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A CERTAIN GRAPH OBTAINED FROM A SET OF SEVERAL POINTS ON A RIEMANN SURFACE

By

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Introduction

0-1. Let M be a compact Riemann surface of genus $g \geq 2$, and let $P_1, P_2, \ldots, P_n$ be distinct points on M. We define the Weierstrass gap set $G(P_1, P_2, \ldots, P_n)$ by

$$G(P_1, P_2, \ldots, P_n) := \{(\gamma_1, \gamma_2, \ldots, \gamma_n) \in \mathbf{N}_0 \times \cdots \times \mathbf{N}_0 \mid \nexists \text{ meromorphic function } f \text{ on } M \text{ whose pole divisor } (f)_\infty \text{ is } \gamma_1 P_1 + \gamma_2 P_2 + \cdots + \gamma_n P_n\},$$

where $\mathbf{N}_0$ is the set of non-negative integers.

When $n = 1, G(P_1)$ is the set of Weierstrass gaps at P_1. One of the essential differences between the case $n = 1$ and the case $n \geq 2$ is that the cardinarity $\# G(P_1)$ is the constant g but $\# G(P_1, \ldots, P_n)$ $(n \geq 2)$ depends on the choice of M and the set of points $\{P_1, \ldots, P_n\}$ on M.

Kim has given formulas for $\# G(P_1, P_2)$ and shown the following inequalities

$$\frac{(g^2 + 3g)}{2} \leq \# G(P_1, P_2) \leq \frac{(3g^2 + g)}{2}.$$

Moreover he has proved that the upper bound $(3g^2 + g)/2$ can be realized if and only if "M is hyperelliptic and $|2P_1| = |2P_2| = g_2^1$" ([3]). The lower bound $(g^2 + 3g)/2$ can be attained by taking general points P_1 and P_2 on arbitrary M. This is stated in [1] without proof, and has been proved by Homma ([2]). He also has translated Kim's formulas into other practical ones, and added several interesting remarks in the case where M is a curve defined over a field of characteristic $p \geq 0$ ([2]). Through their works it seems to be helpful to use a certain type of graph $D^{(n)}$ defined as follows.

Definition 0-2 (Riemann-Roch Graph). Fix positive integers g and n. Let $\mathbf{e}_i$ be the n-tuple $(0, \ldots, 0, 1, 0, \ldots, 0)$ (i.e., the i-th component of $\mathbf{e}_i$ is 1) in $\mathbf{N}_0^n$.

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For an element $(\gamma_1, \dots, \gamma_n) \in \mathbf{N}_0^n$, we also write $\sum_i \gamma_i \mathbf{e}_i$. Let $V^{(n)}$ denote the subset

$$\{\Gamma = (\gamma_1, \dots, \gamma_n) \mid \gamma_i \in \mathbf{N}_0, 0 \le \gamma_1 + \cdots + \gamma_n \le 2g-1\}$$

of $\mathbf{N}_0^n$.

For $\Gamma = \sum_i \gamma_i \mathbf{e}_i \in V^{(n)}$, define $deg\,\Gamma$ by

$$deg\,\Gamma := \sum_i \gamma_i.$$

Let $\Gamma = \sum_i \gamma_i \mathbf{e}_i$ and $\Gamma' = \sum_i \gamma_i' \mathbf{e}_i$ be in $V^{(n)}$. Then we write

$$\Gamma' \le \Gamma \quad \text{if} \quad \gamma_i' \le \gamma_i \quad \text{for } i = 1, 2, \dots, n.$$

Let $E^{(n)}$ denote the subset

$$\{(\Gamma - \mathbf{e}_i)\Gamma \mid \Gamma \in V^{(n)} \text{ and } \Gamma - \mathbf{e}_i \in V^{(n)}\}$$

of $V^{(n)} \times V^{(n)}$, where $\Gamma - \mathbf{e}_i = (\gamma_1, \dots, \gamma_i - 1, \dots, \gamma_n)$ with $\Gamma = (\gamma_1, \dots, \gamma_i, \dots, \gamma_n)$. Let $D^{(n)}$ denote the graph $\{V^{(n)}, E^{(n)}\}$ consisting of $V^{(n)}$ and $E^{(n)}$ as a set of vertices and a set of edges respectively. When $\Gamma' \le \Gamma$, any chain of successive $(deg\,\Gamma - deg\,\Gamma')$ edges

$$\Gamma'\Gamma_1, \Gamma_1\Gamma_2, \Gamma_2\Gamma_3, \dots, \Gamma_{deg\,\Gamma - deg\,\Gamma' - 1}\Gamma$$

is called a path from Γ' to Γ. Of course these paths are not unique even though Γ and Γ' are fixed, but we write $\Gamma'\Gamma$ for them abusively. Moreover, each edge is labeled "0" or "1", which is called the weight of the edge, and the labeling has the following properties.

$*_n)$

- $*_n - 1)$ Let $\Gamma = \sum_i \gamma_i \mathbf{e}_i$ and $\tilde{\Gamma} = \sum_i \tilde{\gamma}_i \mathbf{e}_i$ be in $V^{(n)}$. Assume $\tilde{\Gamma} \ge \Gamma$ and $\gamma_i = \tilde{\gamma}_i > 0$ with some i. If the edge $(\Gamma - \mathbf{e}_i)\Gamma$ is of weight 1, then so is the edge $(\tilde{\Gamma} - \mathbf{e}_i)\tilde{\Gamma}$.
- $*_n - 2)$ Let $O = \sum_i 0\mathbf{e}_i$ and $\Gamma = \sum_i \gamma_i \mathbf{e}_i$ be in $V^{(n)}$ with $deg\,\Gamma = 2g-1$. The number of edges of weight 1 (resp. 0) on any path $O\Gamma$ is $g-1$ (resp. g).

From now on, we will call the above type of graph $(D^{(n)}, *_n)$ a Riemann-Roch graph.

Definition 0-3. Define the gap set $G^{(n)}$ of $(D^{(n)}, *_n)$ by

$G^{(n)} := \{\Gamma \in V^{(n)} \mid \exists i \text{ such that the edge } (\Gamma - \mathbf{e}_i)\Gamma \in E^{(n)} \text{ is of weight } 0\}$.

$H^{(n)}$ denotes the compliment $V^{(n)} \backslash G^{(n)}$ of $G^{(n)}$ in $V^{(n)}$.

REMARK. $O = (0, \dots, 0) \in H^{(n)}$.

0-4. Let M and $\{P_1, \dots, P_n\}$ be as before. Then the following facts on an effective divisor $E = \gamma_1 P_1 + \gamma_2 P_2 + \cdots + \gamma_n P_n$ are known:

1) if $deg\, E = \gamma_1 + \cdots + \gamma_n = 2g - 1$, then $l(E) = h^0(\mathcal{O}(E)) = g$;

2) if P_i is not a base point of the linear system $|E|$, then P_i is not a base point of any linear system

$$|\tilde{\gamma}_1 P_1 + \tilde{\gamma}_2 P_2 + \cdots + \tilde{\gamma}_i P_i + \cdots + \tilde{\gamma}_n P_n|,$$

where $\tilde{\gamma}_k \geq \gamma_k$ $(k = 1, \dots, n)$ and $\tilde{\gamma}_i = \gamma_i$.

Identify each effective divisor $E = \sum_{i=1}^n \gamma_i P_i$ of degree $\leq 2g - 1$ with the vertex $\Gamma = \sum_{i=1}^n \gamma_i \mathbf{e}_i$, and give 1 to the edges $(\Gamma - \mathbf{e}_i)\Gamma$ if and only if P_i is not a base point of $|\sum_{i=1}^n \gamma_i P_i|$. Then we get a Riemann-Roch graph. $D_M(P_1, \dots, P_n)$ denotes this graph. Then the gap set $G^{(n)}$ obtained from $D_M(P_1, \dots, P_n)$ coincides with the Weierstrass gap set $G(P_1, \dots, P_n)$ in 0-1.

0-5. In this paper, we start studying Riemann-Roch graphs $D^{(n)}$ and their gap sets $G^{(n)}$ in general (i.e., they are not necessarily obtained from M and $\{P_1, \dots, P_n\}$).

In particular we will prove that

$$\# G^{(n)} \geq \binom{n+g}{g} - 1$$

and there is a unique graph $D^{(n)}$ satisfying $\# G^{(n)} = \binom{n+g}{g} - 1$, where $\binom{a}{b} = a!/(a-b)!b!$ for integers $a \geq b \geq 0$ (Theorem 2-3).

About upper bounds of $\# G^{(n)}$, we calculate in case $n = 3$, and show that

$$\# G^{(3)} \leq \frac{g(7g^2 + 6g + 5)}{6}$$

and there is a unique graph satisfying $\# G^{(3)} = g(7g^2 + 6g + 5)/6$. Moreover this graph is exactly equal to $D_M(P_1, P_2, P_3)$, where M is hyperelliptic and P_1, P_2, P_3 are satisfying $|2P_1| = |2P_2| = |2P_3| = g_2^1$ (Theorem 3-9).

Finally we try to replace $*_n)$ with another set of conditions in order to study a Riemann-Roch graph in detail(Appendix).

§1

Fix a Riemann-Roch graph $(D^{(n)}, *_n)$. Then we can easily have the following lemma.

Lemma 1-1. *The condition* $* - 2)$ *is equivalent to the following set* $\{A), B), C)\}$ *of conditions.*

A) *Let* Γ *and* Γ' *be in* $V^{(n)}$ *with* $\Gamma \geq \Gamma'$. *Evry path from* Γ' *to* Γ *has the same number of edges of weight* 1.

We will write $[\Gamma'\Gamma]$ *for the number of edges of weight* 1 *on a path* $\Gamma'\Gamma$.

B) *Let* Γ, Γ' *and* Γ'' *be in* $V^{(n)}$ *with* $\Gamma' \leq \Gamma, \Gamma' \leq \Gamma''$, *and* $deg\,\Gamma = deg\,\Gamma'' = 2g - 1$. *Then*

$$[\Gamma'\Gamma''] = [\Gamma'\Gamma].$$

C) *Let* $\Gamma = (2g-1)\mathbf{e}_1$ *and* $O = (0, \ldots, 0)$ *be in* $V^{(n)}$.
Then

$$[O\Gamma] = g - 1.$$

Definition 1-2. For $\Gamma \in V^{(n)}$, define non-negative integers $l(\Gamma)$ and $i(\Gamma)$ by $l(\Gamma) := [O\Gamma] + 1(\geq 1)$ and by $i(\Gamma) := l(\Gamma) - 1 + g - deg\,\Gamma(\geq 0)$ respectively.

Then we have:

Lemma 1-3. *If* Γ *and* Γ' *are in* $V^{(n)}$ *satisfying* $deg\,\Gamma = 2g - 1$ *and* $\Gamma' \leq \Gamma$, *then* $i(\Gamma')$ *is equal to the number of edges of weight* 0 *on a path* $\Gamma'\Gamma$, *and this number does not depend on the choice of a path from* Γ' *to* Γ.

Let $(D^{(n-1)}, *_{n-1})$ be the subgraph of $(D^{(n)}, *_n)$ obtained by identifying $(\gamma_1, \ldots, \gamma_{n-1}) \in V^{(n-1)}$ with $(\gamma_1, \ldots, \gamma_{n-1}, 0) \in V^{(n)}$ and restricting $*_n)$ to $V^{(n-1)}$. Then $G^{(n-1)}$ (resp. $H^{(n-1)}$) of this subgraph $(D^{(n-1)}, *_{n-1})$ is embedded in $G^{(n)}$ (resp. $H^{(n)}$) of $(D^{(n)}, *_n)$ by the same manner as above. We represent the element of $V^{(n-1)}$ by Γ_n (the index n of Γ_n suggests that Γ_n is obtained by omitting the n-th coordinate of some element Γ of $V^{(n)}$). For $\Gamma_n = (\gamma_1, \ldots, \gamma_{n-1}) \in V^{(n-1)}$ and $\gamma \in \mathbf{N}_0, (\Gamma_n, \gamma)$ denotes $(\gamma_1, \ldots, \gamma_{n-1}, \gamma) \in \mathbf{N}_0^n$.

Definition 1-4. For $\Gamma_n = (\gamma_1, \ldots, \gamma_{n-1}) \in V^{(n-1)}$, define a subset Δ_{Γ_n} of $\mathbf{N}_0$ by

$$\Delta_{\Gamma_n} := \{\delta \mid \delta \in \mathbf{N}_0, (\Gamma_n, \delta) \in H^{(n)}\},$$

and define a non-negative integer δ^{Γ_n} by

$$\delta^{\Gamma_n} := \begin{cases} \min\{\delta \mid \delta \in \Delta_{\Gamma_n}\} & \text{if } \Delta_{\Gamma_n} \neq \emptyset \\ 2g - deg\,\Gamma_n(\geq 1) & \text{if } \Delta_{\Gamma_n} = \emptyset. \end{cases}$$

LEMMA 1-5. *Let* Δ_{Γ_n} *and* δ^{Γ_n} *be as above. Then*:

i) δ^{Γ_n} *satisfies* $0 \le \delta^{\Gamma_n} \le 2g - 1 - deg\,\Gamma_n (\le 2g-1)$ *if and only if* $\Delta_{\Gamma_n} \ne \emptyset$;

ii) *if* $\Delta_{\Gamma_n} = \emptyset$, *then* $deg\,\Gamma_n > 0$ *and* $\delta^{\Gamma_n} = 2g - deg\,\Gamma_n \le 2g-1$;

iii) δ^{Γ_n} *satisfies* $\delta^{\Gamma_n} > 0$ *if and only if* $\Gamma_n \in G^{(n-1)}$.

