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A NOTE ON KAEHLERIAN METRICS WITH CERTAIN PROPERTY FOR ∇RIC

By

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0. Introduction

Let M^{2n} be a $2n$ real dimensional Kaehlerian manifold, whose complex structure is given by a parallel tensor field $F=(F_j^i)$ satisfying

$$F_j^r F_r^i = \delta_j^i, \quad g_{ji} F_\ell^j F_k^i = g_{\ell k},$$

where $g=(g_{ji})$ is the Riemannian metric tensor of M^{2n}. Let $K=(K_{kji}^h)$ be the Bochner curvature tensor and $\hat{K}=(K_{kji})$ a tensor given by

$$\begin{aligned}(0.1)\qquad K_{kji} &= \nabla_k R_{ji} - \nabla_j R_{ki} \\ &+ \frac{1}{4(n+1)}(g_{ki}\delta_j^h - g_{ji}\delta_k^h + F_{ki}F_j^h - F_{ji}F_k^h + 2F_{kj}F_i^h)r_h,\end{aligned}$$

where $R=(R_{kji}^h)$ is the Riemannian curvature tensor, $\mathrm{Ric}=(R_{ji})=(R_{kji}^k)$ the Ricci tensor, and $r=R_k^k$ the scalar curvature.

Let us consider the condition

$$(\sharp)\qquad \nabla_k R_{ji} = \frac{1}{4(n+1)}(2r_k g_{ji} + r_j g_{ki} + r_i g_{kj} + \tilde{r}_j F_{ik} + \tilde{r}_i F_{jk}),$$

where $\tilde{r}_j = F_j^h r_h$ and $r_j = \nabla_j r$. This condition gives a necessary and sufficient condition for equality in the inequality

$$\frac{1}{m+1}|dr|^2 \leqq |\nabla\,\mathrm{Ric}|^2$$

which was proved in [2].

If M^{2n} satisfies ($\sharp$) $\hat{K}$ vanishes [2]. But the converse is not true. The example of metric satisfying ($\sharp$) is unknown, except the case where r is constant.

On the other hand, if M^{2n} is Bochner-flat, i.e. $K=0$, then $\hat{K}$ vanishes. The

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examples of non-flat Bochner-flat metric have been found by Tachibana and Liu [1]. The purpose of this paper is to show that Tachibana and Liu's metrics just give examples of non-flat metrics satisfying (♮).

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1. Preliminaries

Throughout this paper, the complex coordinate $\{z^\lambda, z^{\lambda^*}\}$ shall be used, where $z^{\lambda^*} = \bar{z}^\lambda$, the conjugate of z^λ. We adopt the following ranges of indices:

$$1 \leqq i, j, k, \ldots \leqq 2n,$$

$$1 \leqq \lambda, \mu, \nu, \ldots \leqq n, \quad \lambda^* = \lambda + n.$$

With respect to the complex coordinate, the metric tensor g_{ji} and the complex structure F_j^i of M^{2n} satisfies

(1.1)
$$g_{\mu\lambda} = g_{\mu^*\lambda^*} = 0, \quad g_{\mu\lambda^*} = g_{\lambda^*\mu} = \bar{g}_{\lambda\mu^*} = \bar{g}_{\mu^*\lambda},$$

(1.2)
$$F_\mu^\lambda = i\delta_\mu^\lambda, \quad F_\mu^{\lambda^*} = 0,$$
$$F_{\mu\lambda} = g_{\alpha^*\lambda} F_\mu^{\alpha^*} = 0, \quad F_{\mu^*\lambda} = F_{\mu^*}^{\alpha^*} g_{\alpha^*\lambda} = -ig_{\mu^*\lambda},$$

and the Ricci tensor $\mathrm{Ric} = (R_{ji})$ and scalar curvature $r = R_\lambda^\lambda + R_{\lambda^*}^{\lambda^*} = 2R_\lambda^\lambda$ satisfy

(1.3)
$$R_{\mu\lambda} = R_{\mu^*\lambda^*} = 0, \quad R_{\mu^*\lambda} = R_{\lambda\mu^*},$$
$$\tilde{r}_\mu = F_\mu^\alpha r_\alpha = i\delta_\mu^\alpha r_\alpha = ir_\mu,$$
$$\tilde{r}_\mu F_{\nu^*\lambda} = (ir_\mu)(-ig_{\lambda\nu^*}) = r_\mu g_{\lambda\nu^*}.$$

Putting $K_{kjih} = K_{kji}^r g_{rh}$, the Bochner curvature tensor is given by

$$K_{\lambda\mu^*\nu\rho^*} = R_{\lambda\mu^*\nu\rho^*} - \frac{1}{n+2}(g_{\lambda\mu^*}R_{\nu\rho^*} + g_{\lambda\rho^*}R_{\nu\mu^*} + g_{\nu\rho^*}R_{\lambda\mu^*} + g_{\nu\mu^*}R_{\lambda\rho^*}) + \frac{R}{2(n+1)(n+2)}(g_{\lambda\mu^*}g_{\nu\rho^*} + g_{\lambda\rho^*}g_{\nu\mu^*}).$$

