ON THE ESTIMATES FOR HELMHOLZ OPERATOR

Dedicated to Professor Kiyoshi Mochizuki on his sixtieth birthday

By

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Abstract. The formulas of the Bessel functions are applied to obtain the estimates of the limiting absorption principle. As an application we prove a result on smoothing effect for the Schrödinger equation.

1. Introduction

In the present paper, we are mainly concerned with the estimates of the limiting absorption principle. However we also deal with restriction properties of the Fourier transform and smoothing effects for dispersive wave equations. They connect with each other deeply, and it can be said that all of them are the estimates for Fourier multipliers with non-smooth symbols. The purpose of the present paper is to improve the results of the previous paper [9] from such a point of view.

We start with introducing some notations related to the polar coordinate. We suppose throughout this paper that the space dimension n satisfies $n \geq 2$. First let Λ be the Laplace-Beltrami operator on the unit sphere S^{n-1}. Namely

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r} \frac{\partial}{\partial r} + \frac{\Lambda}{r^2},$$

where Δ is the Laplace operator in Euclidean space R^n and $r = |x|$. It is known that the eigenvalues of $-\Lambda$ are

$$\lambda_k = k(k+n-2), \quad k = 0, 1, 2, \dots$$

and the projection on to the eigenspace associated with λ_k in $L^2(S^{n-1})$ can be expressed as follows if $n \geq 3$:

$$(1.1) \qquad H_k f(\omega) = \frac{v+k}{v|S^{n-1}|} \int_{\tilde{\omega} \in S^{n-1}} C_k^v(\omega \cdot \tilde{\omega}) f(\tilde{\omega}) \, d\tilde{\omega},$$

where $v = (n - 2)/2$, $C_k^v(z)$ is the Gegenbauer polynomial of degree k, $\omega \cdot \tilde{\omega} = \sum_{j=1}^{n} \omega_j \tilde{\omega}_j$, $d\tilde{\omega}$ and $|S^{n-1}|$ are respectively Lebesgue surface measure and the area of the unit sphere.

Note that the operator H_k can be applied to the functions of R^n. Namely, we can replace $f(\tilde{\omega})$ by $f(r\tilde{\omega})$. Also fractional power $(I - \Lambda)^\alpha (\alpha \in R)$ can be written as follows.

$$(I - \Lambda)^\alpha = \sum_{k=0}^{\infty} (1 + \lambda_k)^\alpha H_k.$$

The first result of this paper is

THEOREM 1. *Suppose that* $0 < \alpha < 1/2$ *and* $\alpha' > \alpha$. *Then we have*

(1.2) $\| |x|^{\alpha-1}(I - \Lambda)^{(1-2\alpha')/2} |D|^{2\alpha} u \|_{L^2(R^n)} \leq C \| |x|^{1-\alpha}(-\Delta - \zeta) u \|_{L^2(R^n)},$

where the constant C *may depend on* n, α *and* α', *but does not depend on* $u \in C_0^\infty(R^n)$ *and* $\zeta \in C \backslash [0, \infty)$.

The estimate (1.2) is an analogue of the estimate obtained by S. Agmon [1] (see also Kato and Yajima [11]). Recall that Theorem 1.1 in [9] dealt with a restriction property of the Fourier transform in a similar semi-norm as above. Also note that the restriction property and the limiting absoption principle are closely connected. Thus the estimate (1.2) is a variant of Theorem 1.1 in [9] to the limiting absorption principle. The author expects that, in (1.2), α' can be taken so that $\alpha = \alpha'$. However he could not prove it by a technical reason.

We turn to state the next result. In the previous work [9], we have shown some results concerning smoothing effects for the homogeneous initial value problems. The next purpose is to improve it to the inhomogeneous one. Let $u(t, x)$ $((t, x) \in R^{n+1})$ be a solution to the Schrödinger equation as follows:

(1.3) $\begin{cases} i\dfrac{\partial u}{\partial t} + \Delta u = f, \\ u|_{t=0} = 0. \end{cases}$

Then

THEOREM 2. *Suppose that* $0 < \alpha < 1/2$ *and* $\alpha' > \alpha$. *Then, concerning the solutions of the initial value problem* (1.3), *we have*

$\| |x|^{\alpha-1}(I - \Lambda)^{(1-2\alpha')/2} |D|^{2\alpha} u \|_{L^2(R^{n+1})} \leq C \| |x|^{1-\alpha} f \|_{L^2(R^{n+1})},$

where the constant C may depend on n, α and α', but does not depend on $f \in C_0^\infty(\mathbf{R}^{n+1})$.

In Theorem 1.3 of [9], we have dealt with a case of the similar regularity property as above. Theorem 2 says that, compared with the known results (cf., for example T. Kato and K. Yajima [11]), the solution $u(t, x)$ has better property on the smoothness of angular variables.

Next we turn to state the results, which are proved by another approach. The following theorem is an improvement of the result given in section 3 of the previous paper [9].

THEOREM 3. *Suppose that* $n \geq 3$, $1/p - 1/q \leq 2/n$, $(n+1)/2 \leq n/p - 1/q$ *and* $n/q - 1/p \leq (n-3)/2$. *Set*

$$\alpha = 1 - \frac{n}{2}\left(\frac{1}{p} - \frac{1}{q}\right).$$

Then we have

(1.4) $\| |D|^{2\alpha} u \|_{L^q(\mathbf{R}^n)} \leq C \|(-\Delta - \zeta)u\|_{L^p(\mathbf{R}^n)},$

where the constant C *may depend on* n, p *and* q, *but does not depend on* $u \in C_0^\infty(\mathbf{R}^n)$ *and* $\zeta \in C \setminus [0, \infty)$.

Notice that the case $\alpha = 0$ in Theorem 3 is contained in the results of C. Kenig, A. Ruiz and C. D. Sogge [13], the purpose of which is to prove unique continuation theorems for Schrödinger operators. Our approach here is applicable to prove weighted $L^p - L^q$ estimates as follows.

THEOREM 4. *Suppose that* $n \geq 3$, $(n-1)/(2(n+1)) < 1/q \leq 1/p < (n+3)/(2(n+1))$, $s + s' < 2 - n(1/p - 1/q)$, $s/n + s' > -1/p + n/q - (n-3)/2$ *and* $s + s'/n > -n/p + 1/q + (n+1)/2$. *Set*

$$\alpha = 1 - \frac{s+s'}{2} - \frac{n}{2}\left(\frac{1}{p} - \frac{1}{q}\right).$$

Then we have

$$\| |x|^{-s'} |D|^{2\alpha} u \|_{L^q(\mathbf{R}^n)} \leq C \| |x|^s (-\Delta - \zeta)u \|_{L^p(\mathbf{R}^n)},$$

where the constant C *may depend on* n, p, q, s *and* s', *but does not depend on* $u \in C_0^\infty(\mathbf{R}^n)$ *and* $\zeta \in C \setminus [0, \infty]$.