Moreover we have a surjective map

$$\{\Gamma_n \mid \Gamma_n \in G^{(n-1)}\} \to \{\gamma(>0) \mid (O_n, \gamma) \in G^{(n)}\}$$

defined by $\Gamma_n \mapsto (O_n, \delta^{\Gamma_n})$, *where* $O_n = (0, \ldots, 0) \in V^{(n-1)}$.

PROOF. i) This follows from the fact that $\Delta_{\Gamma_n} \ne \emptyset$ is equivalent to $(\Gamma_n, \delta^{\Gamma_n}) \in V^{(n)}$.

ii) If $\Delta_{\Gamma_n} = \emptyset$, then $deg\,\Gamma_n \ge 1$. In fact, $deg\,\Gamma_n = 0$ means $\Gamma_n = O_n$. But O_n is in $H^{(n-1)}$ and $\delta^{O_n} = 0$. Therefore we get ii) by Definition 1-4.

iii) The first half of iii) follows from the fact that $\delta^{\Gamma_n} = 0$ is equivalent to $(\Gamma_n, 0) \in H^{(n)}$ (i.e., $\Gamma_n \in H^{(n-1)}$).

We will prove that the map in iii) is well-defined, that is, $(O_n, \delta^{\Gamma_n}) \in G^{(n)}$ for $\Gamma_n \in G^{(n-1)}$.

Assume that there is a $\Gamma_n \in V^{(n-1)}$ satisfying

$$\delta^{\Gamma_n} > 0 \quad \text{and} \quad (O_n, \delta^{\Gamma_n}) \in H^{(n)}. \qquad \cdots\cdots 1\text{-}5\text{-}1)$$

Then $[(O_n, \delta^{\Gamma_n}) - \mathbf{e}_n, (O_n, \delta^{\Gamma_n})] = 1$.

Thus, by $*_n - 1)$, we have

$$[\{(\Gamma_n, \delta^{\Gamma_n}) - \mathbf{e}_i\} - \mathbf{e}_n, \{(\Gamma_n, \delta^{\Gamma_n}) - \mathbf{e}_i\}] = 1 \qquad \cdots\cdots 1\text{-}5\text{-}2)$$

for all i satisfying $\gamma_i > 0$ and $i \ne n$.

<u>*case*</u> $\Delta_{\Gamma_n} \ne \emptyset$

As $(\Gamma_n, \delta^{\Gamma_n}) \in H^{(n)}$, we have

$$[(\Gamma_n, \delta^{\Gamma_n}) - \mathbf{e}_i, (\Gamma_n, \delta^{\Gamma_n})] = 1 \qquad \cdots\cdots 1\text{-}5\text{-}3)$$

for all i satisfying $1 \le i \le n$ and $\gamma_i > 0$.

Define a subset Θ of $\mathbf{N}_0$ by

$$\Theta := \{\delta \in \mathbf{N}_0 \mid [(\Gamma_n, \delta) - \mathbf{e}_i, (\Gamma_n, \delta)] = 1 \text{ for all } i \text{ satisfying } \gamma_i > 0 \text{ and } i \ne n\}.$$

By 1-5-3, $\Theta \ni \delta^{\Gamma_n}$ and $\Theta \ne \emptyset$. Then we can define a non-negative integer $\tilde{\delta}$ by

$$\tilde{\delta} := \min\{\delta \in \mathbf{N}_0 \mid \delta \in \Theta\}.$$

On this $\tilde{\delta}$, we have

$$[(\Gamma_n,\tilde{\delta}) - \mathbf{e}_i, (\Gamma_n,\tilde{\delta})] = 1 \text{ for all } i \text{ satisfying } 1 \le i \le n \text{ and } \gamma_i > 0. \quad \cdots\cdots 1\text{-}5\text{-}4)$$

(i.e., $\tilde{\delta} \in \Delta_{\Gamma_n}$.)

In fact, this is from the definition of Θ when $i = 1,\dots,n-1$.
If $[(\Gamma_n,\tilde{\delta}) - \mathbf{e}_n, (\Gamma_n,\tilde{\delta})] = 0$, then $[\{(\Gamma_n,\tilde{\delta}) - \mathbf{e}_n\} - \mathbf{e}_i, \{(\Gamma_n,\tilde{\delta}) - \mathbf{e}_n\}] = 1$ for all i satisfying $i \neq n$ and $\gamma_i > 0$ by Lemma 1-1 A). Therefore $\tilde{\delta} - 1 \in \Theta$, and this contradicts to the definition of $\tilde{\delta}$. Hence 1-5-4) is correct when $i = n$. By 1-5-4) and the definition of δ^{Γ_n}, we have $\tilde{\delta} \ge \delta^{\Gamma_n}$.
On the other hand, by Lemma 1-1 A), 1-5-2) and $(\Gamma_n,\delta^{\Gamma_n}) \in H^{(n)}$,

$$[\{(\Gamma_n,\delta^{\Gamma_n}) - \mathbf{e}_n\} - \mathbf{e}_i, \{(\Gamma_n,\delta^{\Gamma_n}) - \mathbf{e}_n\}] = 1$$

for all i satisfying $\gamma_i > 0$ and $i \neq n$.
Hence $\delta^{\Gamma_n} - 1 \in \Theta$ and $\tilde{\delta} \le \delta^{\Gamma_n} - 1$. This is a contradiction. Thus we get $(O_n,\delta^{\Gamma_n}) \in G^{(n)}$.

<u>*case*</u> $\Delta_{\Gamma_n} = \varnothing$

We have $\delta^{\Gamma_n} = 2g - deg\,\Gamma_n$ by Definition 1-4, and $(\Gamma_n,\delta^{\Gamma_n}) - \mathbf{e}_n \in V^{(n)}$.
Assume

$$[\{(\Gamma_n,\delta^{\Gamma_n}) - \mathbf{e}_n\} - \mathbf{e}_i, \{(\Gamma_n,\delta^{\Gamma_n}) - \mathbf{e}_n\}] = 1$$

for all i satisfying $\gamma_i > 0$ and $i \neq n$.
Then by the same way as in the case $\Delta_{\Gamma_n} \neq \varnothing$, we can find a positive integer $\tilde{\delta}$ satisfying $\tilde{\delta} \le 2g - 1 - deg\,\Gamma_n$ and $(\Gamma_n,\tilde{\delta}) \in H^{(n)}$. This contradicts to $\Delta_{\Gamma_n} = \varnothing$. So there is an i satisfying

$$[\{(\Gamma_n,\delta^{\Gamma_n}) - \mathbf{e}_n\} - \mathbf{e}_i, \{(\Gamma_n,\delta^{\Gamma_n}) - \mathbf{e}_n\}] = 0.$$

By Lemma 1-1 B),

$$[\{(\Gamma_n,\delta^{\Gamma_n}) - \mathbf{e}_i\} - \mathbf{e}_n, \{(\Gamma_n,\delta^{\Gamma_n}) - \mathbf{e}_i\}] = 0.$$

Then, by $*_n - 1$),

$$[(O_n,\delta^{\Gamma_n}) - \mathbf{e}_n, (O_n,\delta^{\Gamma_n})] = 0 \quad \text{and} \quad (O_n,\delta^{\Gamma_n}) \in G^{(n)}.$$

Thus our map is well-defined.

Next we will prove the surjectivity of our map.
Fix $(O_n,\gamma) \in G^{(n)} (\gamma > 0)$. Define a subset Δ of $\mathbf{N}_0$ and a positive integer $\tilde{\gamma}_1$ by

$$\Delta := \{\gamma_1 \mid (\gamma_1, 0, \dots, 0, \gamma) \in H^{(n)}\}$$

and by

$$\tilde{\gamma}_1 := \begin{cases} \min\{\gamma_1 \mid \gamma_1 \in \Delta\} & \text{if } \Delta \neq \emptyset \\ 2g - \gamma & \text{if } \Delta = \emptyset \end{cases}$$

respectively.
Let $\tilde{\Gamma}_n = (\tilde{\gamma}_1, 0, \ldots, 0) \in V^{(n-1)}$. Let $\Delta_{\tilde{\Gamma}_n}$ and $\delta^{\tilde{\Gamma}_n}$ be as in Definition 1-4.
We will show $\delta^{\tilde{\Gamma}_n} = \gamma$.

<u>case</u> $\Delta \neq \emptyset$

Since $(\tilde{\Gamma}_n, \gamma)$ is in $H^{(n)}$, we have $\gamma \in \Delta_{\tilde{\Gamma}_n}$. Now assume that γ satisfies

$$\delta^{\tilde{\Gamma}_n} = \min\{\gamma' \mid \gamma' \in \Delta_{\tilde{\Gamma}_n}\} < \gamma.$$

Then, by $*_n - 1)$,

$$[\{(\tilde{\Gamma}_n, \gamma) - \mathbf{e}_n\} - \mathbf{e}_1, \{(\tilde{\Gamma}_n, \gamma) - \mathbf{e}_n\}] = 1. \qquad \cdots\cdots 1\text{-}5\text{-}5)$$

By 1-5-5), Lemma 1-1 A) and $(\tilde{\Gamma}_n, \gamma) \in H^{(n)}$, we have

$$[\{(\tilde{\Gamma}_n, \gamma) - \mathbf{e}_1\} - \mathbf{e}_n, \{(\tilde{\Gamma}_n, \gamma) - \mathbf{e}_1\}] = 1. \qquad \cdots\cdots 1\text{-}5\text{-}6)$$

Define

$$\Phi := \{\gamma_1 \mid [(\gamma_1, 0, \ldots, 0, \gamma) - \mathbf{e}_n, (\gamma_1, 0, \ldots, 0, \gamma)] = 1\}.$$

By 1-5-6), $\tilde{\gamma}_1 - 1 \in \Phi$, and we can define a positive integer $\tilde{\gamma}_1'$ by $\tilde{\gamma}_1' = \min\{\gamma_1 \mid \gamma_1 \in \Phi\}$. Then $\tilde{\gamma}_1' \leq \tilde{\gamma}_1 - 1$. But $(\tilde{\gamma}_1', 0, \ldots, 0, \gamma) \in H^{(n)}$ by the minimality of $\tilde{\gamma}_1'$ and Lemma1-1 A). This is a contradiction. Thus we get $\delta^{\tilde{\Gamma}_n} = \gamma$.

<u>case</u> $\Delta = \emptyset$

If $\Delta_{\tilde{\Gamma}_n} = \emptyset$, then $\delta^{\tilde{\Gamma}_n} = 2g - deg\,\tilde{\Gamma}_n = 2g - \tilde{\gamma}_1 = \gamma$ by the definition of $\delta^{\tilde{\Gamma}_n}$ and $\tilde{\gamma}_1$. Then it is sufficient to show $\Delta_{\tilde{\Gamma}_n} = \emptyset$.
If $\Delta_{\tilde{\Gamma}_n} \neq \emptyset$, then there exists γ' such that $(\tilde{\Gamma}_n, \gamma') \in H^{(n)}$.
Because of $\gamma' < 2g - \tilde{\gamma}_1 = \gamma$ and $*_n - 1)$,

$$[\{(\tilde{\Gamma}_n, \gamma) - \mathbf{e}_n\} - \mathbf{e}_1, \{(\tilde{\Gamma}_n, \gamma) - \mathbf{e}_n\}] = 1.$$

By Lemma 1-1 B),

$$[\{(\tilde{\Gamma}_n, \gamma) - \mathbf{e}_1\} - \mathbf{e}_n, \{(\tilde{\Gamma}_n, \gamma) - \mathbf{e}_1\}] = 1.$$

By the same argument in case $\Delta \neq \emptyset$, there exists an integer $\tilde{\gamma}_1'$ satisfying $\tilde{\gamma}_1' \leq \tilde{\gamma}_1 - 1$ and $(\tilde{\gamma}_1', 0, \ldots, 0, \gamma) \in H^{(n)}$. This is a contradiction. Therefore we get $\Delta_{\tilde{\Gamma}_n} = \emptyset$. □

Definition 1-6. Let $\Gamma_n = (\gamma_1, \dots, \gamma_{n-1}) \in V^{(n-1)}$. Assume $\Delta_{\Gamma_n} = \varnothing$. By the definition of δ^{Γ_n}, $deg\,\Gamma_n + \delta^{\Gamma_n} = 2g$. Hence the n-tuple $(\Gamma_n, \delta^{\Gamma_n})$ is not in $V^{(n)}$. But we define $i(\Gamma_n, \delta^{\Gamma_n})$ and $l(\Gamma_n, \delta^{\Gamma_n})$

$$\text{by} \quad i(\Gamma_n, \delta^{\Gamma_n}) = 0 \quad \text{and} \quad \text{by} \quad l(\Gamma_n, \delta^{\Gamma_n}) = g + 1$$

respectively (See Definition 1-2).

Using the above notations we have the following equalities on $\# G^{(n)}$.

Theorem 1-7.
(1)

$$\# G^{(n)} = \sum_{\Gamma_n \in H^{(n-1)}} i(\Gamma_n) + \sum_{\Gamma_n \in G^{(n-1)}} i(\Gamma_n, \delta^{\Gamma_n}) + \sum_{\Gamma_n \in G^{(n-1)}} \delta^{\Gamma_n}.$$

(2)

$$\# G^{(n)} = \sum_{\Gamma_n \in H^{(n-1)}} l(\Gamma_n) + \sum_{\Gamma_n \in G^{(n-1)}} l(\Gamma_n, \delta^{\Gamma_n}) - \sum_{\Gamma_n \in V^{(n-1)}} deg\,\Gamma_n + (g-1) \times \# V^{(n-1)}$$

$$= \sum_{\Gamma_n \in H^{(n-1)}} l(\Gamma_n) + \sum_{\Gamma_n \in G^{(n-1)}} l(\Gamma_n, \delta^{\Gamma_n}) - \sum_{k=0}^{2g-1} k \binom{n+k-2}{k}$$

$$+ (g-1) \binom{n+2g-2}{2g-1}.$$

Proof. (1) Take $\Gamma_n = (\gamma_1, \dots, \gamma_{n-1}) \in V^{(n-1)}$ and γ with $0 \le \gamma \le 2g - 1 - deg\,\Gamma_n$.