By virtue of (1.1)–(1.3), the condition (♮) is reduced to the following simple form:

(♮′)
$$\nabla_\lambda R_{\mu\nu^*} = \frac{1}{2(n+1)}(r_\lambda g_{\mu\nu^*} + r_\mu g_{\lambda\nu^*}).$$

2. Metrics with Vanishing Bochner Curvature Tensor

Let C^n be the complex number space with complex coordinate $\{z_\lambda\}$. In the following of this paper, we denote coordinates by z_λ instead of z^λ. A real valued holomorphic function $\phi = \phi(z, \bar{z})$ of $\{z_\lambda, \bar{z}_\lambda\}$ gives a Kaehlerian metric $g_{\mu\lambda^*} = \partial^2\phi/\partial z_\mu \partial \bar{z}_\lambda$ to C^n or its subdomain. Under the assumption that ϕ is a function of $t = \sum_{\alpha=1}^n z_\alpha \bar{z}_\alpha$, Tachibana and Liu found $\phi = f(t)$ so that the corresponding Kaehlerian metric has the vanishing Bochner curvature tensor. In this case, the metric tensor $g_{\mu\lambda^*}$, the Christoffel symbols $\Gamma^\nu_{\mu\lambda}$, the Ricci tensor $R_{\mu\lambda^*}$ and the scalar curvature r are as follows ($'$ means differentiation with respect to t):

(2.1) $$g_{\mu\lambda^*} = f'\delta_{\mu\lambda} + f''\bar{z}_\mu z_\lambda,$$

(2.2) $$\Gamma^\nu_{\mu\lambda} = \frac{f''}{f'}(\bar{z}_\mu \delta_{\nu\lambda} + \bar{z}_\lambda \delta_{\nu\mu}) + \sigma z_\nu \bar{z}_\mu \bar{z}_\lambda,$$

where

(2.3) $$\sigma = \frac{f'f''' - 2f''^2}{f'(f' + tf'')},$$

(2.4) $$R_{\mu\lambda^*} = \lambda \bar{z}_\mu z_\lambda + \mu \delta_{\mu\lambda},$$

where

(2.5) $$\lambda = -\frac{(n+1)(f'f''' - f''^2)}{f'^2} - \sigma' t - \sigma = \mu'$$

and

(2.6) $$\mu = -\frac{(n+1)f''}{f'} - \sigma t,$$

(2.7) $$r = \frac{2}{f'}\left(t\lambda + n\mu - \frac{tf''(t\lambda + \mu)}{f' + tf''}\right).$$

For convenience sake we put

(2.8) $$\Delta = \frac{r}{2(n+1)(n+2)}.$$

On account of vanishing Bochner curvature tensor, the function f satisfies the differential equation

(2.9) $$2\sigma f'' = \sigma' f',$$

which induces by integration

$$\sigma = af'^2 \quad (a \text{ is a constant}) \tag{2.10}$$

or equivalently

$$f'f''' - 2f''^2 = af'^3(f' + tf''). \tag{2.11}$$

From this equation, Tachibana and Liu obtained the result: that $f(t)$ gives a non-flat Bochner-flat Kaehlerian metric satisfying $f''(0) = 0$ is equivalent to that f takes one of the following two forms;

$$(\sharp\sharp) \qquad \begin{aligned} f(t) &= \frac{1}{c}\sin^{-1}\left(\frac{c}{b}t\right) + k, \\ f(t) &= \frac{1}{c}\sinh^{-1}\left(\frac{c}{b}t\right) + k, \end{aligned}$$

where b and c are positive constants and k is any constant.

For these metrics the following formulae hold [1]:

$$\sigma' = 2af'f'' \tag{2.12}$$

$$f'' = atf'^3, \tag{2.13}$$

$$f'^2\Delta = -\frac{nf'' + 2f'\sigma t}{n+2}, \tag{2.14}$$

$$\lambda = -(n+2)af'^2(1 + 2at^2f'^2), \tag{2.15}$$

$$\mu = -(n+2)atf'^2 \tag{2.16}$$

$$g_{\mu\nu^*} = f'(\sigma_{\mu\nu} + atf'^2\bar{z}_\mu z_\nu). \tag{2.17}$$

3. Metrics Satisfying the Condition ($\sharp$)

Now we shall show that above-mentioned metrics are examples of the space satisfying ($\sharp$).