This paper is organized as follows: In Section 2 we prove Theorem 1 and Theorem 2. They can be regarded as consequences of a restriction property as in [9] with Hölder continuity with respect to a parameter. Section 3 is devoted to the proofs of Theorem 3 and Theorem 4, the technique of which is quite different from the one of Section 2. In Appendix, we prove a proposition, which is necessary for the proofs. As usual, the letter C will denote a constant that may be different in different equations or inequalities.

Finally, the author would like to express his sincere graditude to the referee, who pointed out an error in the proof of the original manuscript and kindly suggested a way to recover it.

2. Weighted L^2 Estimates

In this section, we shall prove Theorem 1 and Theorem 2. We begin the proofs by introducing some operator related to the restriction of Fourier transform to spheres. For $f, g \in C_0^\infty(R^n)$ and $0 < \alpha < 1/2$, set

$$(2.1) \qquad (A_\alpha(\lambda)f, g) = \frac{\lambda^{\alpha-(1/2)}}{2(2\pi)^n} \int_{|\xi|=\sqrt{\lambda}} \hat{f}(\xi)\overline{\hat{g}(\xi)} \, dS_{\sqrt{\lambda}},$$

where $dS_{\sqrt{\lambda}}$ is surface measure of $S_{\sqrt{\lambda}} = \{\xi \in R^n \mid |\xi| = \sqrt{\lambda}\}$ and $\hat{f}$, $\hat{g}$ are Fourier transforms of f, g respectively. The operator $A_\alpha(\lambda)$ is useful to prove Theorem 1, because the expression by Fourier transform gives

$$(2.2) \qquad |D|^{2\alpha}(-\Delta - \zeta)^{-1} = \int_0^\infty \frac{A_\alpha(\lambda)}{\lambda - \zeta} \, d\lambda.$$

Also the right hand side of (2.1) can be represented by the projection operator H_k in Introduction. Indeed, when $n \geq 3$ we have

$$(2.3) \qquad (A_\alpha(\lambda)f, g) = \frac{\lambda^\alpha}{2} \sum_{k=0}^\infty \int_{|\omega|=1} d\omega \int_0^\infty J_{k+\nu}(\sqrt{\lambda}r)r^{n/2}H_k f(r, \omega) \, dr$$

$$\times \overline{\int_0^\infty J_{k+\nu}(\sqrt{\lambda}\rho)\rho^{n/2}H_k g(\rho, \omega) \, d\rho}$$

where $J_{\nu+k}$ is the Bessel function of order $\nu + k$. This equality comes from the classical formulas of the Bessel functions as follows (see A. Eldélyi et al. [6] and T. Hoshiro [9]):

$$\int_{|\xi|=\sqrt{\lambda}} e^{i(y-x)\xi}\, dS_{\sqrt{\lambda}} = \lambda^{(n-1)/2} \int_{|\omega|=1} e^{i(y-x)\sqrt{\lambda}\omega}\, d\omega$$

$$= \lambda^{(n-1)/2}(2\pi)^{n/2} \frac{J_\nu(\sqrt{\lambda}|x-y|)}{|\sqrt{\lambda}|x-y||},$$

$$(2.4) \qquad \frac{J_\nu(|x-y|)}{|x-y|^\nu} = 2^\nu \Gamma(\nu) \sum_{k=0}^\infty (\nu+k) \frac{J_{\nu+k}(r)}{r^\nu} \frac{J_{\nu+k}(\rho)}{\rho^\nu} C_k^\nu(\omega_1 \cdot \omega_2),$$

$$(x = r\omega_1, \quad y = \rho\omega_2)$$

and

$$\frac{\nu+k}{\nu|S^{n-1}|} \int_{|\omega|=1} C_k^\nu(\omega_1 \cdot \omega) C_\ell^\nu(\omega \cdot \omega_2)\, d\omega = \begin{cases} C_k^\nu(\omega_1 \cdot \omega_2), & (k = \ell), \\ 0, & (k \neq \ell). \end{cases}$$

Even if $n = 2$ the equality (2.3) holds, because the representation of the operator H_k should be replaced by

$$H_k f(\omega) = \begin{cases} \dfrac{1}{2\pi} \displaystyle\int_{|\tilde{\omega}|=1} f(\tilde{\omega})\, d\tilde{\omega}, & k = 0, \\ \dfrac{1}{\pi} \displaystyle\int_{|\tilde{\omega}|=1} \cos k(\omega \cdot \tilde{\omega}) f(\tilde{\omega})\, d\tilde{\omega}, & k = 1, 2, \dots \end{cases}$$

and the addition formula (2.4) should be replaced by

$$J_0(|x-y|) = J_0(r)J_0(\rho) + 2\sum_{k=0}^\infty J_k(r)J_k(\rho) \cos k(\omega_1 \cdot \omega_2),$$

$$(x = r\omega_1, y = \rho\omega_2).$$

The representation (2.3) tells us that the operator $A_\alpha(\lambda)$ is uniformly bounded and (locally) Hölder continuous with respect to $\lambda > 0$ in certain operator norm. Precisely we have

LEMMA 2.1.

(i) *For* $f, g \in C_0^\infty(\mathbf{R}^n)$,

$$(2.5) \qquad |(H_k A_\alpha(\lambda)f, g)| \leq C(1+k)^{2\alpha-1}\| |x|^{1-\alpha} H_k f\|_{L^2(\mathbf{R}^n)} \| |x|^{1-\alpha} H_k g\|_{L^2(\mathbf{R}^n)},$$

where the constant C may depend on n and α, but does not depend on f, g, k and λ.

(ii) *Let θ be a real number satisfying $0 \leq \theta < 1 - 2\alpha$. Then, for $f, g \in C_0^\infty(\mathbf{R}^n)$,*

(2.6) $|(H_k(A_\alpha(\lambda) - A_\alpha(\mu))f, g)| \leq C_\theta(1 + k)^{2\alpha - 1}$

$$\times |(1 + k)|\lambda - \mu|/\max(\lambda, \mu)|^{\theta/2} \| |x|^{1-\alpha} H_k f \|_{L^2(\mathbf{R}^n)} \| |x|^{1-\alpha} H_k g \|_{L^2(\mathbf{R}^n)},$$

where the constant C_θ may depend on n, α, and θ, but does not depend on f, g, k, λ and μ.