Suppose $\Gamma_n \in H^{(n-1)}$ first. By $*_n - 1)$, we can see that $(\Gamma_n, \gamma) \in G^{(n)}$ if and only if "$\gamma > 0$ and $[(\Gamma_n, \gamma) - \mathbf{e}_n, (\Gamma_n, \gamma)] = 0$".
Then, by Lemma 1-3,

$$\# \{\gamma \mid (\Gamma_n, \gamma) \in G^{(n)}\} = i(\Gamma_n) \quad \text{for} \quad \Gamma_n \in H^{(n-1)}. \qquad \cdots\cdots 1\text{-}7\text{-}1)$$

Next suppose $\Gamma_n \in G^{(n-1)}$.
If $\gamma \ge \delta^{\Gamma_n}$, then $[(\Gamma_n, \gamma) - \mathbf{e}_i, (\Gamma_n, \gamma)] = 1$ for $i = 1, \dots, n-1$. Thus we have

$$(\Gamma_n, \gamma) \in G^{(n)} \text{ if and only if } \begin{cases} \text{“}0 \le \gamma < \delta^{\Gamma_n}\text{”} \\ \text{or} \\ \text{“}\gamma \ge \delta^{\Gamma_n} \quad \text{and} \quad [(\Gamma_n, \gamma - 1), (\Gamma_n, \gamma)] = 0\text{”}. \end{cases}$$

Therefore, by Lemma 1-3,

$$\#\{\gamma \mid (\Gamma_n, \gamma) \in G^{(n)}\} = i(\Gamma_n, \delta^{\Gamma_n}) + \delta^{\Gamma_n} \quad \text{for } \Gamma_n \in G^{(n-1)}. \qquad \cdots\cdots 1\text{-}7\text{-}2)$$

Thus we have the equation (1) by 1-7-1 and 1-7-2.

(2) This follows from $l(\Gamma) = i(\Gamma) + 1 + deg\,\Gamma - g$, $\# V^{(n-1)} = \binom{n+2g-2}{2g-1}$ and

$$\sum_{\Gamma_n \in V^{(n-1)}} deg\,\Gamma_n = \sum_{k=0}^{2g-1} k \binom{n+k-2}{k}. \qquad \square$$

§2. The lower bound of $\# G^{(n)}$

In this section we will determine the lower bound of $\# G^{(n)}$, and show that there is a unique graph $(D^{(n)}, *_n)$ which attains the lower bound of $\# G^{(n)}$.

Let the notation be as in §1. First we will prove the following lemma.

LEMMA 2-1. *Let $\Gamma = (\gamma_1, \dots, \gamma_n) \in V^{(n)}$. Assume $\gamma_i > 0$ and $[\Gamma - \mathbf{e}_i, \Gamma] = 1$ for some i. Then there exists $\Gamma' = (\gamma'_1, \gamma'_2, \dots, \gamma'_n) \in H^{(n)}$ that satisfies $\Gamma' \leq \Gamma$ and $\gamma'_i = \gamma_i$.*

PROOF. We may assume $i = 1$. Define

$$\gamma'_2 := \min\{\gamma \mid [(\gamma_1, \gamma, \gamma_3, \dots, \gamma_n) - \mathbf{e}_1, (\gamma_1, \gamma, \gamma_3, \dots, \gamma_n)] = 1\}$$

for the above $\Gamma = (\gamma_1, \gamma_2, \gamma_3, \dots, \gamma_n)$.

Then

$[(\gamma_1, \gamma'_2, \gamma_3, \dots, \gamma_n) - \mathbf{e}_2, (\gamma_1, \gamma'_2, \gamma_3, \dots, \gamma_n)] = 1.$

In fact, if

$[(\gamma_1, \gamma'_2, \gamma_3, \dots, \gamma_n) - \mathbf{e}_2, (\gamma_1, \gamma'_2, \gamma_3, \dots, \gamma_n)] = 0,$

then $[\{(\gamma_1, \gamma'_2, \gamma_3, \dots, \gamma_n) - \mathbf{e}_2\} - \mathbf{e}_1, \{(\gamma_1, \gamma'_2, \gamma_3, \dots, \gamma_n) - \mathbf{e}_2\}] = 1$ by Lemma 1-1 A). This contradicts to the definition of γ'_2.

Next define

$$\gamma'_3 := \min\{\gamma \mid [(\gamma_1, \gamma'_2, \gamma, \gamma_4, \dots, \gamma_n) - \mathbf{e}_1, (\gamma_1, \gamma'_2, \gamma, \gamma_4, \dots, \gamma_n)] = [(\gamma_1, \gamma'_2, \gamma, \gamma_4, \dots, \gamma_n) - \mathbf{e}_2, (\gamma_1, \gamma'_2, \gamma, \gamma_4, \dots, \gamma_n)] = 1\}.$$

Then

$$[(\gamma_1, \gamma'_2, \gamma'_3, \gamma_4, \dots, \gamma_n) - \mathbf{e}_3, (\gamma_1, \gamma'_2, \gamma'_3, \gamma_4, \dots, \gamma_n)] = 1$$

by the same reason as above. After repeating these procedures, we get the Γ' that we want. $\square$

Next we will define a filtration of $G^{(n)}$ by

$$G^{(n)} = A_0^{(n)} \supset A_1^{(n)} \supset A_2^{(n)} \supset \cdots \supset A_k^{(n)} \supset \cdots \supset A_{g-1}^{(n)} \supset A_g^{(n)} = \varnothing,$$

where

$$A_k^{(n)} := \{\Gamma \mid i(\Gamma) \geq k, \Gamma \in G^{(n)}\}.$$

For each k, define subsets $B_k^{(n)}$ and $C_k^{(n)}$ of $A_k^{(n)}$ by

$$B_k^{(n)} = \{\Gamma \mid \Gamma = (\Gamma_n, \gamma) \in G^{(n)}, \Gamma_n \in H^{(n-1)}, i(\Gamma) \geq k\}$$

and by

$$C_k^{(n)} = \{\Gamma \mid \Gamma = (0_n, \gamma) \in G^{(n)}, i(\Gamma) \geq k\}$$

respectively, where $0_n = (0, \dots, 0) \in H^{(n-1)}$. Then we have

$$B_0^{(n)} \supset B_1^{(n)} \supset B_2^{(n)} \cdots \supset B_k^{(n)} \supset \cdots \supset B_{g-1}^{(n)} \supset B_g^{(n)},$$

$$C_0^{(n)} \supset C_1^{(n)} \supset C_2^{(n)} \cdots \supset C_k^{(n)} \supset \cdots \supset C_{g-1}^{(n)} \supset C_g^{(n)}$$

and

$$A_k^{(n)} \supset B_k^{(n)} \supset C_k^{(n)} \ (k = 0, \dots, g).$$

$a_k^{(n)}$ and $b_k^{(n)}$ denote $\# A_k^{(n)}$ and $\# B_k^{(n)}$ respectively.

Then we have the following lemma.

Lemma 2-2. i) $b_k^{(n)} \geq g - k$ *for* $k = 0, \dots, g$.
Moreover $b_k^{(n)} = g - k$ *if and only if* $B_k^{(n)} = C_k^{(n)}$.

ii) *The following conditions are equivalent*:

a) $b_0^{(n)} = g$;

b) $b_k^{(n)} = g - k$ *for* $k = 0, 1, \dots, g$;

c) $i(\Gamma_n) = 0$ *for* $\Gamma_n \in H^{(n-1)} \backslash \{O_n\}$;

d) *take* $\tilde{\Gamma}_n \in V^{(n-1)}$ *with* $deg\, \tilde{\Gamma}_n = 2g - 1$. *Then the first g edges of any path from* O_n *to* $\tilde{\Gamma}_n$ *are of weight* 0;

e) $G^{(n-1)} = \{\Gamma_n \in V^{(n-1)} \mid 0 < deg\, \Gamma_n \leq g\}$.

Proof. i) By Lemma 1-3, we have $\# C_k^{(n)} = g - k$ $(k = 0, \dots, g)$. Then i) follows from $B_k(n) \supset C_k(n)$ $(k = 1, \dots, g)$.

ii) *a)* $\Leftrightarrow$ *b)*

We can easily see that

$$\begin{aligned} b_0^{(n)} = g \quad &\Leftrightarrow \quad B_0^{(n)} = C_0^{(n)} \\ &\Leftrightarrow \quad B_k^{(n)} = C_k^{(n)} (k = 0, \dots, g) \\ &\Leftrightarrow \quad b_k^{(n)} = g - k. \end{aligned}$$

b) $\Leftrightarrow$ *c)*

If $b_k^{(n)} > g - k$ for some k, then there exists $\Gamma = (\Gamma_n, \gamma) \in G^{(n)}$ with $\Gamma_n \in H^{(n-1)} \backslash \{0_n\}$ and $i(\Gamma) \geq k$. By Lemma 1-3, $i(\Gamma_n) \geq k + 1$. Thus we have *b)* $\Leftarrow$ *c)*, and vice versa.

c) $\Rightarrow$ *d)*

Suppose c) to be true. Fix a path $0_n \tilde{\Gamma}_n$ with $deg\, \tilde{\Gamma}_n = 2g - 1$. We denote this path by $\mathscr{P}$. Take a vertex $\Gamma_n = (\gamma_1, \dots, \gamma_i, \dots, \gamma_{n-1}) \neq 0_n$ on $\mathscr{P}$ that satisfies $\gamma_i > 0$ and $[\Gamma_n - \mathbf{e}_i, \Gamma_n] = 1$ for some $1 \leq i \leq n - 1$. Then there exists $\Gamma'_n = (\gamma'_1, \dots, \gamma'_i, \dots, \gamma'_{n-1}) \in H^{(n-1)} \backslash \{0_n\}$ that satisfies $\Gamma'_n \leq \Gamma_n$ and $\gamma_i = \gamma'_i$ by Lemma 2-1.

Since $i(\Gamma'_n) = 0$ by c), there is no edge of weight 0 on any path $\Gamma'_n \tilde{\Gamma}_n$. So there is no edge of weight 0 between Γ_n and $\tilde{\Gamma}_n$ on $\mathscr{P}$. By $*_n - 2)$ we get d).

d) $\Rightarrow$ *e)*

By $*_n - 2)$, d) implies that $\Gamma_n \in G^{(n-1)}$ if and only if $deg\, \Gamma_n \leq g$.

e) $\Rightarrow$ *c)*

e) is equivalent to the fact that $\Gamma_n \in H^{(n-1)} \backslash \{0_n\}$ if and only if $deg\, \Gamma_n > g$. This implies c). □

Now we will show the main theorem of this section.

THEOREM 2-3. i) *For $n \geq 2$, the following conditions are equivalent:*
(1) $G^{(n)} = \{\Gamma \mid 0 < deg\, \Gamma \leq g\}$;
(2) $a_0^{(n)} = \# G^{(n)}$ *is minimal for all types of* $(D^{(n)}, *_n)$;
(3) *For each* $k (= 0, \dots, g - 1)$, $a_k^{(n)}$ *is minimal for all types of* $(D^{(n)}, *_n)$.
ii) *The lower bound of* $\# G^{(n)}$ *is*

$$\binom{n+g}{g} - 1,$$

which is only attainable by a unique graph defined by (1).

Proof. Let $(D^{(n)}, *_n)$ be an arbitrary Riemann-Roch graph, and let $(D^{(n-1)}, *_{n-1})$ be the subgraph of it as before. Since $i(\Gamma_n) = k$ for $\Gamma_n \in A_k^{(n-1)} \backslash A_{(k+1)}^{(n-1)}$, we have

$$\#\{\gamma > 0 \mid [(\Gamma_n, \gamma - 1), (\Gamma_n, \gamma)] = 0, deg\, \Gamma_n + \gamma \leq 2g - 1\} = k.$$

Of course $(\Gamma_n, \gamma) \in G^{(n)}$ if $[(\Gamma_n, \gamma - 1), (\Gamma_n, \gamma)] = 0$. Watching $(\Gamma_n, 0) \in G^{(n)}$ for $\Gamma_n \in G^{(n-1)}$, we have

$$\#\{\gamma \geq 0 \mid i(\Gamma_n, \gamma) \geq 0, (\Gamma_n, \gamma) \in G^{(n)}\} = \#\{\gamma \mid (\Gamma_n, \gamma) \in G^{(n)}\} \geq k + 1$$

$$\#\{\gamma \geq 0 \mid i(\Gamma_n, \gamma) \geq 1, (\Gamma_n, \gamma) \in G^{(n)}\} \geq k$$

I_k $\qquad\qquad\qquad\qquad$ ......

$$\#\{\gamma \geq 0 \mid i(\Gamma_n, \gamma) \geq k, (\Gamma_n, \gamma) \in G^{(n)}\} \geq 1$$

$$\text{for } \Gamma_n \in A_k^{(n-1)} \backslash A_{(k+1)}^{(n-1)} \qquad (k = 0, 1, \dots, g - 1).$$

By using I_k for $k = 0, \dots, g - 1$, we have

$$a_0^{(n)} \geq (a_0^{(n-1)} - a_1^{(n-1)}) + 2(a_1^{(n-1)} - a_2^{(n-1)}) + \cdots$$

$$+ (g - 1)(a_{g-2}^{(n-1)} - a_{g-1}^{(n-1)}) + g a_{g-1}^{(n-1)} + b_0^{(n)}$$

$$a_1^{(n)} \geq (a_1^{(n-1)} - a_2^{(n-1)}) + \cdots + (g - 2)(a_{g-2}^{(n-1)} - a_{g-1}^{(n-1)}) + (g - 1)a_{g-1}^{(n-1)} + b_1^{(n)}$$

....

$$a_{g-1}^{(n)} \geq a_{g-1}^{(n-1)} + b_{g-1}^{(n)},$$

and then

II $\qquad a_k^{(n)} \geq a_k^{(n-1)} + \cdots\cdots\cdots + a_{g-1}^{(n-1)} + b_k^{(n)} \quad (k = 0, 1, \dots, g - 1).$

Remark. All the equalities of II) hold if and only if all the equalities of I_k hold for all $\Gamma_n \in G^{(n-1)}$.

