *Let I be an ideal of R , and x_1, \dots, x_n a regular sequence in I on R . Then $n \leq \text{ht}(I)$.*

PROOF. Assume that the assertion holds for $n - 1$. We may assume that $P = I$ is a prime ideal. Set $T = \{kx_n + s \mid k \geq 0, s \in R - P\}$. Suppose that $T \cap (x_1, \dots, x_{n-1}) \neq \emptyset$, say $kx_n + s \in (x_1, \dots, x_{n-1})$. By the choice of s and k_n , we have $k \geq 1$ and $(k - 1)x_n + s \in (x_1, \dots, x_{n-1})$. Thus $x_n + s \in (x_1, \dots, x_{n-1})$; a contradiction. Hence $T \cap (x_1, \dots, x_{n-1}) = \emptyset$. Then there exists a prime ideal Q such that $Q \supset (x_1, \dots, x_{n-1})$ and $Q \cap T = \emptyset$. By the assumption, we have $ht(Q) \geq n - 1$, and hence $ht(P) \geq n$.

Let R be a Noetherian semigroup with maximal ideal M . If $G(M, R) = \dim(R)$, then R is called a Macaulay semigroup. Let A be an R -module, and S an additive system in R . If, for $a_1, a_2 \in A$ and $s_1, s_2 \in S$, we have $a_1 + s_2 + s = a_2 + s_1 + s$ for some $s \in S$, we define $a_1 - s_1 = a_2 - s_2$. Thus $A_S = \{a - s \mid a \in A, s \in S\}$ is an R_S -module. If P is a prime ideal of R , then A_{R-P} is denoted by A_P .

THEOREM 14. *Let A be an R -module, and x_1, \dots, x_n a regular sequence on A . Let S be an additive system in R such that $(x_1, \dots, x_n) + A_S \subseteq A_S$. Then x_1, \dots, x_n is a regular sequence in R_S on A_S .*

Theorem 14 implies the following,

THEOREM 15. *Let P be a prime ideal of R , and I an ideal contained in P . Then $G(I, P) \leq G(I_P, R_P)$.*

THEOREM 16. *Let R be a Macaulay semigroup such that (R, R) satisfies property (*). Then we have $G(I, R) = ht(I)$ for every ideal I of R .*

PROOF. Suppose the contrary. Let P be a maximal member in the set of all ideals I with $G(I, R) < ht(I)$. Then P is a prime ideal by Theorem 6. By Theorem 11, there exists a prime ideal Q containing P such that $G(Q, R) = 1 + G(P, R)$. Then $G(Q, R) < ht(P) + 1 < ht(Q) + 1$, and hence $G(Q, R) < ht(Q)$; a contradiction.

Let P be a prime ideal of R . Then the minimum of $n + 1$ such that there exists a saturated chain of prime ideals $P \supseteq P_1 \supseteq \dots \supseteq P_n$ of R is called the little height of P , and is denoted by $lh(P)$. If R satisfies the following conditions (1) and (2), we say that R satisfies the saturated chain condition:

- (1) $lh(P) = ht(P)$ for every prime ideal P of R .
- (2) For all prime ideals P, Q with $P \supseteq Q$, any two saturated chains from P to Q have the same length.

THEOREM 17. *Let R be a Noetherian semigroup such that (R_P, R_P) satisfies property (*) for every prime ideal P . Then we have $G(P, R) \leq lh(P)$ for every prime ideal P .*

PROOF. We may assume that $M = P$ is a maximal ideal of R . Let $lh(M) = m$, and assume that the assertion holds for $m - 1$. There exists a prime ideal Q with $lh(Q) = m - 1$. Then $G(Q + R_Q, R_Q) \leq m - 1$ by the assumption, and hence $G(Q, R) \leq m - 1$. Then $G(M, R) = 1 + G(Q, R)$ by Theorem 11, and hence $G(M, R) \leq m$.