REMARK. The inequality (2.5) implies

$$|(A_\alpha(\lambda)f, g)| \leq C' \| |x|^{1-\alpha} f \|_{L^2(\mathbf{R}^n)} \| |x|^{1-\alpha} (I - \Lambda)^{\alpha - (1/2)} g \|_{L^2(\mathbf{R}^n)},$$

by taking the summation with respect to k (notice that $\lambda_k = O(k^2)$ as $k \to \infty$). Also the inequality (2.6) implies

$$|((A_\alpha(\lambda) - A_\alpha(\mu))f, g)|$$

$$\leq C'_\theta |\lambda - \mu|/\max(\lambda, \mu)|^{\theta/2} \| |x|^{1-\alpha} f \|_{L^2(\mathbf{R}^n)} \| |x|^{1-\alpha} (I - \Lambda)^{\alpha - (1/2) + (\theta/4)} g \|_{L^2(\mathbf{R}^n)}.$$

PROOF. First notice that

$$(H_k A_\alpha(\lambda)f, g) = \frac{\lambda^\alpha}{2} \int_{|\omega|=1} d\omega \int_0^\infty J_{k+v}(\sqrt{\lambda} r) r^{n/2} H_k f(r, \omega) \, dr$$

$$\times \overline{\int_0^\infty J_{k+v}(\sqrt{\lambda} \rho) \rho^{n/2} H_k g(\rho, \omega) \, d\rho}.$$

Hence by Schwarz inequality, we obtain

$$|(H_k A_\alpha(\lambda)f, g)| \leq \frac{\lambda^\alpha}{2} \left(\int_0^\infty J_{v+k}(\sqrt{\lambda} r)^2 r^{2\alpha - 1} \, dr \right)$$

$$\times \left(\int_{|\omega|=1} \int_0^\infty r^{n-1+2-2\alpha} |H_k f(r, \omega)|^2 \, dr d\omega \right)^{1/2}$$

$$\times \left(\int_{|\omega|=1} \int_0^\infty r^{n-1+2-2\alpha} |H_k g(r, \omega)|^2 \, dr d\omega \right)^{1/2}$$

$$\leq \frac{1}{2} \int_0^\infty J_{v+k}(r)^2 r^{2\alpha - 1} \, dr \cdot \| |x|^{1-\alpha} H_k f \|_{L^2(\mathbf{R}^n)} \| |x|^{1-\alpha} H_k g \|_{L^2(\mathbf{R}^n)}.$$

We quote now the formula of Weber-Schaftheitlin (see A. Erdélyi *et al.* [6]), namely

$$\int_0^\infty J_{v+k}(r)^2 r^{2\alpha-1} dr = \frac{\Gamma(1-2\alpha)\Gamma(v+k+\alpha)}{2^{1-2\alpha}\Gamma(1-2\alpha)^2\Gamma(v+k+1-\alpha)}.$$

Hence the above integral is asymptotically $O((1+k)^{2\alpha-1})$ as $k \to \infty$. Thus the inequality (2.5) follows.

The proof of (2.6) is similar to the above. Notice that

$$(H_k(A_\alpha(\lambda) - A_\alpha(\mu))f, g)$$

$$= \int_{|\omega|=1} d\omega \left(\int_0^\infty \{\lambda^{\alpha/2} J_{v+k}(\sqrt{\lambda}r) - \mu^{\alpha/2} J_{v+k}(\sqrt{\mu}r)\} r^{n/2} H_k f(r, \omega) dr \right.$$

$$\times \int_0^\infty \lambda^{\alpha/2} J_{v+k}(\sqrt{\lambda}\rho) \rho^{n/2} H_k g(\rho, \omega) d\rho$$

$$+ \int_0^\infty \mu^{\alpha/2} J_{v+k}(\sqrt{\mu}r) r^{n/2} H_k f(r, \omega) dr$$

$$\left. \times \int_0^\infty \{\lambda^{\alpha/2} J_{v+k}(\sqrt{\lambda}\rho) - \mu^{\alpha/2} J_{v+k}(\sqrt{\mu}\rho)\} \rho^{n/2} H_k g(\rho, \omega) d\rho \right).$$

Hence by a similar argument as in the proof of (2.5), we obtain

$$|(H_k(A_\alpha(\lambda) - A_\alpha(\mu))f, g)| \le C \cdot B(\alpha, k; \lambda, \mu)\| |x|^{1-\alpha} H_k f\|_{L^2(\mathbf{R}^n)} \| |x|^{1-\alpha} H_k g\|_{L^2(\mathbf{R}^n)},$$

where

$$B(\alpha, k; \lambda, \mu) = \left(\int_0^\infty J_{v+k}(r)^2 r^{2\alpha-1} dr \right)^{1/2}$$

$$\times \left(\int_0^\infty |\lambda^{\alpha/2} J_{v+k}(\sqrt{\lambda}\rho) - \mu^{\alpha/2} J_{v+k}(\sqrt{\mu}\rho)|^2 \rho^{2\alpha-1} d\rho \right)^{1/2}.$$

Moreover observe that

$$\int_0^\infty |\lambda^{\alpha/2} J_{v+k}(\sqrt{\lambda}\rho) - \mu^{\alpha/2} J_{v+k}(\sqrt{\mu}\rho)|^2 \rho^{2\alpha-1} d\rho$$

$$= 2\left\{ \int_0^\infty J_{v+k}(\rho)^2 \rho^{2\alpha-1} d\rho - (\lambda\mu)^{\alpha/2} \int_0^\infty J_{v+k}(\sqrt{\lambda}\rho) J_{v+k}(\sqrt{\mu}\rho) \rho^{2\alpha-1} d\rho \right\}$$

and recall the classical formula as follows (see A. Eldélyi [6]):

(2.7) $\displaystyle\int_0^\infty J_a(\lambda\rho)J_b(\mu\rho)\rho^{-c}\,d\rho$

$$= \frac{\lambda^a}{2^c \cdot \mu^{a-c+1}} \cdot \frac{\Gamma(1/2(a+b-c+1))}{\Gamma(a+1)\Gamma(1/2(-a+b+c+1))}$$

$$\times F\left(\frac{a+b-c+1}{2}, \frac{a-b-c+1}{2}; a+1; \frac{\lambda^2}{\mu^2}\right)$$

if $a > 0$, $b > 0$, $a+b+1 > c > 0$ and $0 < \lambda \le \mu$.