To prove the theorem we use the follwing Lemma.

Lemma 2-4. (1) $b_0^{(n)}, \dots, b_{g-1}^{(n)}$ *are minimal if and only if*

$$G^{(n-1)} = \{\Gamma_n \mid 0 < deg\, \Gamma_n \leq g\}.$$

(2) *Assume* $G^{(n-1)} = \{\Gamma_n \mid 0 < deg\, \Gamma_n \leq g\}$. *Then the following conditions are equivalent*:

a) *the first equality in each* $I_k (0 \leq k \leq g-1)$ *holds*;
b) *all the equalities in each* $I_k (0 \leq k \leq g-1)$ *hold*;
c) $\delta^{\Gamma_n} = g + 1 - deg\,\Gamma_n$ *for* $\Gamma_n \in G^{n-1}$;
d) $G^{(n)} = \{\Gamma \mid 0 < deg\,\Gamma \leq g\}$.

PROOF. (1) This follows from Lemma 2-2.
(2) $b) \Rightarrow c)$
Assume $\delta^{\Gamma_n} > g + 1 - deg\,\Gamma_n$ for some $\Gamma_n \in A_k^{(n-1)} \backslash A_{k+1}^{(n-1)}$. $i(\Gamma_n) = k \geq 0$. By Lemma 2-2 d), $i(\Gamma_n) = g - deg\,\Gamma_n$.
Hence there is $\tilde{\gamma}$ satisfying

$$[(\Gamma_n, \tilde{\gamma} - 1), (\Gamma_n, \tilde{\gamma})] = 1 \quad \text{and} \quad 0 < \tilde{\gamma} \leq g + 1 - deg\,\Gamma_n.$$

But $(\Gamma_n, \tilde{\gamma}) \in G^{(n)}$ because of $\delta^{\Gamma_n} > \tilde{\gamma}$. Then

$$\#\{\gamma \mid i(\Gamma_n, \gamma) \geq 0, (\Gamma_n, \gamma) \in G^{(n)}\} \geq k + 2.$$

$c) \Rightarrow d)$
Suppose c) to be true. By Lemma 1-5 iii) and $\{\delta^{\Gamma_n} \mid \Gamma_n \in G^{(n-1)}\} = \{1, \ldots, g\}$, we have

$$(O_n, k) \in G^{(n)} \quad \text{if and only if} \quad 1 \leq k \leq g.$$

First we will show

$$[\Gamma - \mathbf{e}_n, \Gamma] = 1$$

for $\Gamma = (\gamma_1, \ldots, \gamma_n) \in V^{(n)}$ with $deg\,\Gamma \geq g + 1$ and $\gamma_n > 0$.
If $\gamma_n \geq g + 1$, then $[\Gamma - \mathbf{e}_n, \Gamma] = 1$ by $(O_n, \gamma_n) \in H^{(n)}$ and $*_n - 1)$. When $\gamma_n \leq g$, take $\Gamma' = (\gamma'_1, \ldots, \gamma'_{n-1}, \gamma_n) = (\Gamma'_n, \gamma_n)$ with $deg\,\Gamma' = g + 1$ and $\Gamma' \leq \Gamma$. Then $deg\,\Gamma'_n \leq g, \Gamma'_n \in G^{(n-1)}$ and $\gamma_n = g + 1 - deg\,\Gamma'_n = \delta^{\Gamma'_n}$ by c). Also by $*_n - 1)$ and the definition of $\delta^{\Gamma'_n}$, we have $[\Gamma - \mathbf{e}_n, \Gamma] = 1$.
Next we will show

$$[\Gamma - \mathbf{e}_1, \Gamma] = 1$$

for $\Gamma = (\gamma_1, \ldots, \gamma_n) \in V^{(n)}$ with $deg\,\Gamma \geq g + 1$ and $\gamma_1 > 0$.
When $\gamma_1 \geq g + 1, [\Gamma - \mathbf{e}_1, \Gamma] = 1$ as above. When $\gamma_1 \leq g$, take $\Gamma' = (\gamma_1, \gamma'_2, \ldots, \gamma'_n)$ satisfying $\Gamma' \leq \Gamma$ and $deg\,\Gamma' = g + 1$. Put $\Gamma' = (\tilde{\Gamma}_n, \gamma'_n)$, then $\gamma'_n = \delta^{\tilde{\Gamma}_n}$ and $[\Gamma' - \mathbf{e}_1, \Gamma'] = 1$. Thus we have $[\Gamma - \mathbf{e}_1, \Gamma] = 1$ by $*_n - 1)$.
This argument is also effective when the index 1 is replaced with $i \neq 1$. Thus if Γ satisfies $deg\,\Gamma \geq g + 1$, then $[\Gamma - \mathbf{e}_i, \Gamma] = 1$ $(0 \leq i \leq n)$.

The implications $d) \Rightarrow a)$ and $a) \Rightarrow b)$ are easy. □

Proof of Theorem 2-3. i)
We prove this theorem by induction on n.
Now we assume that

$$a_k^{(n-1)}(k=0,\dots,g-1) \text{ are minimal if } G^{(n-1)}=\{\Gamma_n \mid 0<deg\,\Gamma_n\le g\}.\dots\star_{n-1})$$

By our assumption $\star_{n-1}$) and Lemma 2-4 (1), the right hand side of each inequality of II is minimal if and only if

$$G^{(n-1)}=\{\Gamma_n \mid 0<deg\,\Gamma_n\le g\}.$$

Moreover, when $G^{(n-1)}=\{\Gamma_n \mid 0<deg\,\Gamma_n\le g\}$, all the equalities of II hold if and only if

$$G^{(n)}=\{\Gamma \mid 0<deg\,\Gamma\le g\}$$

by Lemma 2-4 (2) and Remark before Lemma 2-4.
Thus $a_k^{(n)}(k=0,\dots,g-1)$ are minimal if and only if

$$G^{(n)}=\{\Gamma \mid 0<deg\,\Gamma\le g\}$$

under the assumption $\star_{n-1}$).

When $n=2$, $\# G^{(1)}=g$ and $a_k^{(1)}=g-k$ $(k=0,\dots,g-1)$ for any type of $D^{(1)}$. Then the assumption $\star_1$ is satisfied, and we get Theorem 2-3. □

Example 2-5. Let M be a hyperelliptic curve and $P_1,P_2,\dots,P_n$ be non-Wierestrass points satisfying $|P_i+P_j|\ne g_2^1(1\le i,j\le n)$. Then

$$G_M(P_1,\dots,P_n)=\{\Gamma \mid 0<deg\,\Gamma\le g\}.$$

In fact this can be easily seen by the same calculation done by Kim([3]) in case $n=2$.

§3. The upper bound of $\# G^{(3)}$

In this section we determine the upper bound of $\# G^{(3)}$.

Let $(D^{(n)},*_n)$ be a Riemann-Roch graph and let $(D^{(n-1)},*_{n-1})$ be its subgraph as in §1. The subsets of vertices

$$V^{(n)}\supset V^{(n-1)}\supset\cdots\supset V^{(1)},$$

$$G^{(n)}\supset G^{(n-1)}\supset\cdots\supset G^{(1)}$$

and

$$H^{(n)}\supset H^{(n-1)}\supset\cdots\supset H^{(1)}$$

are also as in §1.

Define

$$G_i := \{x \mid xe_i \in G^{(n)}\} \quad \text{and} \quad H_i := \{n \mid 0 \le n \le 2g-1\} \backslash G_i$$

respectively.

REMARK. H_1 and G_1 coincide with $H^{(1)}$ and $G^{(1)}$ respectively.

LEMMA 3-1. *Fix a Riemann-Roch graph* $(D^{(2)}, *_2)$. *For* $\alpha \in V^{(1)}$, *let* $\beta(\alpha)$ *be the non-negative integer* δ^α *defined in* 1-4

$$\left(i.e.,\ \beta(\alpha) = \delta^\alpha = \begin{cases} min\{\beta \mid (\alpha,\beta) \in H^{(2)}\}\ (\le 2g-1-\alpha) & if\ \{\beta \mid (\alpha,\beta) \in H^{(2)}\} \neq \varnothing \\ 2g-\alpha & if\ \{\beta \mid (\alpha,\beta) \in H^{(2)}\} = \varnothing \end{cases} \right).$$

Then

i) *For* $\alpha \in G_1$, $\beta(\alpha)$ *is in* G_2. *Moreover the map* $\beta(*) : G_1 \to G_2$ *defined by* $\beta(\alpha)$ *is one to one.*

ii) *For* $\alpha \in G_1$, *we have*

$$\{\beta | [(\alpha-1,\beta),(\alpha,\beta)] = 1\} \neq \varnothing \quad \text{if and only if} \quad \{\beta \mid (\alpha,\beta) \in H^{(2)}\} \neq \varnothing$$

and

$$\beta(\alpha) = \begin{cases} \min\{\beta \mid [(\alpha-1,\beta),(\alpha,\beta)] = 1\}(\le 2g-1-\alpha) & if\ \{\beta \mid [(\alpha-1,\beta),(\alpha,\beta)]=1\} \neq \varnothing \\ 2g-\alpha & if\ \{\beta \mid [(\alpha-1,\beta),(\alpha,\beta)]=1\} = \varnothing. \end{cases}$$

iii) *For* $\beta \in G_2$, *we have*

$$\{\alpha \mid [(\alpha,\beta-1),(\alpha,\beta)] = 1\} \neq \varnothing \quad if\ and\ only\ if \quad \{\alpha \mid (\alpha,\beta) \in H^{(2)}\} \neq \varnothing\}.$$

If $\alpha(*) : G_2 \to G_1$ *be the inverse map of* $\beta(*)$ *in* i), *then*

$$\alpha(\beta) =_{*)} \begin{cases} \min\{\alpha \mid (\alpha,\beta) \in H^{(2)}\} & if \quad \{\alpha \mid (\alpha,\beta) \in H^{(2)}\} \neq \varnothing \\ 2g-\beta & if \quad \{\alpha \mid (\alpha,\beta) \in H^{(2)}\} = \varnothing \end{cases}$$

$$=_{**)} \begin{cases} \min\{\alpha \mid [(\alpha,\beta-1),(\alpha,\beta)] = 1\} & if \quad \{\alpha \mid [(\alpha,\beta-1),(\alpha,\beta)] = 1\} \neq \varnothing \\ 2g-\beta & if \quad \{\alpha \mid [(\alpha,\beta-1),(\alpha,\beta)] = 1\} = \varnothing. \end{cases}$$

PROOF. i) This follows from Lemma 1-5 iii) and $\# G_1 = \# G_2 = g$.

ii) Fix $\alpha \in G_1$.

Put

$$\beta' = \begin{cases} \min\{\beta \mid [(\alpha-1,\beta),(\alpha,\beta)] = 1\}(\le 2g-1-\alpha) & \text{if } \{\beta \mid [(\alpha-1,\beta),(\alpha,\beta)]=1\} \neq \varnothing \\ 2g-\alpha & \text{if } \{\beta \mid [(\alpha-1,\beta),(\alpha,\beta)]=1\}=\varnothing. \end{cases}$$

Assume $\{\beta \mid [(\alpha-1,\beta),(\alpha,\beta)] = 1\} \neq \varnothing$.

Then we have

$$[(\alpha, \beta' - 1), (\alpha, \beta')] = 1.$$

In fact, if $[(\alpha, \beta' - 1), (\alpha, \beta')] = 0$, then

$$[(\alpha - 1, \beta' - 1), (\alpha, \beta' - 1)] = 1$$

by 1-1 A). This contradicts to the definition of β'. Thus

$$\beta\prime \in \{\beta \mid (\alpha, \beta) \in H^{(2)}\}.$$

Consequently we have

$$\{\beta \mid (\alpha, \beta) \in H^{(2)}\} \neq \varnothing \quad \text{and} \quad \beta' \geq \beta(\alpha).$$

Conversely, if $\{\beta \mid (\alpha, \beta) \in H^{(2)}\} \neq \varnothing$, then obviously

$$\{\beta \mid [(\alpha - 1, \beta), (\alpha, \beta)] = 1\} \neq \varnothing \quad \text{and} \quad \beta' \leq \beta(\alpha).$$

Thus we have

$$\{\beta \mid [(\alpha - 1, \beta), (\alpha, \beta)] = 1\} \neq \varnothing \quad \text{if and only if} \quad \{\beta \mid (\alpha, \beta) \in H^{(2)}\} \neq \varnothing,$$

and

$$\beta(\alpha) = \beta'.$$

iii) Fix $\beta \in G_2$. By the same way as in ii), we have

$$\{\alpha \mid [(\alpha, \beta - 1), (\alpha, \beta)] = 1\} \neq \varnothing \quad \text{if and only if} \quad \{\alpha \mid (\alpha, \beta) \in H^{(2)}\} \neq \varnothing,$$

and

$$\min\{\alpha \mid [(\alpha, \beta - 1), (\alpha, \beta)] = 1\} = \min\{\alpha \mid (\alpha, \beta) \in H^{(2)}\}$$

if $\{\alpha \mid [(\alpha, \beta - 1), (\alpha, \beta)] = 1\} \neq \varnothing$.
Thus we get the second equality $**$).