First we calculate each term of

$$\nabla_\lambda R_{\mu\nu^*} = \partial_\lambda R_{\mu\nu^*} - \Gamma^\rho_{\lambda\mu} R_{\rho\nu^*} - \Gamma^{\rho^*}_{\lambda\nu^*} R_{\mu\rho^*}.$$

From (2.4) and $\mu' = \lambda$, the first term is

$$\partial_\lambda R_{\mu\nu^*} = \lambda' \bar{z}_\lambda \bar{z}_\mu z_\nu + \lambda(\bar{z}_\mu \delta_{\lambda\nu} + \bar{z}_\lambda \delta_{\mu\nu}). \tag{3.1}$$

From (2.2), (2.9), (2.10) and (2.13), we have

$$\Gamma^{\rho}_{\lambda\mu} = af'^2\{z_\rho \bar{z}_\lambda \bar{z}_\mu + t(\bar{z}_\lambda \delta_{\rho\mu} + \bar{z}_\mu \delta_{\rho\lambda})\}, \tag{3.2}$$

from which the second term is

$$\begin{aligned}\Gamma^{\rho}_{\lambda\mu} R_{\rho\nu^*} &= \sum_\rho af'^2\{z_\rho \bar{z}_\lambda \bar{z}_\mu + t(\bar{z}_\lambda \delta_{\rho\mu} + \bar{z}_\mu \delta_{\rho\lambda})\}\{\lambda \bar{z}_\rho z_\nu + \mu \delta_{\rho\nu}\} \\ &= af'^2\{(\lambda t + \mu)\bar{z}_\lambda \bar{z}_\mu z_\nu + 2\lambda t \bar{z}_\lambda \bar{z}_\mu z_\nu + \mu t(\bar{z}_\mu \delta_{\lambda\nu} + \bar{z}_\lambda \sigma_{\mu\nu})\}.\end{aligned} \tag{3.3}$$

Hence

$$\nabla_\lambda R_{\mu\nu^*} = \{\lambda' - af'^2(3\lambda t + \mu)\}\bar{z}_\lambda \bar{z}_\mu z_\nu + \{\lambda - af'^2 \mu t\}(\bar{z}_\lambda \delta_{\mu\nu} + \bar{z}_\mu \delta_{\lambda\nu}) \tag{3.4}$$

from (3.1) and (3.3) on account of $\Gamma^{\rho^*}_{\lambda\nu^*} = 0$, where insides of { } is calculated as follows: differentiating (2.15), and using (2.13), we have

$$\begin{aligned}\lambda' &= -(n+2)a\{2f'f''(1 + 2at^2 f'^2) + f'^2 2a(2tf'^2 + 2t^2 f'f'')\} \\ &= -2(n+2)a^2 tf'^4(3 + 4at^2 f'^2),\end{aligned} \tag{3.5}$$

and from (2.15) and (2.16)

$$3\lambda t + \mu = -2(n+2)atf'^2(2 + 3at^2 f'^2). \tag{3.6}$$

Hence

$$\begin{aligned}\lambda' - af'^2(3t\lambda + \mu) &= -2(n+2)a^2 tf'^4(3 + 4at^2 f'^2) \\ &\quad + 2(n+2)a^2 tf'^4(2 + 3at^2 f'^2) \\ &= -2(n+2)a^2 tf'^4(1 + at^2 f'^2),\end{aligned} \tag{3.7}$$

and from (2.15) and (2.16),

$$\begin{aligned}\lambda - af't\mu &= -(n+2)af'^2(1 + 2at^2 f'^2) + (n+2)a^2 t^2 f'^4 \\ &= -(n+2)af'^2(1 + at^2 f'^2).\end{aligned} \tag{3.8}$$

Substituting (3.7) and (3.8) into (3.4), we obtain

$$\begin{aligned}\nabla_\lambda R_{\mu\nu^*} &= -2(n+2)a^2 tf'^4(1 + at^2 f'^2)\bar{z}_\lambda \bar{z}_\mu z_\nu \\ &\quad - (n+2)af'^2(1 + at^2 f'^2)(\bar{z}_\lambda \delta_{\mu\nu} + \bar{z}_\mu \delta_{\lambda\nu}) \\ &= -(n+2)af'^2(1 + at^2 f'^2)(2atf'^2 \bar{z}_\lambda \bar{z}_\mu z_\nu + \bar{z}_\lambda \delta_{\mu\nu} + \bar{z}_\mu \delta_{\lambda\nu}).\end{aligned} \tag{3.9}$$

Now we shall calculate the right hand side of ($\sharp'$) for our metrics. Substituting (2.10) and (2.13) in (2.14), we have

$$\Delta = -atf',$$

from which

$$r = 2(n+1)(n+2)\Delta = -2(n+1)(n+2)atf'. \tag{3.10}$$

Differentiating (3.10) by z_λ, and taking account of (2.13) we have

$$\begin{aligned} r_\lambda &= -2(n+1)(n+2)a(\bar{z}_\lambda f' + tf''\bar{z}_\lambda) \\ &= -2(n+1)(n+2)af'(1 + at^2 f'^2)\bar{z}_\lambda. \end{aligned} \tag{3.11}$$

Substituting (3.11) and (2.17), we can see that the right hand side of ($\sharp'$) coincides to that of (3.9).

Thus we conclude that the metrics given by ($\sharp\sharp$) satisfy ($\sharp$).

References

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