Hence we obtain

$$B(\alpha, k; \lambda, \mu) \le C(1+k)^{(3\alpha/2)-1}\left| F(\nu+k+\alpha, \alpha; \nu+k+1; 1) \right.$$

$$\left. - \left(\frac{\min(\lambda,\mu)}{\max(\lambda,\mu)}\right)^{(\nu+k+\alpha)/2} F\left(\nu+k+\alpha, \alpha; \nu+k+1; \frac{\min(\lambda,\mu)}{\max(\lambda,\mu)}\right) \right|^{1/2}$$

where the constant C does not depend on λ, μ and k.

Thus the remaining task is to show that

$$\left| F(\nu+k+\alpha, \alpha; \nu+k+1; 1) \right.$$

$$\left. - \left(\frac{\min(\lambda,\mu)}{\max(\lambda,\mu)}\right)^{(\nu+k+\alpha)/2} F\left(\nu+k+\alpha, \alpha; \nu+k+1; \frac{\min(\lambda,\mu)}{\max(\lambda,\mu)}\right) \right|$$

$$\le C_\theta'(1+k)^\alpha |(1+k)|\lambda - \mu|/\max(\lambda,\mu)|^\theta.$$

The proof for the case $\theta = 0$ is easy, because $0 < (\min(\lambda,\mu)/\max(\lambda,\mu)) \le 1$, and

$$F\left(\nu+k+\alpha, \alpha; \nu+k+1; \frac{\min(\lambda,\mu)}{\max(\lambda,\mu)}\right)$$

$$\le F(\nu+k+\alpha, \alpha; \nu+k+1; 1)$$

$$= \frac{\Gamma(\nu+k+1)}{\Gamma(\nu+k+1-\alpha)} \cdot \frac{\Gamma(1-2\alpha)}{\Gamma(1-\alpha)}$$

$$\le C(1+k)^\alpha.$$

(Note that, $F(a,b;c;z)$ is monotone increasing for $0 \le z \le 1$, and $F(a,b;c;1) = \Gamma(c)\Gamma(c-a-b)/(\Gamma(c-a)\Gamma(c-b))$ if $c > 0$ and $c-a-b > 0$ are satisfied.)

Let us now abbreviate

$$F(1) = F(v + k + \alpha, \alpha; v + k + 1; 1),$$

$$F(z) = F(v + k + \alpha, \alpha; v + k + 1; z),$$

and

$$z = \frac{\min(\lambda, \mu)}{\max(\lambda, \mu)}.$$

Then, since

$$|F(1) - z^{(v+k+\alpha)/2}F(z)| \leq |F(1) - F(z)| + F(z)|1 - z^{(v+k+\alpha)/2}|$$

$$\leq |F(1) - F(z)| + F(1)|1 - z^{(v+k+\alpha)/2}|,$$

the assertion comes from

$$|F(1) - F(z)| \leq C_\theta''(1 + k)^{\alpha+\theta}|1 - z|^\theta$$

and

$$|1 - z^{(v+k+\alpha)/2}| \leq C_\theta'''(1 + k)^\theta|1 - z|^\theta.$$

The latter inequality for $1/2 \leq z \leq 1$ can be seen quite easily, because the left hand side is not larger than 2 and $C(1 + k)|1 - z|$ for $1/2 \leq z \leq 1$. To show the former one, recall Euler's integral representation:

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \int_0^1 t^{b-1}(1 - t)^{c-b-1}(1 - tz)^{-a} dt.$$

Also notice that

$$|(1 - tz)^{-a} - (1 - t)^{-a}| \leq |tz - t|^\theta \cdot \frac{2^{1-\theta}a^\theta}{(1 - t)^{a+\theta}} \quad \text{for} \quad 0 < t < 1, \quad \frac{1}{2} \leq z \leq 1$$

holds if $a > 0$ and $0 \leq \theta \leq 1$. Hence we have

$$|F(1) - F(z)| \leq \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \int_0^1 t^{b-1}(1 - t)^{c-b-1}|(1 - tz)^{-a} - (1 - t)^{-a}| dt$$

$$\leq \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)} 2^{1-\theta}a^\theta|1 - z|^\theta \int_0^1 t^{b+\theta-1}(1 - t)^{c-b-1-a-\theta} dt$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \frac{\Gamma(b + \theta)\Gamma(c - a - b - \theta)}{\Gamma(c - a)} \cdot 2^{1-\theta}a^\theta|1 - z|^\theta,$$

where $a = v + k + \alpha$, $b = \alpha$ and $c = v + k + 1$. Note that

$$\frac{\Gamma(c)\Gamma(b + \theta)\Gamma(c - a - b - \theta)}{\Gamma(b)\Gamma(c - b)\Gamma(c - a)} \leq C(1 + k)^\alpha$$

if $c - a - b - \theta = 1 - 2\alpha - \theta > 0$, and

$$a^\theta \leq C'(1 + k)^\theta.$$

Thus we see that the former inequality holds for $1/2 \leq z \leq 1$. The proof for $0 < z < 1/2$ will be similar to that of the case $\theta = 0$. $\qquad\square$

Let us choose $0 < \alpha < 1/2$ and $\alpha' > \alpha$. Define the function spaces as follows:

$$F_\alpha = \{f \in \mathscr{S}'(\mathbf{R}^n) | \, |x|^{1-\alpha} f \in L^2(\mathbf{R}^n)\}$$

and

$$G_{\alpha,\alpha'} = \{f \in \mathscr{S}'(\mathbf{R}^n) | \, |x|^{\alpha-1}(I - \Lambda)^{(1-2\alpha')/2} f \in L^2(\mathbf{R}^n)\}.$$

Lemma 2.1 shows that the operator $A_\alpha(\lambda)$ is a bounded operator from F_α to $G_{\alpha,\alpha'}$, and moreover, in the operator norm, it is (locally) Hölder continuous with respect to $\lambda > 0$ (take $\theta > 0$ so that $2\alpha - 1 + (\theta/2) \leq 2\alpha' - 1$). Roughly speaking, these facts and relation (2.2) imply Theorem 1. As the first step, we shall prove the following proposition:

PROPOSITION 2.2. *Suppose that $0 < \alpha < 1/2$. Then we have*

$$|(|D|^{2(\alpha-1)}f, f)| \leq C\|(I - \Lambda)^{(2\alpha-1)/4}|x|^{1-\alpha}f\|^2_{L^2(\mathbf{R}^n)},$$

where the constant C does not depend on $f \in C_0^\infty(\mathbf{R}^n)$.