THEOREM 18. *Let R be a Macaulay semigroup such that (R, R) satisfies property (*). Then R_S is a Macaulay semigroup for every additive system S of R .*

PROOF. There exists a prime ideal P of R such that $N = P + R_S$ is a maximal ideal of R_S . Then we have $G(P, R) = ht(P)$ by Theorem 16. It follows that $ht(N) \leq G(N, R_S)$ by Theorem 15, and $G(N, R_S) \leq ht(N)$ by Theorem 13.

LEMMA 19 ([M3, Theorem 1]). *Let R be a Noetherian semigroup, and x a nonunit of R . If P is a minimal prime ideal over (x) , then $ht(P) = 1$.*

Let S be a g -monoid and R a submonoid of S . If x is an element of S , then the submonoid $R + \mathbf{Z}_0s$ of S is denoted by $R[s]$. Let X be an indeterminate over R . Then the g -monoid $R + \mathbf{Z}_0X$ is denoted by $R[X]$, and is called the polynomial semigroup of X over R .

LEMMA 20 ([TM]). (1) *Assume that R satisfies the ascending chain condition on radical ideals, and let I be an ideal. Then there exists only a finite number of prime ideals minimal over I .*

(2) *Assume that R satisfies the ascending chain condition on radical ideals. If R has an infinite number of prime ideals of height 1, then their intersection is empty.*

(3) *Let R be a g -monoid with $G = q(R)$, and let $u \in R$. Then every prime ideal of R contains u if and only if $G = R[-u]$.*

If $q(R)$ is generated by one element over R as a monoid, that is, if $q(R)$ is of the form $R[x]$ for some $x \in q(R)$, then R is called a G -semigroup. Lemmas 7 and 20 imply the following,

THEOREM 21. *Any Noetherian semigroup is a G -semigroup.*

LEMMA 22 ([TM]). (1) Let P be a prime ideal of R of height 1. Then $P + R[X]$ is a prime ideal of $R[X]$ of height 1.

(2) Let P be a prime ideal of R with $n = ht(P)$. Let Q be a prime ideal of $R[X]$ properly containing $P + R[X]$ and with $P = Q \cap R$. Then $ht(P + R[X]) = n$ and $ht(Q) = n + 1$.

QUESTION. If R is Noetherian, what are conditions for $(R[X], R[X])$ to satisfy property (*)?

THEOREM 23. Let R be a Noetherian semigroup such that $(R[X], R[X])$ satisfies property (*). Then R is a Macaulay semigroup if and only if $R[X]$ is a Macaulay semigroup.

PROOF. Let M be a maximal ideal of R with $n = ht(M)$. The necessity: Let x_1, \dots, x_n be a regular sequence in M on R . Then x_1, \dots, x_n, X is a regular sequence in $R[X]$ on $R[X]$. Hence $R[X]$ is a Macaulay semigroup by Theorem 13 and Lemma 22. The sufficiency: Let X, f_1, \dots, f_n be a regular sequence in $R[X]$ on $R[X]$. Set $f_i = a_i + k_i X$ for each i (for $a_i \in R$ and $k_i \geq 0$). Then we have $k_i = 0$ for each i . It follows that a_1, \dots, a_n is a regular sequence on R .

LEMMA 24 ([M3, Theorem 1]). Let R be a Noetherian semigroup, and I an n -generated proper ideal. Let P be a prime ideal minimal over I . Then $ht(P) \leq n$.

Let P be a prime ideal, and I an ideal contained in P . The maximum of n so that there exists a chain of prime ideals $P \supseteq P_1 \supseteq \dots \supseteq P_n \supset I$ is called the height of P/I , and is denoted by $ht(P/I)$.

THEOREM 25. Let R be a Noetherian semigroup, P a prime ideal, and I an n -generated ideal contained in P . Then $ht(P) \leq n + ht(P/I)$.