Next we will show the first equality $*$).
Assume $\{\alpha \mid [(\alpha, \beta - 1), (\alpha, \beta)] = 1\} \neq \varnothing$.

Put

$$\tilde{\alpha} = \min\{\alpha \mid [(\alpha, \beta - 1), (\alpha, \beta)] = 1\} = \min\{\alpha \mid (\alpha, \beta) \in H^{(2)}\}.$$

Then $\tilde{\alpha} \leq 2g - 1 - \beta$ and $\beta(\tilde{\alpha}) \leq \beta$.

Now assume $\beta(\tilde{\alpha}) < \beta$. Then

$$[(\tilde{\alpha} - 1, \beta - 1), (\tilde{\alpha}, \beta - 1)] = 1$$

by $*_2 - 1$), and

$$[(\tilde{\alpha} - 1, \beta - 1), (\tilde{\alpha} - 1, \beta)] = 1$$

by Lemma 1-1 A) and $(\tilde{\alpha}, \beta) \in H^{(2)}$.
This contradicts to the minimality of $\tilde{\alpha}$. Thus we have $\beta(\tilde{\alpha}) = \beta = \beta(\alpha(\beta))$. By i) of this lemma we get $\tilde{\alpha} = \alpha(\beta)$.

Next assume that $\{\alpha \mid [(\alpha, \beta - 1), (\alpha, \beta)] = 1\} = \emptyset$. If $2g - 1 - \alpha(\beta) \geq \beta = \beta(\alpha(\beta))$, then $(\alpha(\beta), \beta(\alpha(\beta))) \in H^{(2)}$. This contradicts to the above assumption. Since $\alpha(\beta) + \beta(\alpha(\beta)) \leq 2g$ (Lemma 1-5), $\alpha(\beta) = 2g - \beta$.

Then we get the equality $*$). □

REMARK. At first the map $\beta(*)$ was introduced by Kim in case $D^{(2)} = D_M(P, Q)$.

Formula (2) in Theorem 1-7 for $n = 3$ and $n = 2$ can be written as follows.

LEMMA 3-2 (Corollary of Theorem 1-7).
(1) *Let* $(\alpha, \beta) \in V^{(2)}$. *We write* $\delta^{\alpha\beta}$ *for* $(\alpha, \beta) \in V^{(2)}$. *Then*

$$\# G^{(3)} = \sum_{(\alpha,\beta) \in H^{(2)}} l(\alpha, \beta) + \sum_{(\alpha,\beta) \in G^{(2)}} l(\alpha, \beta, \delta^{\alpha\beta}) - \frac{g(2g+1)(g+1)}{3},$$

where $l(\alpha, \beta, \delta^{\alpha\beta}) = g + 1$ *if* $\alpha + \beta + \delta^{\alpha\beta} = 2g$.
(2)

$$\# G^{(2)} = \frac{g(g-1)}{2} + \sum_{\alpha \in G_1} l(\alpha, \beta(\alpha)) \leq \frac{(3g^2 + g)}{2},$$

where $l(\alpha, \beta(\alpha)) = g + 1$ *if* $\alpha + \beta(\alpha) = 2g$.
Moreover $\# G^{(2)} = (3g^2 + g)/2$ *if and only if* $\beta(\alpha) = 2g - \alpha$ *for all* $\alpha \in G_1$.

PROOF. (2) This follows from $\{l(\alpha) \mid \alpha \in H^{(1)} = H_1\} = \{1, 2, \ldots, g\}$. □

DEFINITION 3-3. Let $(D^{(3)}, *_3)$ be a Riemann-Roch graph. $(D^{(2)}, *_2)$ is the subgraph of $(D^{(3)}, *_3)$, and $(D^{(1)}, *_1)$ is the subgraph of $(D^{(2)}, *_2)$ as before. Define subsets S, T and R of $V^{(2)}$ as follows.

$S := \{(\alpha, \beta) \in G^{(2)} \mid (\alpha, \beta, \gamma) \in G^{(3)}$ for any $\gamma \leq 2g - 1 - \alpha - \beta\}$.

$T := \{(u, v) \in V^{(2)} \mid 0 \leq u + v \leq 2g - 2, [(u, v), (u + 1, v)] = [(u, v), (u, v + 1)] = 0\}$.

$R := \{(a, b) \in V^{(2)} \mid 0 \leq a + b \leq 2g - 2, [(a, b, 2g - 2 - a - b), (a, b, 2g - 1 - a - b)] = 0\}$.

(N.B., $(u + 1, v) \in G_2$ and $(u, v + 1) \in G_2$ for $(u, v) \in T$).

Lemma 3-4.

(1)

$$R = \{(a,b) \in V^{(2)} \mid [(a,b,2g-2-a-b),(a,b,2g-1-a-b)] = 0\}$$

$$= \{(a,b) \in V^{(2)} \mid [(a,b,2g-2-a-b),(a+1,b,2g-2-a-b)] = 0\}$$

$$= \{(a,b) \in V^{(2)} \mid [(a,b,2g-2-a-b),(a,b+1,2g-2-a-b)] = 0\}.$$

(2)

$$S = \{(\alpha,\beta) \in G^{(2)} \mid l(\alpha,\beta,\delta^{\alpha\beta}) = g+1\} = \{(\alpha,\beta) \in G^{(2)} \mid \delta^{\alpha\beta} = 2g-\alpha-\beta\}.$$

Proof. (1) This follows from Lemma 1-1 B).
(2) This follows from the definition of S and Definition 1-6. □

By Lemma 3-4 (1), $[(a,b),(a,b+1)] = [(a,b),(a+1,b)] = 0$ for $(a,b) \in R$. Then there is a natural inclusion $\varphi : R \to T$(i.e., $(u,v) = \varphi(a,b) = (a,b)$) and $\# R \leq \# T$.

To estimate the cardinarities of S and T, we use the following number $r(\beta(*))$ defined by Homma.

Definition 3-5 (Homma [2]). Let $G_1 = \{\alpha_1 < \alpha_2 < \cdots < \alpha_g\}$, and let $G_2 = \{\beta_1 < \beta_2 < \cdots < \beta_g\}$. Define a non-negative integer $r(\beta(*))$ by

$$r(\beta(*)) := \#\{(i,j) \mid \alpha_i < \alpha_j \text{ (i.e.,} i<j) \quad \text{and} \quad \beta(\alpha_i) > \beta(\alpha_j)\}.$$

Lemma 3-6. *Let $(D^{(3)}, *_3)$ be a Riemann-Roch graph, and let S and T be as above. Then*

(1)

$$T = \{(u,v) \in V^{(2)} \mid u+1 \in G_1, v+1 \in G_2, 0 \leq u+v \leq 2g-2, \beta(u+1) \geq v+1 \text{ and } \alpha(v+1) \geq u+1\}.$$

(2)

$$\# T = r(\beta(*)) + \#(G_1) = r(\beta(*)) + g \leq \frac{g(g+1)}{2}.$$

And the equality $\# T = g(g+1)/2$ holds if and only if

$$\beta(\alpha_i) = \beta_{g+1-i}, \quad 1 \leq i \leq g.$$

(3) $\#S \leq g(g+1)$.
If the equality $\#S = g(g+1)$ *holds, then*

$$G_1 = G_2 = G_3 = \{1, 3, 5, \ldots, 2g-1\} \quad \textit{and} \quad \beta(\alpha) = 2g - \alpha.$$

In this case, $(D^{(2)}, *_2)$ *is defined by*

$$\text{“}[(u-1, v), (u, v)] = 0 \quad \textit{if and only if} \quad u \quad \textit{is odd”}$$

and

$$\text{“}[(u, v-1), (u, v)] = 0 \quad \textit{if and only if} \quad v \quad \textit{is odd.”}$$

Therefore we have $G^{(2)} = \{(u, v) \in V^{(2)} \mid u \textit{ or } v \textit{ is odd}\}$ *and* $l(\alpha, \beta(\alpha)) = g + 1$ *for* $\alpha \in G_1$.

PROOF. (1) By Lemma 3-1 ii),

$$\text{“}[(u, v), (u+1, v)] = 0 \quad \text{if and only if} \quad v < \beta(u+1)\text{”}$$

for $u + 1 \in G_1$, and by Lemma 3-1 iii),

$$\text{“}[(u, v), (u, v+1)] = 0 \quad \text{if and only if} \quad u < \alpha(v+1)\text{”}$$

for $v + 1 \in G_2$. Thus we get (1).

(2) For $(u, v) \in T$, put $x = u + 1$ and $y = v + 1$. Then $x \in G_1$, $y \in G_2$, $\beta(x) \geq y$ and $\alpha(y) \geq x$. Since $\alpha(*) = \beta^{-1}(*)$ on G_2, there exists a unique $x' \in G_1$ satisfying $\beta(x') = y$ and $\alpha(y) = x'$. Thus

$$\begin{aligned}\#T &= \#\{(x, y) \mid x \in G_1, y \in G_2, y < \beta(x) \quad \text{and} \quad x < \alpha(y)\} \\ &\quad + \#\{(x, y) \mid x \in G_1, \beta(x) = y\} \\ &= \#\{(x, x') \mid x \in G_1, x' \in G_1, x' > x, \beta(x') < \beta(x)\} + \#\{(x, \beta(x)) \mid x \in G_1\},\end{aligned}$$

and we have $\#T = r(\beta(*)) + g$.
Homma ([2]) has shown that

$$0 \leq r(\beta(*)) \leq \frac{g(g-1)}{2}$$

and

$$\text{“}r(\beta(*)) = \frac{g(g-1)}{2} \quad \text{if and only if} \quad \beta(\alpha_i) = \beta_{g+1-i} \ (1 \leq i \leq g)\text{”}.$$

Thus we get (2).

(3) Assume

$$\begin{aligned}[(\alpha-1,\beta,2g-1-\alpha-\beta),(\alpha,\beta,2g-1-\alpha-\beta)] \\ =[(\alpha,\beta-1,2g-1-\alpha-\beta),(\alpha,\beta,2g-1-\alpha-\beta)] \\ =1.\end{aligned}$$

for $(\alpha,\beta)\in S$.
Let

$$\gamma_0:=\min\{\gamma\,|\,[(\alpha-1,\beta,\gamma),(\alpha,\beta,\gamma)]=[(\alpha,\beta-1,\gamma),(\alpha,\beta,\gamma)]=1\}.$$

Then $\gamma_0\leq 2g-1$, and $[(\alpha,\beta,\gamma_0-1),(\alpha,\beta,\gamma_0)]=1$ by Lemma 1-1 A) and the minimality of γ_0. This implies that (α,β,γ_0) is in $H^{(3)}$. This contradicts to $(\alpha,\beta)\in S$. Then for $(\alpha,\beta)\in S$, we have

$$[(\alpha-1,\beta,2g-1-\alpha-\beta),(\alpha,\beta,2g-1-\alpha-\beta)]=0$$

$b)$ or

$$[(\alpha,\beta-1,2g-1-\alpha-\beta),(\alpha,\beta,2g-1-\alpha-\beta)]=0.$$

$b)$ means that

$$(\alpha-1,\beta)\quad\text{or}\quad(\alpha,\beta-1)\quad\text{is in } R\quad\text{for } (\alpha,\beta)\in S. \qquad\cdots\cdots 3\text{-}6\text{-}1)$$

On the other hand, by Lemma 3-4 (1) and $*_3-1)$,

$$(a+1,b)\quad\text{and}\quad(a,b+1)\quad\text{are in } S\quad\text{for } (a,b)\in R. \qquad\cdots\cdots 3\text{-}6\text{-}2)$$

Then we can consider the one-to-two correspndence $(a,b)\to\{(a+1,b),(a,b+1)\}$ from R to S by 3-6-2), and $\#S\leq 2\times\#R$ by 3-6-1). Therefore, by (2) of this lemma, we have

$$\#S\leq 2\times\#R\leq 2\times\#T\leq 2\times\frac{g(g+1)}{2}=g(g+1).$$

Thus we get the former half of (3).

Moreover we have

$$\#S=g(g+1)\quad\text{if and only if}\quad\begin{cases} a)\ \#T=\#R=\dfrac{g(g+1)}{2} \\ b)\ \text{one and only one of } (\alpha-1,\beta)\quad\text{or}\quad(\alpha,\beta-1) \\ \text{is in } R \text{ for } (\alpha,\beta)\in S.\end{cases}$$

Now assume $\#S=g(g+1)$, and let $G_3=\{\gamma_1<\gamma_2,\dots,<\gamma_g\}$. We will show that $\alpha_i+\beta(\alpha_i)$ $(i=1,\dots,g)$ is constant.

Claim

$$\alpha_i + \beta(\alpha_i) = \alpha(\beta_{g-i+1}) + \beta_{g-i+1}$$
$$= 2g - \gamma_1 + 1 \quad \text{for all } i.$$

PROOF OF CLAIM. By Lemma 3-1 ii) and $*_3 - 1$), we have

$$[(\alpha_j - 1, \beta(\alpha_j) - 1), (\alpha_j, \beta(\alpha_j) - 1)]$$
$$= [(\alpha_i - 1, \beta(\alpha_j) - 1), (\alpha_i, \beta(\alpha_j) - 1)] = 0. \qquad \cdots\cdots 3\text{-}6\text{-}3)$$

for $j \geq i$.