PROOF. As in the proof of Lemma 2.1, our task is to estimate $(H_k|D|^{2(\alpha-1)}f, f)$. First it follows from (2.1) and (2.3) that

$$(H_k|D|^{2(\alpha-1)}f, f)$$

$$= \int_0^\infty (H_k A_\alpha(\lambda)f, f)\frac{d\lambda}{\lambda}$$

$$= \frac{1}{2}\int_{|\omega|=1} d\omega \int_0^\infty \int_0^\infty K(\alpha, k; r, \rho)r^{n/2}H_k f(r, \omega)\rho^{n/2}\overline{H_k f(\rho, \omega)} \, dr d\rho$$

where

$$K(\alpha, k; r, \rho) = \int_0^\infty \lambda^{\alpha-1}J_{v+k}(\sqrt{\lambda}r)J_{v+k}(\sqrt{\lambda}\rho) \, d\lambda.$$

Moreover, if $0 < \alpha < (1/2)$, by the formula (2.7) we have

$$|K(\alpha, k; r, \rho)| \leq C(1 + k)^{2\alpha - 1} |\max(r, \rho)|^{-2\alpha}.$$

Thus

$$|(H_k |D|^{2(\alpha - 1)} f, f)| \leq C(1 + k)^{2\alpha - 1} \int_{|\omega| = 1} d\omega \int_0^\infty \int_0^\infty K_\alpha(r, \rho) |H_k f(r, \omega)| \, |H_k f(\rho, \omega)|$$

$$\times r^{(n-1)/2 + 1 - \alpha} \rho^{(n-1)/2 + 1 - \alpha} dr d\rho$$

where

$$K_\alpha(r, \rho) = r^{\alpha - (1/2)} \rho^{\alpha - (1/2)} |\max(r, \rho)|^{-2\alpha}.$$

Let us note that $K_\alpha(r, \rho)$ satisfies $K_\alpha(\lambda r, \lambda \rho) = \lambda^{-1} K_\alpha(r, \rho)$ for $\lambda > 0$ and

$$\int_0^\infty |K_\alpha(1, \rho)| \frac{d\rho}{\sqrt{\rho}} = \int_0^1 \rho^{\alpha - (1/2)} \frac{d\rho}{\sqrt{\rho}} + \int_1^\infty \rho^{\alpha - (1/2)} \rho^{-2\alpha} \frac{d\rho}{\sqrt{\rho}} < \infty,$$

if $0 < \alpha < 1$. Hence by Hardy-Littlewood-Pólya inequality we obtain

$$|(H_k |D|^{2(\alpha - 1)} f, f)| \leq C(1 + k)^{2\alpha - 1} \left(\int_{|\omega| = 1} \int_0^\infty |H_k f(r, \omega)|^2 r^{n - 1 + 2(1 - \alpha)} dr d\omega \right)^{1/2}$$

$$\times \left(\int_{|\omega| = 1} \int_0^\infty |H_k f(\rho, \omega)|^2 \rho^{n - 1 + 2(1 - \alpha)} \, d\rho d\omega \right)^{1/2}$$

$$= C(1 + k)^{2\alpha - 1} \| |x|^{1 - \alpha} H_k f \|_{L^2(\mathbf{R}^n)}^2.$$

Taking the summation of the above inequality with respect to k immediately implies the result of Proposition 2.2. $\qquad \qquad \square$

Proposition 2.2 shows that the operator $|D|^{2\alpha} (-\Delta - \zeta)^{-1}$ is a bounded operator from F_α to $G_{\alpha, \alpha'}$ for $\zeta \notin [0, \infty)$. Indeed it is easy to see that

$$|(|D|^{2\alpha} (-\Delta - \zeta)^{-1} f, g)| \leq \frac{1}{(2\pi)^n} \int \left| \frac{|\xi|^{2\alpha}}{|\xi|^2 - \zeta} \right| |\hat{f}(\xi)| \, |\hat{g}(\xi)| \, d\xi$$

$$\leq C \int |\xi|^{2(\alpha - 1)} |\hat{f}(\xi)| \, |\hat{g}(\xi)| \, d\xi$$

$$\leq C(|D|^{2(\alpha - 1)} (I - \Lambda)^{(1 - 2\alpha)/2} f, f)^{1/2} (|D|^{2(\alpha - 1)}$$

$$\times (I - \Lambda)^{(2\alpha - 1)/2} g, g)^{1/2}$$

$$\leq C' \| |x|^{1 - \alpha} f \|_{L^2(\mathbf{R}^n)} \| |x|^{1 - \alpha} (I - \Lambda)^{(2\alpha - 1)/2} g \|_{L^2(\mathbf{R}^n)}.$$

Hence the remaining task is to show that the value of $\| \, |D|^{2\alpha} \cdot (-\Delta - (1 \pm i\varepsilon))^{-1} \|_{\mathscr{L}(F_\alpha, G_{\alpha, \alpha'})}$ is bounded even if $\varepsilon \to 0$. This is because the scaling argument induces that

$$\sup_{0 < \varepsilon < \infty} \| \, |D|^{2\alpha}(-\Delta - (\lambda \pm i\varepsilon))^{-1} \|_{\mathscr{L}(F_\alpha, G_{\alpha, \alpha'})}$$

is bounded and the bound does not depend on $\lambda > 0$.

Now let us decompose the operator $|D|^{2\alpha}(\Delta - (1 + i\varepsilon))^{-1}$ as follows

$$(2.8) \qquad |D|^{2\alpha}(-\Delta - (1 + i\varepsilon))^{-1}$$

$$= \int_0^\infty \chi(\lambda) \frac{A_\alpha(\lambda)}{\lambda - (1 + i\varepsilon)} \, d\lambda + \int_0^\infty (1 - \chi(\lambda)) \frac{A_\alpha(\lambda)}{\lambda - (1 + i\varepsilon)} \, d\lambda$$

$$= R_1 + R_2,$$

where $\chi(\lambda) \in C_0^\infty[1/2, 2]$ satisfies $\chi(\lambda) \equiv 1$ in a neighborhood of $\lambda = 1$. We remark that the value of $\|R_2\|_{\mathscr{L}(F_\alpha, G_{\alpha, \alpha'})}$ remains bounded even if $\varepsilon \to 0$. This can be easily seen by the similar argument as above. To estimate the value of $\|R_1\|_{\mathscr{L}(F_\alpha, G_{\alpha, \alpha'})}$, we quote now the following proposition, the proof of which will be given in Appendix.