PROOF. Let $I = (a_1, \dots, a_n)$ with $a_i \neq a_j$ (for $i \neq j$), and set $ht(P/I) = k$. For any prime ideal Q , the cardinality of $Q \cap \{a_1, \dots, a_n\}$ is called the capacity of Q , and is denoted by $c(Q)$. Let $ht(P) = l + 1$, and let $P \supseteq P_1 \supseteq \dots \supseteq P_l$ be a chain of prime ideals. Then $\sum_1^l c(P_i)$ is called the capacity of the chain. We will show $l + 1 \leq k + n$ by the induction on the number $\sum_1^l c(P_i)$. Thus, if $\sum_1^l c(P_i) = nl$, then $l \leq k$. Hence $l + 1 \leq k + n$. Assume that $0 = c(P_l) = \dots = c(P_\alpha) < c(P_{\alpha-1})$. The case that $ht(P_\alpha) = 1$: Since $ht(P_{\alpha-1}) = 2$, $P_{\alpha-1}$ is not a prime ideal minimal over (a_1) . Hence there exists a prime ideal Q with $a_1 \in Q$ such that $P_{l-1} \supseteq Q$.

Then the capacity of the chain $P \supseteq P_1 \supseteq \cdots \supseteq P_{l-1} \supseteq Q$ is larger than that of the chain $P \supseteq P_1 \supseteq \cdots \supseteq P_{l-1} \supseteq P_l$. By the induction hypothesis, we have $l + 1 \leq k + n$. The case that $ht(P_\alpha) > 1$: If $P_{\alpha+1}$ is a prime ideal minimal over $(P_{\alpha-1}, a_1)$, then P_α is a prime ideal minimal over $P_{\alpha-1}$; a contradiction. Hence $P_{\alpha+1}$ is not a prime ideal minimal over $(P_{\alpha-1}, a_1)$. Then there exists a prime ideal Q such that $P_{\alpha+1} \supseteq Q \supseteq P_{\alpha-1}$. Then the capacity of the chain $P \supseteq P_1 \supseteq \cdots \supseteq P_{\alpha+1} \supseteq Q \supseteq P_{\alpha-1} \supseteq \cdots \supseteq P_l$ is larger than that of the chain $P \supseteq P_1 \supseteq \cdots \supseteq P_{\alpha+1} \supseteq P_\alpha \supseteq P_{\alpha-1} \supseteq \cdots \supseteq P_l$. By the induction hypothesis, we have $l + 1 \geq k + n$.

THEOREM 26. *Let R be a Noetherian semigroup, P a prime ideal, and $x \in P$. Then $ht(P/(x)) = ht(P) - 1$.*

By Theorem 26, we have the following,

THEOREM 27. *Let R be a Noetherian semigroup, M a maximal ideal of R , and $x \in M$. Let $k = ht(M/(x))$, and x_1, \dots, x_k be elements of M such that $x_1 \notin Z(R/(x))$, $x_2 \notin Z(R/(x, x_1))$, \dots , $x_{k-1} \notin Z(R/(x, x_1, \dots, x_{k-1}))$. Then R is a Macaulay semigroup.*

LEMMA 28 ([TM]). *Let R be a Noetherian semigroup with maximal ideal M , and A a finitely generated R -module. Assume that $A = (a_1, \dots, a_r, M + A)$. Then $M = (a_1, \dots, a_r)$.*

Let R be a Noetherian semigroup with maximal ideal M . If M is generated by a finite subset $\{a_1, \dots, a_n\}$ of R , and If M is not generated by any proper subset of $\{a_1, \dots, a_n\}$, then $\{a_1, \dots, a_n\}$ is called a minimal generators of M . Let $\{a_1, \dots, a_n\}$ and $\{b_1, \dots, b_m\}$ be two set of minimal generators of M . Then each a_i is contained in the ideal $(b_{m(i)})$ for some $m(i)$, and each b_j is contained in some $(a_{n(i)})$. Then it follows that $i = n(m(i))$ and $j = m(n(j))$ for all i and j . Hence $n = m$. The cardinality of a minimal generators of M is called the V-dimension of R , and is denoted by $V(R)$.