By (2) of this lemma, we have

$$\beta(\alpha_i) = \beta_{g+1-i} > \beta(\alpha_j) = \beta_{g+1-j} \quad \text{with } j > i.$$

Since $[(\alpha_i - 1, \beta(\alpha_i) - 1), (\alpha_i - 1, \beta(\alpha_i))] = 0$,

$$[(\alpha_i - 1, \beta(\alpha_j) - 1), (\alpha_i - 1, \beta(\alpha_j))] = 0 \quad \text{for } j \geq i. \qquad \cdots\cdots 3\text{-}6\text{-}4)$$

By 3-6-3) and 3-6-4) $(\alpha_i - 1, \beta(\alpha_j) - 1) \in T = R$, and $(\alpha_i, \beta(\alpha_j) - 1) \in S$ for all $j \geq i$. Since $2g - \alpha - \beta = \delta^{\alpha\beta} \in G_3$ for $(\alpha, \beta) \in S$ by Lemma 3-4(2), we have

$$2g - \alpha_i - \beta(\alpha_j) + 1 \in G_3 \quad \text{with } j \geq i.$$

As $\alpha_i < \alpha_j$ and $\beta(\alpha_i) > \beta(\alpha_j)$ $(j > i)$, we have

$$\gamma_k = 2g - \alpha_{g-i+1} - \beta(\alpha_{g-i+k}) + 1 \quad \text{with } k = 1, \ldots, i.$$

In particular

$$\gamma_1 = 2g - \alpha_{g-i+1} - \beta(\alpha_{g-i+1}) + 1.$$

Then Claim has been proved.

Assume $\alpha_{i+1} = \alpha_i + 1$, for some i. By Claim, $\beta(\alpha_i) = \beta(\alpha_{i+1}) + 1$. Then

$$(\alpha_i, \beta(\alpha_{i+1}) - 1) = (\alpha_{i+1} - 1, \beta(\alpha_{i+1}) - 1) \in T = R$$

and

$$(\alpha_i - 1, \beta(\alpha_{i+1})) = (\alpha_i - 1, \beta(\alpha_i) - 1) \in T = R.$$

But the condition b) of $\# S = g(g+1)$ means that $(a+1, b-1)$ is not in R if (a, b) is in R. Then

$$\alpha_{i+1} \neq \alpha_i + 1 \quad \text{and} \quad \beta_{i+1} \neq \beta_i + 1 \quad \text{for all } i.$$

Since $\beta(\alpha_i) = \beta_{g-i+1}$, we also have

$$G_1 = \{\alpha_k = 2k - 1 \mid 1 \le k \le g - 1\}, \quad G_2 = \{\beta_k = 2k - 1 \mid 1 \le k \le g - 1\}$$

and $\beta(\alpha) = 2g - \alpha$ for $\alpha \in G_1$.
Using Lemma 3-1 ii), iii) and $*_3 - 1$), we get the graph $(D^{(2)}, *_2)$ mentioned at the end of (3). □

Proposition 3-7. *Assume* $\# S = g(g+1)$.
Then $(D^{(3)}, *_3)$ *is defined by*

α)

$$[(\alpha - 1, \beta, \gamma), (\alpha, \beta, \gamma)] = 0 \quad \textit{if and only if} \begin{cases} \text{“}\alpha \textit{ is odd and } \alpha + \beta + \gamma \neq 2g - 1\text{”} \\ \textit{or} \\ \text{“}\alpha+\beta+\gamma=2g-1 \textit{ and } \beta, \gamma \textit{ are even”}, \end{cases}$$

β)

$$[(\alpha, \beta - 1, \gamma), (\alpha, \beta, \gamma)] = 0 \quad \textit{if and only if} \begin{cases} \text{“}\beta \textit{ is odd and } \alpha + \beta + \gamma \neq 2g - 1\text{”} \\ \textit{or} \\ \text{“}\alpha+\beta+\gamma=2g - 1 \textit{ and } \alpha, \gamma \textit{ are even”} \end{cases}$$

and

γ)

$$[(\alpha, \beta, \gamma - 1), (\alpha, \beta, \gamma)] = 0 \quad \textit{if and only if} \begin{cases} \text{“}\gamma \textit{ is odd and } \alpha + \beta + \gamma \neq 2g - 1\text{”} \\ \textit{or} \\ \text{“}\alpha+\beta+\gamma=2g-1 \textit{ and } \alpha, \beta \textit{ are even”}. \end{cases}$$

In this case,

$$S = \{(\alpha, \beta) \mid 1 \le \alpha + \beta \le 2g - 1 \textit{ and } \alpha + \beta \textit{ is odd}\}$$

and

$$G^{(2)} \backslash S = \{(\alpha, \beta) \mid 2 \le \alpha + \beta \le 2g - 2, \alpha \textit{ and } \beta \textit{ are odd}\}.$$

Moreover, $\delta^{(\alpha\beta)} = 2g - 1 - \alpha - \beta$ *and* $l(\alpha, \beta, \delta^{(\alpha\beta)}) = g$ *for* $(\alpha, \beta) \in G^{(2)} \backslash S$.

Proof. By Lemma 3-6(3) and the proof of it, we can see that

$$R = T = \{(\alpha, \beta) \in V^{(2)} \mid \alpha \text{ and } \beta \text{ are even}, 0 \le \alpha + \beta \le 2g - 2\},$$

$$S = \{(\alpha, \beta) \mid 1 \le \alpha + \beta \le 2g - 2 \text{ and } \alpha + \beta \text{ odd}\}$$

and

$$G^{(2)} \backslash S = \{(\alpha, \beta) \mid 2 \le \alpha + \beta \le 2g - 2, \alpha \text{ and } \beta \text{ are odd}\}.$$

Then, by Lemma 3-4(1),

$$(\alpha-1,\beta+1)\in R \quad\text{and}\quad [(\alpha-1,\beta+1,2g-2-\alpha-\beta),(\alpha,\beta+1,2g-2-\alpha-\beta)]=0$$

for $(\alpha,\beta)\in G^{(2)}\backslash S$.
By $*_3-1)$,

$$[(\alpha-1,\beta,\gamma),(\alpha,\beta,\gamma)]=0 \text{ (i.e., } (\alpha,\beta,\gamma)\in G^{(3)}) \qquad \cdots\cdots 3\text{-}7\text{-}1)$$

for every γ with $0\le\gamma\le 2g-\alpha-\beta-2$ and $(\alpha,\beta)\in G^{(2)}\backslash S$. Therefore we get $\delta^{\alpha\beta}\ge 2g-\alpha-\beta-1$. Since $(\alpha,\beta)\in G^{(2)}\backslash S$ and $\delta^{\alpha\beta}\le 2g-\alpha-\beta-1$, we have

$$\delta^{\alpha\beta}=2g-\alpha-\beta-1 \quad\text{and}\quad l(\alpha,\beta,\delta^{\alpha\beta})=g.$$

Then we get the latter half of this lemma.

Let α and β be odd and even respectively. If $\tilde{\gamma}=2g-1-\alpha-\beta\ge 0$, then $(\alpha,\beta)\in S$ and $(\alpha,\beta,\tilde{\gamma})\in G^{(3)}$. But $[(\alpha,\beta-1,\tilde{\gamma}),(\alpha,\beta,\tilde{\gamma})]=[(\alpha,\beta,\tilde{\gamma}-1),(\alpha,\beta,\tilde{\gamma})]=1$ because β and $\tilde{\gamma}$ are even. Then

$$[(\alpha-1,\beta,\gamma),(\alpha,\beta,\gamma)]=0 \qquad \cdots\cdots 3\text{-}7\text{-}2)$$

for $0\le\gamma\le 2g-1-\alpha-\beta$.

Let both α and β be odd. If $\tilde{\gamma}=2g-1-\alpha-\beta\ge 0$, then $(\alpha,\beta)\in G^{(2)}\backslash S$ and $\delta^{\alpha\beta}=\tilde{\gamma}$. Hence $(\alpha,\beta,\tilde{\gamma})\in H^{(3)}$ and

$$[(\alpha-1,\beta,\tilde{\gamma}),(\alpha,\beta,\tilde{\gamma})]=1. \qquad \cdots\cdots 3\text{-}7\text{-}3)$$

By 3-7-1), 3-7-2), 3-7-3) and $*_3-1)$, we get the statement α). β) can be proved by the same way as in case α). The statement γ) follows from α), $*_3-2)$ and $*_3-1)$. □

Lemma 3-8. (1) *The first term* $\sum_{(\alpha\beta)\in H^{(2)}} l(\alpha,\beta)$ *of the equation of Lemma* 3-2(1) *satisfies*

$$\sum_{(\alpha\beta)\in H^{(2)}} l(\alpha,\beta)=\frac{g(g+1)(5g+1)}{6}+\frac{\sum_{\alpha\in G_1}\{-l(\alpha,\beta(\alpha))^2+l(\alpha,\beta(\alpha))\}}{2}.$$

(2) *The second term* $\sum_{(\alpha\beta)\in G^{(2)}} l(\alpha,\beta,\delta^{\alpha\beta})$ *of* 3-2(1) *satisfies*

$$\sum_{(\alpha\beta)\in G^{(2)}} l(\alpha,\beta,\delta^{\alpha\beta})\le g(g+1)+g\times \#G^{(2)},$$

and the equality holds if and only if $\#S=g(g+1)$.

(3)

$$\# G^{(3)} \leq \frac{g(g+1)(g+5)}{6} + g \times \# G^{(2)} + \frac{\sum_{\alpha \in G_1} \{-l(\alpha, \beta(\alpha))^2 + l(\alpha, \beta(\alpha))\}}{2},$$

and the equality holds if and only if $\# S = g(g+1)$.

Proof. (1) Let

$$A = \sum_{\alpha \in H_1} \left(\sum_{\beta \text{ s.t. } (\alpha\beta) \in H^{(2)}} l(\alpha, \beta) \right) \quad \text{and} \quad B = \sum_{\alpha \in G_1} \left(\sum_{\beta \text{ s.t. } (\alpha\beta) \in H^{(2)}} l(\alpha, \beta) \right).$$

Then

$$\sum_{(\alpha\beta) \in H^{(2)}} l(\alpha, \beta) = A + B.$$

We can calculate A and B as follows.

$$\begin{aligned} A &= \sum_{\alpha \in H^{(1)}} \{l(\alpha, 0) + (l(\alpha, 0) + 1) + \cdots + g\} \\ &= \sum_{\alpha \in H^{(1)}} \frac{(g - l(\alpha) + 1)(g + l(\alpha))}{2} \\ &= \frac{\sum_{k=1}^{g} \{(g - k + 1)(g + k)\}}{2} = \frac{g(g+1)(2g+1)}{6}. \end{aligned}$$

$$\begin{aligned} B &= \sum_{\alpha \in G^{(1)}} \left(\sum_{\beta \text{ s.t. } (\alpha, \beta) \in H^{(2)}} l(\alpha, \beta) \right) \\ &= \sum_{\alpha \in G^{(1)}} \{l(\alpha, \beta(\alpha)) + (l(\alpha, \beta(\alpha) + 1) + \cdots + g\} \\ &= \frac{\sum_{\alpha \in G^{(1)}} \{-l(\alpha, \beta(\alpha))^2 + l(\alpha, \beta(\alpha))\}}{2} + \frac{g^2(g+1)}{2}. \end{aligned}$$

Adding A and B, we get the equation in (1).

(2) Splitting $G^{(3)}$ into two subsets S and $G^{(3)} \backslash S$, we have

$$\begin{aligned} \sum_{(\alpha\beta) \in G^{(2)}} l(\alpha, \beta, \delta^{\alpha\beta}) &= \sum_{(\alpha\beta) \in S} l(\alpha, \beta, \delta^{\alpha\beta}) + \sum_{(\alpha\beta) \in G^{(2)} \backslash S} l(\alpha, \beta, \delta^{\alpha\beta}) \\ &\leq \# S \times (g+1) + (\# G^{(2)} - \# S) \times g \\ &\leq g(g+1) + g \times \# G^{(2)} \quad \text{(by Lemma 3-6 (3)).} \end{aligned}$$

□

THEOREM 3-9. *Let* $(D^{(3)}, *_3)$ *be a Riemann-Roch graph, and let* $G^{(3)}$ *be its gap set.*
Then

$$\# G^{(3)} \leq \frac{g(7g^2 + 6g + 5)}{6},$$

and the equality holds if and only if $(D^{(3)}, *_3)$ *is the graph defined as in Proposition 3-7.*

PROOF. Substituting (2) of Lemma 3-2 for $\# G^{(2)}$ in the inequality of lemma 3-8 (3), we have

$$\# G^{(3)} \leq_{(1)} \frac{g(4g^2 + 3g + 5)}{6} + \sum_{\alpha \in G_1} \{-l(\alpha, \beta(\alpha))^2 + (2g+1) l(\alpha, \beta(\alpha))\}$$

$$\leq_{(2)} \frac{g(7g^2 + 6g + 5)}{6}.$$

As

$$-l(\alpha, \beta(\alpha))^2 + (2g+1) l(\alpha, \beta(\alpha)) = -\left\{ l(\alpha, \beta(\alpha)) - \left(g + \frac{1}{2}\right) \right\}^2 + g^2 + g + \frac{1}{4},$$

the second equality (2) holds if and only if $l(\alpha, \beta(\alpha)) = g$ or $g+1$ for each $\alpha \in G_1$. If the first equality (1) holds, then $\# S = g(g+1)$ and $(D^{(2)}, *_2)$ is the graph defined in Lemma 3-6 (3). That is,

$$G_1 = G_2 = \{1, 3, 5, \ldots, 2g-1\},$$

$$G^{(2)} = \{(\alpha, \beta) \mid 1 \leq \alpha + \beta \leq 2g - 1, \alpha \text{ or } \beta \text{ is odd}\},$$

$$\beta(\alpha) = 2g - \alpha \quad \text{and} \quad l(\alpha, \beta(\alpha)) = g + 1 \quad \text{for } \alpha \in G_1.$$

Thus the equality (1) implies the equality (2), and then $\# G^{(3)} = g(7g^2 + 6g + 5)/6$ holds if and only if the equality (1) holds. So we have the graph discribed in Proposition 3-8. □

EXAMPLE 3-10. The graph in Theorem 3-9 is exactly the graph $G_M(P_1, P_2, P_3)$ with hyperelliptic M and $|2P_1| = |2P_2| = |2P_3| = g_2^1$. This is also from the same calculation done by Kim in case $n = 2$.