PROPOSITION 2.3. *Let F_1 and F_1 be Banach spaces. Suppose that $A(\lambda)$ is a $\mathscr{L}(F_1, F_2)$ valued Hölder continuous function of $\lambda > 0$, support of which is contained in $[1/2, 2]$. Set*

$$(2.9) \qquad B(\zeta) = \int_0^\infty \frac{A(\lambda)}{\lambda - \zeta} \, d\lambda.$$

Then the set $\{B(1 + i\varepsilon)\}_{\varepsilon > 0}$ is bounded in $\mathscr{L}(F_1, F_2)$.

Notice that $A(\lambda) = A_\alpha(\lambda)\chi(\lambda)$ is a $\mathscr{L}(F_\alpha, G_{\alpha, \alpha'})$ valued Hölder continuous function because of Lemma 2.1. Thus Theorem 1 follows for $\operatorname{Im} \zeta > 0$. Also the proof for $\operatorname{Im} \zeta < 0$ is completely similar. This finishes the proof of Theorem 1. $\square$

PROOF OF THEOREM 2. We are in a position to prove Theorem 2. The argument here is essentially due to C. Kenig, G. Ponce and L. Vega [12]. First let us set

$$u_\varepsilon(t, x) = -(2\pi)^{-n-1} \iint e^{ix\cdot\xi + it\tau} \frac{1}{2} \left(\frac{1}{|\xi|^2 - (\tau + i\varepsilon)} + \frac{1}{|\xi|^2 - (\tau - i\varepsilon)} \right) \hat{f}(\tau, \xi) \, d\tau d\xi$$

$$= -(2\pi)^{-n-1} \int e^{it\tau} \frac{1}{2} \{ (-\Delta - (\tau + i\varepsilon))^{-1} + (-\Delta - (\tau - i\varepsilon))^{-1} \} \tilde{f}(\tau, \cdot) \, d\tau$$

where $\tilde{f}(\tau, \cdot)$ is the partial Fourier transform with respect to t. We are going to show that

(2.10)
$$\int_0^\infty \| \, |D|^{2\alpha} u_\varepsilon(t, \cdot)\|^2_{G_{\alpha, \alpha'}} dt \le C \int_0^\infty \|f(t, \cdot)\|^2_{F_\alpha} dt$$

with a constant C independent of $\varepsilon > 0$. Hence $u(t, x) = \lim_{\varepsilon \downarrow 0} u_\varepsilon(t, x)$ exists, and enjoys the same estimate as (2.10). Also it is easy to see that

(2.11)
$$u(0, x) = \frac{1}{2} \int_{-\infty}^\infty \operatorname{sgn} s \cdot e^{is\Delta} f(s, x) \, ds.$$

The idea to show (2.10) is to regard the map $f \mapsto u_\varepsilon$ as a Fourier multiplier with an operator-valued symbol. Set now

$$m_\varepsilon(\tau) = -\frac{|D|^{2\alpha}}{2}((-\Delta - (\tau + i\varepsilon))^{-1} + (-\Delta - (\tau - i\varepsilon))^{-1}).$$

Then from Theorem 1, it follows

$$\|m_\varepsilon(\tau)f\|_{G_{\alpha, \alpha'}} \le C\|f\|_{F_\alpha},$$

where the constant C does not depend on f, ε and τ. Hence by Planchérel's theorem we obtain

$$\int_{-\infty}^\infty \| \, |D|^{2\alpha} u_\varepsilon(t, \cdot)\|^2_{G_{\alpha, \alpha'}} dt = \frac{1}{2\pi} \int_{-\infty}^\infty \|m_\varepsilon(\tau)\tilde{f}(\tau, \cdot)\|^2_{G_{\alpha, \alpha'}} d\tau$$

$$\le \frac{C^2}{2\pi} \int_{-\infty}^\infty \|\tilde{f}(\tau, \cdot)\|^2_{F_\alpha} d\tau$$

$$= C^2 \int_{-\infty}^\infty \|f(t, \cdot)\|^2_{F_\alpha} dt.$$

We are in a position to finish the proof of Theorem 2. By duality argument it follows from Theorem 1.2 (i) of [9] and the expression (2.11) that $f \in F_\alpha$ implies $|D|^\alpha (I - \Lambda)^{(1-2\alpha)/4} u(0, x) \in L^2(R^n)$ (notice that the operator $|D|^\alpha (I - \Lambda)^{(1-2\alpha)/4}$ commutes with $i\partial_t + \Delta$). Finally applying Theorem 1.2 of [9] again and using (2.10), we obtain Theorem 2. $\qquad \square$

3. $L^p - L^q$ Estimates

In this section we shall prove Theorem 3 and Theorem 4. Both of them are results on the estimates for $|D|^{2\alpha}(-\Delta - \zeta)^{-1}$. So the proofs will become similar to the one of Theorem 1.

At first, let us show that the estimate (1.4) holds if the parameter ζ is away from the spectrum $[0, \infty)$. This immediately comes from Hardy-Littlewood-Sobolev inequality. Indeed, by the similar argument as in the previous section, we obtain

$$|(|D|^{2\alpha}(-\Delta - \zeta)^{-1}f, g)| \leq C \int |\xi|^{2(\alpha-1)} |\hat{f}(\xi)| \, |\hat{g}(\xi)| \, d\xi$$

$$\leq C(|D|^{n(1-(2/p))}f, f)^{1/2}(|D|^{n((2/q)-1)}g, g)^{1/2}$$

$$\leq C\|f\|_{L^p(\mathbf{R}^n)}\|g\|_{L^{q'}(\mathbf{R}^n)},$$

(q' is defined by $1/q + 1/q' = 1$) if $1 < p < 2 < q < \infty$ and $1/p - 1/q = 2(1 - \alpha)/n$, which are more generous than the assumptions of Theorem 3.

Thus the remaining task is to consider the behavior of the estimate when ζ approaches the spectrum $[0, \infty)$. Notice that the scaling argument allows to devote ourselves to the case $\zeta = 1 \pm i\varepsilon$ ($\varepsilon > 0$). Also on the decomposition (2.8), the similar argument as in the previous section gives that the value of $\|R_2\|_{\mathscr{L}(L^p(\mathbf{R}^n), L^q(\mathbf{R}^n))}$ remains bounded even if $\varepsilon \to 0$. On the other hand, concerning R_1, the expression (2.3) does not seem to be useful for the proof. Instead we make some preparations.