THEOREM 29. *Let R be a Noetherian semigroup with maximal ideal M , and let $x \in M - 2M$. Let r be the minimum number so that there exist x_1, \dots, x_r with $(x, x_1, \dots, x_r) = M$. Then $r = V(R) - 1$.*

For a Noetherian semigroup R , we have $V(R) \geq dim(R)$ by Lemma 24. A Noetherian semigroup R is called a regular semigroup if $V(R) = dim(R)$.

THEOREM 30. *Let R be a Noetherian semigroup with maximal ideal M . Assume that M is generated by a regular sequence a_1, \dots, a_k on R . Then $k = \dim(R) = V(R)$, and R is a regular semigroup.*

PROOF. We have $k \leq G(M, R) \leq ht(M)$ by Theorem 13. Also we have $ht(M) \leq V(R) \leq k$ by Lemma 24.

THEOREM 31. *Let R be a regular semigroup with maximal ideal M , and $x \in M - 2M$. Put $ht(M/(x)) = k$. Then there exist x_1, \dots, x_k such that $M = (x, x_1, \dots, x_k)$.*

PROOF. We have $\dim(R) = k + 1$ by Theorem 26. By Theorem 29, there exist x_1, \dots, x_k such that $M = (x, x_1, \dots, x_k)$.

THEOREM 32. *Let R be a Noetherian semigroup with maximal ideal M , and let $x \in M - 2M$. Put $ht(M/(x)) = k$. Assume that there exist elements x_1, \dots, x_k such that $M = (x, x_1, \dots, x_k)$. Then R is a regular semigroup.*

PROOF. Because $ht(M) = k + 1$ by Theorem 26.

THEOREM 33. *Let R be a regular semigroup of dimension n with maximal ideal M . Let $M = (x_1, \dots, x_n)$. Then $x_i \notin 2M$ for each i , and x_1, \dots, x_n is a regular sequence on R .*

PROOF. By Lemma 28, we have $x_i \notin 2M$. It follows that x_1, \dots, x_n is a complete representatives of irreducible elements of R . Suppose that $x_k \in Z(R/(x_1, \dots, x_{k-1}))$. There exists $y \in M - (x_1, \dots, x_{k-1})$ such that $x_k + y \in (x_1, \dots, x_{k-1})$. Let P be a prime ideal minimal over (x_1, \dots, x_{k-1}) . Then there exists $l \geq k$ such that $x_l \in P$. There exist irreducible elements a_1, \dots, a_{n-k} of R such that $M = (P, a_1, \dots, a_{n-k})$. Then we have $ht(M) \leq ht(P) + n - k \leq k - 1 + n - k = n - 1$, namely $ht(M) \leq n - 1$; a contradiction.

Theorem 33 implies the following,

THEOREM 34. *Any regular semigroup is a Macaulay semigroup.*

THEOREM 35. *Let R be a Noetherian semigroup such that R_P is regular for every prime ideal P of R . Then $R[X]_Q$ is regular for every prime ideal Q of $R[X]$.*

PROOF. Set $R[X] = T$. We may assume that R is a regular semigroup with maximal ideal M , and $R \cap N = M$ for a prime ideal N of T . Then M is generated by a regular sequence a_1, \dots, a_n . If $M + R[X] = N$, then our assertion holds. If $N \not\supseteq M + R[X]$, then $N = (a_1, \dots, a_n, X)$, and a_1, \dots, a_n, X is a regular sequence on $R[X]$. Theorem 30 completes the proof.

THEOREM 36. *Let R be a Macaulay semigroup such that (R, R) satisfies property (*). Let I be a proper ideal of height n , which can be generated by n elements x_1, \dots, x_n . Then $P = Z(R/I)$ is a prime ideal of R , has height n and a minimal prime over I .*

PROOF. We see that P is a prime ideal of R ([TM]). Theorem 16 implies that $G(I, R) = n$. Theorem 9 implies that x_1, \dots, x_n is a maximal regular sequence in I on R . Theorem 6 implies that x_1, \dots, x_n is a maximal regular sequence in P on R . Then Theorem 16 implies that $ht(P) = n$.

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