REMARK 3-11. When $n = 2$, the graph which attains the maximal value of $\# G^{(2)}$ is not unique. For example, if

$$G_1 = \{\alpha_1, \dots, \alpha_g\} = \{1, 2, 3, \dots, g\},$$

$$G_2 = \{\beta_1, \dots, \beta_g\} = \{g, g+1, \dots, 2g-1\}\}$$

and $\beta(\alpha_i) = 2g - \alpha_i$, then this graph attains the maximal value by Lemma 3-2, and this graph does not come from any Riemann surfaces.

§. Appendix

Lemma 3-1 shows that a map $\beta(*) : V_1 \to V_2$ with some conditions completely determine a Riemann-Roch graph in case $n = 2$. In this section we study the structure of $(D^{(n)}, *_n)$ in detail when $n \geq 3$, and try to find some means, similar to $\beta(*)$, of construction of $(D^{(n)}, *_n)$.

A-I

First we survey a given $(D^{(n)}, *_n)$.

Definition A-1. Fix a Riemann-Roch graph $(D^{(n)}, *_n)$. Assume $n \geq 3$. Let i and j $(1 \leq i, j \leq n, i \neq j)$ be fixed. Take an $(n-2)$-tuple

$$\Gamma_{ij} = (\gamma_1, \dots, \gamma_{i-1}, \gamma_{i+1}, \dots, \gamma_{j-1}, \gamma_{j+1}, \dots, \gamma_n) \in \mathbf{N}_0^{n-2},$$

and we identify Γ_{ij} with the n-tuple

$$\sum_{k \neq i,j} \gamma_k \mathbf{e}_k = (\gamma_1, \dots, \gamma_{i-1}, 0, \gamma_{i+1}, \dots, \gamma_{j-1}, 0, \gamma_{j+1}, \dots, \gamma_n) \in \mathbf{N}_0^n.$$

We also write Γ_{ij} for this vertex.

For fixed Γ_{ij}, define a subset $G_i^{\Gamma_{ij}}$ of $\mathbf{N}_0$ by

$$G_i^{\Gamma_{ij}} := \{\gamma \mid \gamma > 0, \Gamma = \Gamma_{ij} + \gamma \mathbf{e}_i \in V^{(n)} \quad \text{and} \quad [\Gamma - \mathbf{e}_i, \Gamma] = 0\}.$$

For $\gamma \in \mathbf{N}_0$ with $0 \leq \gamma \leq 2g - deg\,\Gamma_{ij} - 1$, define a non-negative integer $\gamma_j^{\Gamma_{ij}}(\gamma)$ by:

i) for $\gamma \notin G_i^{\Gamma_{ij}}$,
$\gamma_j^{\Gamma_{ij}}(\gamma) := 0$;

\# ii) for $\gamma \in G_i^{\Gamma_{ij}}$,

a) $\gamma_j^{\Gamma_{ij}}(\gamma) := 2g - deg\,\Gamma_{ij} - \gamma (> 0)$ if $\Delta_j(\Gamma_{ij}, \gamma) = \emptyset$

b) $\gamma_j^{\Gamma_{ij}}(\gamma) := \min\{\alpha \mid \alpha \in \Delta_j(\Gamma_{ij}, \gamma)\} (> 0)$ if $\Delta_j(\Gamma_{ij}, \gamma) \neq \emptyset$,

where

$$\Delta_j(\Gamma_{ij}, \gamma) := \{\alpha \mid [(\Gamma - \mathbf{e}_i, \Gamma] = 1 \quad \text{with} \quad \Gamma = \Gamma_{ij} + \gamma \mathbf{e}_i + \alpha \mathbf{e}_j \in V^{(n)} \quad \text{and} \quad \gamma > 0\}.$$

REMARK. i) For $\gamma \in G_i^{\Gamma_{ij}}$, $1 \le \gamma_j^{\Gamma_{ij}}(\gamma) \le 2g - deg\,\Gamma_{ij} - 1$. (see the proof of Lemma 3-1).

ii) If $\Gamma_{ij} = (0, \dots, 0)$ (write 0_{ij}), then $G_i^{0_{ij}} = \{\gamma \mid \gamma \mathbf{e}_i \in G^{(n)}\}$. We wrote G_i for $G_i^{0_{ij}}$ in §.3.

LEMMA A-2. *Fix Γ_{ij}. For γ with $0 \le \gamma \le 2g - deg\,\Gamma_{ij} - 1$, put $\tilde{\gamma} = \gamma_j^{\Gamma_{ij}}(\gamma)$ and $\Gamma = \Gamma_{ij} + \gamma \mathbf{e}_i + \tilde{\gamma}\mathbf{e}_j$.*
If $0 < \tilde{\gamma} < 2g - deg\,\Gamma_{ij} - \gamma$, then

$$\gamma > 0, \quad [\Gamma - \mathbf{e}_j, \Gamma] = 1 \quad \textit{and} \quad [\{\Gamma - \mathbf{e}_i\} - \mathbf{e}_j, \{\Gamma - \mathbf{e}_i\}] = 0.$$

PROOF. As $\tilde{\gamma} > 0, \gamma$ must be positive. By the definition of $\tilde{\gamma} = \gamma_j^{\Gamma_{ij}}(\gamma)$,

$$[\Gamma - \mathbf{e}_i, \Gamma] = 1 \quad \text{and} \quad [\{\Gamma - \mathbf{e}_j\} - \mathbf{e}_i, \{\Gamma - \mathbf{e}_j\}] = 0.$$

By Lemma 1-1 A), we get this lemma. □

The system of maps

$$\Big\{\tilde{\gamma}_j^{\Gamma_{ij}} : \{\gamma \mid 0 \le \gamma \le 2g - 1 - deg\,\Gamma_{ij}\} \\ \rightarrow \{\gamma \mid 0 \le \gamma \le 2g - 1 - deg\,\Gamma_{ij}\} \mid \Gamma_{ij} \in V^{(n)}, 1 \le i, j \le n\Big\}$$

have the following properties.

LEMMA A-3. *Fix a Riemann-Roch graph $(D^{(n)}, *_n)$. Let Γ_{ij} be as in Definition A-1. Then*

i)

$$\# G_i^{\Gamma_{ij}} = \# G_j^{\Gamma_{ij}} = i(\Gamma_{ij}).$$

ii) *$\gamma_j^{\Gamma_{ij}}$ induces a bijection from $G_i^{\Gamma_{ij}}$ to $G_j^{\Gamma_{ij}}$, and its inverse map is*

$$(\gamma_j^{\Gamma_{ij}})^{-1} = \gamma_i^{\Gamma_{ij}}.$$

iii) *Let $\Gamma'_{ij} = \sum_{k \ne i,j} \gamma'_k \mathbf{e}_k$ be another $(n-2)$-tuple with $\Gamma_{ij} \le \Gamma'_{ij}$, then*

$$G_i^{\Gamma_{ij}} \supset G_i^{\Gamma'_{ij}}$$

and

$$\gamma_j^{\Gamma_{ij}}(\gamma) \ge \gamma_j^{\Gamma'_{ij}}(\gamma)$$

for γ with $0 \le \gamma \le 2g - 1 - deg\,\Gamma'_{ij}$.

Moreover if $G_i^{\Gamma_{ij}} = G_i^{\Gamma'_{ij}}$, *then*

$$\gamma_j^{\Gamma_{ij}} = \gamma_j^{\Gamma'_{ij}}.$$

Proof. i) This can be easily proved by Lemma 1-3.

ii) Put $\tilde{\gamma} = \gamma_j^{\Gamma_{ij}}(\gamma)$ and $\Gamma = \Gamma_{ij} + \gamma\mathbf{e}_i + \tilde{\gamma}\mathbf{e}_j$ for $\gamma \in G_i^{\Gamma_{ij}}$. Then $\gamma > 0$ and $\tilde{\gamma} > 0$.

First we will show $\tilde{\gamma} \in G_j^{\Gamma_{ij}}$.

Assume $\tilde{\gamma} \notin G_j^{\Gamma_{ij}}$.

<u>case</u> $\Delta_j(\Gamma_{ij}, \gamma) \neq \varnothing$ $(i.e., \Gamma \in V^{(n)})$

By $\tilde{\gamma} \notin G_j^{\Gamma_{ij}}$, we have $[\{\Gamma_{ij} + \tilde{\gamma}\mathbf{e}_j\} - \mathbf{e}_j, \{\Gamma_{ij} + \tilde{\gamma}\mathbf{e}_j\}] = 1$. Then, by $*_n - 1)$,

$$[\{\Gamma - \mathbf{e}_i\} - \mathbf{e}_j, \{\Gamma - \mathbf{e}_i\}] = [\Gamma - \mathbf{e}_j, \Gamma] = 1.$$

On the other hand $[\Gamma - \mathbf{e}_i, \Gamma] = 1$ by # ii-b).

Thus, by Lemma 1-1 A), we have

$$[\{\Gamma - \mathbf{e}_j\} - \mathbf{e}_i, \{\Gamma - \mathbf{e}_j\}] = 1.$$

But this contradicts to the definition # ii-b).

<u>case</u> $\Delta_j(\Gamma_{ij}, \gamma) = \varnothing$ $(i.e., \Gamma \notin V^{(n)})$

$deg(\Gamma - \mathbf{e}_j) = 2g - 1$ and then $\Gamma - \mathbf{e}_j \in V^{(n)}$. We have

$$[\{\Gamma - \mathbf{e}_j\} - \mathbf{e}_i, \{\Gamma - \mathbf{e}_j\}] = 0.$$

On the other hand, by $\tilde{\gamma} \notin G_j^{\Gamma_{ij}}$ and $*_n - 1)$,

$$[\{\Gamma_{ij} + \tilde{\gamma}\mathbf{e}_j\} - \mathbf{e}_j, \{\Gamma_{ij} + \tilde{\gamma}\mathbf{e}_j\}] = [\{\Gamma - \mathbf{e}_i\} - \mathbf{e}_j, \{\Gamma - \mathbf{e}_i\}] = 1.$$

Then, by Lemma 1-1 B),

$$[\{\Gamma - \mathbf{e}_j\} - \mathbf{e}_i, \{\Gamma - \mathbf{e}_j\}] = 1.$$

This is also a contradiction. Thus $\tilde{\gamma} \in G_j^{\Gamma_{ij}}$ in any case.

Next we will show $(\gamma_i^{\Gamma_{ij}})^{-1} = \gamma_j^{\Gamma_{ij}}$.

<u>case</u> $\Delta_j(\Gamma_{ij}, \gamma) \neq \varnothing$

By Lemma A-2 and $*_n - 1)$, we have

$$[\{\Gamma_{ij} + \delta\mathbf{e}_i + \tilde{\gamma}\mathbf{e}_j\} - \mathbf{e}_j, \{\Gamma_{ij} + \delta\mathbf{e}_i + \tilde{\gamma}\mathbf{e}_j\}] = 0$$

for any δ with $0 \leq \delta \leq \gamma - 1$, and $\Delta_i(\Gamma_{ij}, \tilde{\gamma}) \ni \gamma$. Thus we have

$$\gamma_i^{\Gamma_{ij}}(\tilde{\gamma}) = \gamma = (\gamma_j^{\Gamma_{ij}})^{-1}(\tilde{\gamma})$$

by the definiton of $\gamma_i^{\Gamma_{ij}}(\tilde{\gamma})$.

<u>case</u> $\Delta_j(\Gamma_{ij}, \gamma) = \emptyset$

Using Lemma 1-1 B) and # ii-a), we also have $\Delta_i(\Gamma_{ij}, \tilde{\gamma}) = \emptyset$ and

$$\gamma_i^{\Gamma_{ij}}(\tilde{\gamma}) = 2g - deg\,\Gamma_{ij} - \tilde{\gamma} = \gamma = (\gamma_j^{\Gamma_{ij}})^{-1}(\tilde{\gamma}).$$

iii) $G_i^{\Gamma_{ij}} \supset G_i^{\Gamma'_{ij}}$ and $\gamma_j^{\Gamma_{ij}}(\gamma) \geq \gamma_j^{\Gamma'_{ij}}(\gamma)$ follow from $*_n - 1$).

Next assume $G_i^{\Gamma_{ij}} = G_i^{\Gamma'_{ij}}$. Then $i(\Gamma_{ij}) = i(\Gamma'_{ij})$. By Lemma 1-3 and $*_n - 1$), we have

$$[\Gamma_{ij} + \alpha\mathbf{e}_i + \beta\mathbf{e}_j, \Gamma'_{ij} + \alpha\mathbf{e}_i + \beta\mathbf{e}_j] = deg\,\Gamma'_{ij} - deg\,\Gamma_{ij} \cdots\cdots \natural$$

for $\alpha \geq 0$ and $\beta \geq 0$.

Fix γ with $1 \leq \gamma \leq 2g - 1 - deg\,\Gamma'_{ij}$.

Put $\tilde{\gamma}' = \gamma_j^{\Gamma'_{ij}}(\gamma), \tilde{\Gamma} = \Gamma_{ij} + \gamma\mathbf{e}_i + \tilde{\gamma}'\mathbf{e}_j$ and $\tilde{\Gamma}' = \Gamma'_{ij} + \gamma\mathbf{e}_i + \tilde{\gamma}'\mathbf{e}_j$. Then $\tilde{\Gamma} \leq \tilde{\Gamma}'$ and $[\tilde{\Gamma}' - \mathbf{e}_i, \tilde{\Gamma}'] = 1$.