At first, let $K_{z,\zeta}(x - y)$ denote the kernel of the operator $1/\Gamma(z) \cdot (-\Delta - \zeta)^{z-(n/2)}$. Then

LEMMA 3.1. *Suppose that the both ζ and ζ' are contained in Γ_+ or Γ_-, where*

$$\Gamma_\pm = \{\zeta \in C \mid \mathrm{Im}\,\zeta \gtrless 0, \tfrac{1}{2} \leq |\zeta| \leq 2\}.$$

Then, for $-1/2 < \mathrm{Re}\,z < 1/2$ and $\max(0, 2\,\mathrm{Re}\,z) \leq \mu < \mathrm{Re}\,z + 1/2$, there exists a constant C such that

$$(3.1) \qquad\qquad |K_{z,\zeta}(x)| \leq C \cdot \frac{e^{C|\mathrm{Im}\,z|}}{|x|^\mu}$$

and

$$(3.2) \qquad\qquad |K_{z,\zeta}(x) - K_{z,\zeta'}(x)| \leq Ce^{C|\mathrm{Im}\,z|} \cdot \frac{|\zeta - \zeta'|^\theta}{|x|^\mu},$$

where $\theta = \mathrm{Re}\,z + (1/2) - \mu$.

PROOF. First let us note that the kernel $K_{z,\zeta}(\xi)$ has the following expression:

$$K_{z,\zeta}(x) = \frac{(\sqrt{-\zeta})^z}{\pi^{n/2} \cdot 2^{n-1-z} \cdot \Gamma((n/2) - z)\Gamma(z)} \cdot \frac{K_z(\sqrt{-\zeta}|x|)}{|x|^z},$$

where $K_z(\omega)$ is the modified Bessel function of the second kind (see I. M. Gelfand and G. E. Shilov [7]). Also recall that, if $\operatorname{Re} w \geq 0$,

$$|K_z(w)| \leq C e^{C|\operatorname{Im} z|} |w|^{-1/2} \quad \text{for } |w| \geq 1$$

and

$$|K_z(w)| \leq C \frac{1}{|z|} e^{C|\operatorname{Im} z|} |w|^{-|\operatorname{Re} z|} \quad \text{for } |w| \leq 1.$$

(see C. Kenig, A. Ruiz and C. D. Sogge [13].) These facts immediately imply (3.1).

To show (3.2), we quote now the classical formula of the Bessel functions as follows:

$$K_z'(w) = \frac{1}{2}(K_{z-1}(w) - K_{z+1}(w)).$$

Hence we obtain

$$|K_{z,\zeta}(x) - K_{z,\zeta'}(x)| \leq |\zeta - \zeta'| \sup_{\zeta \in \Gamma_\pm} \left| \frac{\partial}{\partial \zeta} K_{z,\zeta}(x) \right|$$

$$\leq \begin{cases} C e^{C|\operatorname{Im} z|} |\zeta - \zeta'| |x|^{-\mu_1}, & |x| \leq 1, \\ C e^{C|\operatorname{Im} z|} |\zeta - \zeta'| |x|^{-\mu_2}, & |x| \geq 1, \end{cases}$$

where $\mu_1 = \max(0, 2\operatorname{Re} z)$ and $\mu_2 = \operatorname{Re} z - (1/2)$. Also it is clear that

$$|K_{z,\zeta}(x) - K_{z,\zeta'}(x)| \leq |K_{z,\zeta}(x)| + |K_{z,\zeta'}(x)|$$

$$\leq \begin{cases} C e^{C|\operatorname{Im} z|} |x|^{-\mu_1}, & |x| \leq 1, \\ C e^{C|\operatorname{Im} z|} |x|^{-\mu_3}, & |x| \geq 1, \end{cases}$$

where $\mu_3 = \operatorname{Re} z + 1/2$. Interpolating these estimates by Hölder's inequality, we obtain (3.2). $\qquad\qquad\square$

The inequalities in Lemma 3.1 induce that the resolvent operator $R(\zeta) = (-\Delta - \zeta)^{-1}$ is Hölder continuous with respect to $\zeta \in \Gamma_+$ (or $\zeta \in \Gamma_-$) in certain operator topology. Precisely,

PROPOSITION 3.2. *Suppose that $n \geq 3$. Also suppose that the exponents p and q satisfy the assumptions in Theorem 3. Then we have*

$$(3.3) \qquad\qquad \|R(\zeta)u\|_{L^q(\mathbf{R}^n)} \leq C\|u\|_{L^p(\mathbf{R}^n)},$$

and

$$(3.4) \qquad \|[R(\zeta) - R(\zeta')]u\|_{L^q(R^n)} \leq C|\zeta - \zeta'|^\theta \|u\|_{L^p(R^n)},$$

for some $\theta > 0$, *where the constant* C *does not depend on* $u \in C_0^\infty(R^n)$ *and* $\zeta, \zeta' \in \Gamma_\pm$.

PROOF. Lemma 3.1 and Hardy-Littlewood-Sobolev inequality induce that, for $-1/2 < \mathrm{Re}\, z < 1/2$ and $\zeta, \zeta' \in \Gamma_\pm$,

$$\left\| \frac{1}{\Gamma(z)}(-\Delta - \zeta)^{z-(n/2)} u \right\|_{L^q(R^n)} \leq C e^{C|\mathrm{Im}\, z|} \|u\|_{L^p(R^n)}$$

and

$$\left\| \frac{1}{\Gamma(z)}[(-\Delta - \zeta)^{z-(n/2)} - (-\Delta - \zeta')^{z-(n/2)}]u \right\|_{L^q(R^n)} \leq C e^{C|\mathrm{Im}\, z|} |\zeta - \zeta'|^\theta \|u\|_{L^p(R^n)},$$

where $1 < p < q < \infty$, $\max(0, 2\mathrm{Re}\, z) \leq n(1 - 1/p + 1/q) < \mathrm{Re}\, z + 1/2$ and $\theta = \mathrm{Re}\, z + 1/2 - n(1 - 1/p + 1/q)$.

One the other hand, since

$$|(|\xi|^2 - \zeta)^{i\gamma}| \leq e^{\pi|\gamma|} \quad \text{for } \zeta \in C, \gamma \in R,$$

Plancherel's theorem gives

$$\left\| \frac{1}{\Gamma(n/2 + i\gamma)}(-\Delta - \zeta)^{i\gamma} u \right\|_{L^2(R^n)} \leq C e^{C|\gamma|} \|u\|_{L^2(R^n)}.$$

These estimates and complex interpolation theorem by E. M. Stein induce the estimates (3.3) and (3.4) with p and q satisfying

$$\frac{2}{n - 2\mathrm{Re}\, z}\left(1 - \frac{1}{n}\left(\mathrm{Re}\, z + \frac{1}{2}\right)\right) < \frac{1}{p} - \frac{1}{q} \leq \frac{2}{n - 2\mathrm{Re}\, z}\left(1 - \frac{1}{n}\max(0, 2\mathrm{Re}\, z)\right)$$

and

$$\frac{1}{2} \cdot \frac{n - 2 - 2\mathrm{Re}\, z}{n - 2\mathrm{Re}\, z} < \frac{1}{q} < \frac{1}{p} < \frac{1}{2} \cdot \frac{n + 2 - 2\mathrm{Re}\, z}{n - 2\mathrm{Re}\, z}.$$

(The assumption $n \geq 3$ is necessary for the interpolation.) Taking the foliation when $\mathrm{Re}\, z$ goes from $-1/2$ to $1/2$, we obtain Proposition 3.2. $\qquad \square$

Now we are in a position to finish the proof of Theorem 3. First recall the following relation (see *e.g.* S. Agmon [1]):

$$A_0(\lambda) = \frac{1}{2\pi i} \lim_{\varepsilon \downarrow 0}[R(\lambda + i\varepsilon) - R(\lambda - i\varepsilon)].$$

Hence it follows from Proposition 3.2 that $A_0(\lambda)$ is Hölder continuous with respect to $\lambda \in [1/2, 2]$ in operator topology of $\mathscr{L}(L^p(\mathbf{R}^n), L^q(\mathbf{R}^n))$, if p and q satisfy the assumptions of Theorem 3. This implies so does for $A_\alpha(\lambda) = \lambda^\alpha A_0(\lambda)$. Thus the argument in the last part of previous section gives that $\|R_1\|_{\mathscr{L}(L^p(\mathbf{R}^n), L^q(\mathbf{R}^n))}$ is bounded even if $\varepsilon \to 0$. This finishes the proof of Theorem 3. $\square$

PROOF OF THEOREM 4. Here we explain the difference between the proofs of Theorem 3 and Theorem 4 briefly. It is observed before that the estimate for $A_0(\lambda)$ comes from those for $1/\Gamma(n/2 + i\gamma)(-\Delta - \zeta)^{i\gamma}(\gamma \in \mathbf{R})$ and $1/\Gamma(z) \cdot (-\Delta - \zeta)^{z-(n/2)}(-1/2 < \mathrm{Re}\, z < 1/2)$. Moreover the latter estimate follows from the fact that the kernel of the operator $1/\Gamma(z)(-\Delta - \zeta)^{z-(n/2)}$ can be estimated by the one of Riesz operator. Then we have used Hardy-Littlewood-Sobolev inequality. Instead, in the proof of Theorem 4, we use the following fractional estimate by E. M. Stein and G. Weiss [16]. This is the difference.

PROPOSITION 3.3 (E. M. Stein and G. Weiss). *Let*

$$T_\mu f(x) = \int_{\mathbf{R}^n} \frac{f(y)}{|x - y|^\mu}\, dy,$$

for $0 < \mu < n$. *Suppose that* $1 < p < q < \infty$, $s < n/p'$, $s' < n/q$, $s + s' \geq 0$ *and*

$$\frac{1}{q} = \frac{1}{p} + \frac{\mu + s + s'}{n} - 1.$$

Then

$$\| |x|^{-s'} T_\mu f\|_{L^q(\mathbf{R}^n)} \leq C\| |x|^s f\|_{L^p(\mathbf{R}^n)},$$

where the constant C does not depend on $f \in C_0^\infty(\mathbf{R}^n)$, *but may depend on* p, q, s, s' *and* n.

Appendix

Here we shall prove Proposition 2.3 by the method of harmonic analysis. First denote $\langle , \rangle$ by the coupling between F_2 and F_2^*. Define $\Phi(\lambda)(\lambda > 0)$ and $\Psi(\zeta)(\zeta \in C, \mathrm{Im}\, \zeta > 0)$ respectively by $\Phi(\lambda) = \langle A(\lambda)f, g\rangle$ and $\Psi(\zeta) = \langle B(\zeta)f, g\rangle$ for $f \in F_1$ and F_2^*. Notice that the relation (2.9) implies

$$\text{(A.1)} \qquad\qquad \Psi(\zeta) = \int_0^\infty \frac{\Phi(\lambda)}{\lambda - \zeta} \, d\lambda.$$

Moreover, for $0 < \theta < 1$, denote $\|\Phi\|_\theta$ by

$$\|\Phi\|_\theta = \sup_{\lambda \neq \lambda'} \frac{|\Phi(\lambda) - \Phi(\lambda')|}{|\lambda - \lambda'|^\theta}.$$

Then assumptions in Proposition 2.3 imply that $\Phi(\lambda)$ is a compactly supported continuous function satisfying $\|\Phi\|_\theta < \infty$.

Observe now that the relation (A.1) tells that the operation $\Phi(\lambda) \mapsto \Psi_\varepsilon(\mu) = \Psi(\mu + i\varepsilon)(\mu \in R)$ can be written by Fourier multiplier with symbol

$$a_\varepsilon(\xi) = \begin{cases} 0, & \xi > 0, \\ 2\pi i e^{\varepsilon \xi}, & \xi < 0. \end{cases}$$

Since the set $\{a_\varepsilon(\xi)\}_{\varepsilon > 0}$ is bounded in Mihlin-Hörmander's class (cf. L. Hörmander [8] page 243 (7.9.8)), the multiplier theorem of the Lipschitz class (cf. L. Hörmander [8] Theorem 7.9.6) can be applied. Thus there exists a constant C_θ independent of $\varepsilon > 0$ such that

$$\|\Psi_\varepsilon\|_\theta \leq C_\theta \|\Phi\|_\theta.$$

On the other hand, since supp $\Phi \subset [1/2, 2]$, it is clear that

$$|\Psi_\varepsilon(4)| = \left| \int_0^\infty \frac{\Phi(\lambda)}{\lambda - (4 + i\varepsilon)} \, d\lambda \right| \leq \int_{1/2}^2 \left| \frac{1}{\lambda - (4 + i\varepsilon)} \right| d\lambda \times \sup_\lambda |\Phi(\lambda)|$$

$$\leq \frac{3}{4} \sup_\lambda |\Phi(\lambda)|.$$

Thus we obtain

$$|\Psi_\varepsilon(1)| \leq |\Psi_\varepsilon(4)| + |\Psi_\varepsilon(4) - \Psi_\varepsilon(1)|$$

$$\leq \frac{3}{4} \sup_\lambda |\Phi(\lambda)| + 3^\theta C_\theta \|\Phi\|_\theta,$$

and this implies that the set $\{\langle B(1 + i\varepsilon)f, g\rangle\}_{\varepsilon > 0}$ is bounded. Note that, in the above argument, all the constants are independent of $f \in F_1$ and $g \in F_2^*$. This proves Proposition 2.3.

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