Therefore, by $\natural$,

$$[\tilde{\Gamma} - \mathbf{e}_i, \tilde{\Gamma}'] = [\tilde{\Gamma} - \mathbf{e}_i, \tilde{\Gamma}' - \mathbf{e}_i] + [\tilde{\Gamma}' - \mathbf{e}_i, \tilde{\Gamma}'] = (deg\,\Gamma'_{ij} - deg\,\Gamma_{ij}) + 1 = [\tilde{\Gamma}, \tilde{\Gamma}'] + 1.$$

On the other hand, since

$$[\tilde{\Gamma} - \mathbf{e}_i, \tilde{\Gamma}'] = [\tilde{\Gamma} - \mathbf{e}_i, \tilde{\Gamma}] + [\tilde{\Gamma}, \tilde{\Gamma}'],$$

we have $[\tilde{\Gamma} - \mathbf{e}_i, \tilde{\Gamma}] = 1$ and $\gamma_j^{\Gamma_{ij}}(\gamma) \leq \tilde{\gamma}' = \gamma_j^{\Gamma'_{ij}}(\gamma)$. □

Also we can have the following proposition from $*_n - 1$).

Proposition A-4. *Let* $\Gamma = \sum_{i=1}^n \gamma_i\mathbf{e}_i$ *be in* $V^{(n)}$. *Let* Γ_{kn} $(k \neq n)$ *be the* $(n-2)$*-tuple that satisfies* $\Gamma = \Gamma_{kn} + \gamma_k\mathbf{e}_k + \gamma_n\mathbf{e}_n$.

i) *Assume* $\gamma_i > 0$ *for some* i $(\neq n)$. *Then*

$$[\Gamma - \mathbf{e}_i, \Gamma] = 1 \quad \textit{if and only if} \quad \gamma_n^{\Gamma_{in}}(\gamma_i) \leq \gamma_n.$$

ii) *Assume* $\gamma_n > 0$. *Then, for any* k $(\neq n)$,

$$[\Gamma - \mathbf{e}_n, \Gamma] = 1 \quad \textit{if and only if} \quad \gamma_k^{\Gamma_{kn}}(\gamma_n) \leq \gamma_k.$$

This proposition and Proposition A-3 ii) imply that $D^{(n)}$ with $*_n$ is exactly decided by the system

$$\Big\{\{\gamma_n^{\Gamma_{in}} \mid \{\gamma \mid 0 \leq \gamma \leq 2g - 1 - deg\,\Gamma_{in}\} \\ \rightarrow \{\gamma_n \mid 0 \leq \gamma \leq 2g - 1 - deg\,\Gamma_{in}\}\} \mid \Gamma_{in} \in V^{(n)}, 1 \leq i \leq n-1\Big\}.$$

A-II

Let $D^{(n)}$ be as before, but we do not assume the condition $*_n$ on it. Regarding $D^{(n-1)}$ as the subgraph of $D^{(n)}$ by the natural way (i.e., $(\gamma_1, \dots, \gamma_{n-1}) \leftrightarrow (\gamma_1, \dots, \gamma_{n-1}, 0)$), and assume that $D^{(n-1)}$ is equipped with the condition $*_{n-1}$. We will investigate how we can build up $*_n$, which induces the given $*_{n-1}$.

Definition A-5. i) Let Γ_{in} and Γ'_{in} be as in Definition A-1. We define a subset $\tilde{G}_i^{\Gamma_{in}}$ of $\{\gamma \mid 1 \le \gamma \le 2g - 1 - deg\,\Gamma_{in}\}$ by

$$\tilde{G}_i^{\Gamma_{in}} := \{\gamma \mid [\{\Gamma_{in} + \gamma \mathbf{e}_i\} - \mathbf{e}_i, \{\Gamma_{in} + \gamma \mathbf{e}_i\}] = 0 \text{ by } *_{n-1}\}.$$

If $\Gamma_{in} \le \Gamma'_{in}$, then we can see from $*_{n-1} - 1)$ that

C-0) $$\tilde{G}_i^{\Gamma_{in}} \supseteq \tilde{G}_i^{\Gamma'_{in}}.$$

ii) Assume that there is a system of maps

$$\Big\{\tilde{\gamma}_n^{\Gamma_{in}} : \{\gamma \mid 0 \le \gamma \le 2g - 1 - deg\,\Gamma_{in}\} \to \{\gamma \mid 0 \le \gamma \le 2g - 1 - deg\,\Gamma_{in}\} \mid \Gamma_{in} \in V^{(n)}, 1 \le i \le n - 1\Big\}.$$

satisfying

α) $\tilde{\gamma}_n^{\Gamma_{in}}(\gamma) = 0$ if $\gamma \notin \tilde{G}_i^{\Gamma_{in}}$.

β) $\tilde{\gamma}_n^{\Gamma_{in}}$ is an injective map from $\tilde{G}_i^{\Gamma_{in}}$ into $\{\gamma \mid 1 \le \gamma \le 2g - 1 - deg\,\Gamma_{in}\}$.

Define a map $\tilde{\gamma}_i^{\Gamma_{in}}$ on $\{\gamma \mid 0 \le \gamma \le 2g - 1 - deg\,\Gamma_{in}\}$ by

γ)
$$\tilde{\gamma}_i^{\Gamma_{in}}(\gamma) = (\tilde{\gamma}_n^{\Gamma_{in}})^{-1}(\gamma) \quad \text{for} \quad \gamma \in \tilde{\gamma}_n^{\Gamma_{in}}(\tilde{G}_i^{\Gamma_{in}})$$
$$\tilde{\gamma}_i^{\Gamma_{in}}(\gamma) = 0 \qquad \text{for} \quad \gamma \notin \tilde{\gamma}_n^{\Gamma_{in}}(\tilde{G}_i^{\Gamma_{in}}).$$

Moreover they assume to be satisfied the following conditions (C-1), C-2), C-3)).

C-1) If $\Gamma_{in} \le \Gamma'_{in}$, then

$$\tilde{\gamma}_n^{\Gamma_{in}}(\gamma) \ge \tilde{\gamma}_n^{\Gamma'_{in}}(\gamma) \quad \text{on} \quad \{\gamma \mid 0 \le \gamma \le 2g - 1 - deg\,\Gamma'_{in}\}.$$

(N.B. C-1) is equivalent to $\tilde{\gamma}_n^{\Gamma_{in}}(\gamma) \ge \tilde{\gamma}_n^{\Gamma'_{in}}(\gamma)$ on $\tilde{G}_i^{\Gamma'_{in}}$ by C-0), α) and β).)

C-2) $\tilde{\gamma}_i^{\Gamma_{in}}(\gamma) \ge \tilde{\gamma}_i^{\Gamma'_{in}}(\gamma)$ on $\{\gamma \mid 0 \le \gamma \le 2g - 1 - deg\,\Gamma'_{in}\}$.

(N.B. C-2) is equivalent to

$$\begin{cases} \tilde{\gamma}_n^{\Gamma_{in}}(\tilde{G}_i^{\Gamma_{in}}) \supset \tilde{\gamma}_n^{\Gamma'_{in}}(\tilde{G}_i^{\Gamma'_{in}}) \\ \text{and} \\ \tilde{\gamma}_i^{\Gamma_{in}}(\gamma) \geq \tilde{\gamma}_i^{\Gamma'_{in}}(\gamma) \quad \text{on} \quad \tilde{\gamma}_n^{\Gamma'_{in}}(\tilde{G}_i^{\Gamma'_{in}}). \end{cases}$$

In fact, if C-2) holds and there exists $\gamma \in \tilde{G}_i^{\Gamma'_{in}}$ satisfying $\tilde{\gamma}_n^{\Gamma'_{in}}(\gamma) \notin \tilde{\gamma}_n^{\Gamma_{in}}(\tilde{G}_i^{\Gamma_{in}})$, then $\tilde{\gamma}_i^{\Gamma'_{in}}\tilde{\gamma}_n^{\Gamma'_{in}}(\gamma) \leq \tilde{\gamma}_i^{\Gamma_{in}}\tilde{\gamma}_n^{\Gamma'_{in}}(\gamma) = 0$ by γ). Hence $\gamma \leq 0$. This is a contradiction.)

C-3) For $\Gamma = \sum_{i=1}^{n} \gamma_i \mathbf{e}_i \in V^{(n)}$ and $1 \leq k, l \leq n-1, \Gamma_{kn}$ and Γ_{ln} are as in Proposition A-4. Then

$$\gamma_k < \tilde{\gamma}_k^{\Gamma_{kn}}(\gamma_n) \quad \text{if and only if} \quad \gamma_l < \tilde{\gamma}_l^{\Gamma_{ln}}(\gamma_n).$$

Now we put the weight 0 or 1 on each edge in $E^{(n)}$ according to the following set R of rules $R1), \ldots, Rn)$.

$R-i)(i = 1, \ldots, n-1)$ Let $\Gamma = (\gamma_1, \ldots, \gamma_n) \in V^{(n)}$ with $\gamma_i > 0$.

$$[\Gamma - \mathbf{e}_i, \Gamma] = 1 \quad \Leftrightarrow \quad \tilde{\gamma}_n^{\Gamma_{in}}(\gamma_i) \leq \gamma_n.$$

$R)$

$R-n)$ Let $\Gamma = (\gamma_1, \ldots, \gamma_n) \in V^{(n)}$ with $\gamma_n > 0$.

$$[\Gamma - \mathbf{e}_n, \Gamma] = 1 \quad \Leftrightarrow \quad \tilde{\gamma}_k^{\Gamma_{kn}}(\gamma_n) \leq \gamma_k \quad \text{for some } k \neq n.$$

$$(\Leftrightarrow \quad \tilde{\gamma}_k^{\Gamma_{kn}}(\gamma_n) \leq \gamma_k \quad \text{for all } k \neq n \text{ by C-3)}).$$

Because of C-3), the weight of each edge is well defined by $R-i)$.

Definition A-6. $(D^{(n)}, R)$ denotes the graph such that each edge has weight 0 or 1 according to R), and define $G_i^{\Gamma_{ik}}$ and $\gamma_k^{\Gamma_{ik}}(*)$ by the same way as in Definition A-1.

(i.e., Let $\Gamma = (\gamma_1, \ldots, \gamma_n) \in V^{(n)}$ and put $\Gamma = \Gamma_{ij} + \gamma_i \mathbf{e}_i + \gamma_j \mathbf{e}_j$ for fixed i and j $(1 \leq i, j \leq n, i \neq j)$, then

$G_i^{\Gamma_{ij}} := \{\gamma \mid 0 < \gamma \leq 2g - deg\,\Gamma_{ij} - 1 \text{ and } [\{\Gamma_{ij} + \gamma \mathbf{e}_i\} - \mathbf{e}_i, \{\Gamma_{ij} + \gamma \mathbf{e}_i\}] = 0 \text{ by } R\}$.

For $0 \leq \gamma \leq 2g - deg\,\Gamma_{ij} - 1$, we define a non-negative integer $\gamma_j^{\Gamma_{ij}}(\gamma)$ by

i) For $\gamma \notin G_i^{\Gamma_{ij}}$, $\gamma_j^{\Gamma_{ij}}(\gamma) = 0$.

$\#'$ ii) For $\gamma \in G_i^{\Gamma_{ij}}$,

a) $\gamma_j^{\Gamma_{ij}}(\gamma) := 2g - deg\,\Gamma_{ij} - \gamma \ (\geq 1)$ if $\Delta_j(\Gamma_{ij}, \gamma) = \emptyset$

b) $\gamma_j^{\Gamma_{ij}}(\gamma) := \min\{\alpha \mid \alpha \in \Delta_j(\Gamma_{ij}, \gamma)\}$ if $\Delta_j(\Gamma_{ij}, \gamma) \neq \emptyset$, where

$$\Delta_j(\Gamma_{ij},\gamma)=\{\alpha\,|\,[\{\Gamma_{ij}+\gamma\mathbf{e}_i+\alpha\mathbf{e}_j\}-\mathbf{e}_i,\{\Gamma_{ij}+\gamma\mathbf{e}_i+\alpha\mathbf{e}_j\}]=1 \text{ by } R\}.\Big)$$

Lemma A-7. (1) *For* $1\le i\le n-1$, *we have*

$$\tilde{G}_i^{\Gamma_{in}}=G_i^{\Gamma_{in}},\quad \tilde{\gamma}_n^{\Gamma_{in}}(\tilde{G}_i^{\Gamma_{in}})=G_n^{\Gamma_{in}},\quad \tilde{\gamma}_n^{\Gamma_{in}}(*)=\gamma_n^{\Gamma_{in}}(*)\quad \textit{and}\quad \tilde{\gamma}_i^{\Gamma_{in}}(*)=\gamma_i^{\Gamma_{in}}(*).$$

(2) *Let* $1\le i,k\le n-1$ *and* $i\ne k$.
For $\Gamma=\Gamma_{ik}+\gamma_i\mathbf{e}_i+\gamma_k\mathbf{e}_k\in V^{(n)}$ *with* $\gamma_i>0$,

$$\gamma_k^{\Gamma_{ik}}(\gamma_i)\le\gamma_k\quad \textit{if and only if}\quad [\Gamma-\mathbf{e}_i,\Gamma]=1.$$

Proof. (1) By α) and β) in Definition A-5, $\gamma\in\tilde{G}_i^{\Gamma_{in}}$ is equivalent to $\tilde{\gamma}_n^{\Gamma_{in}}(\gamma)>0$. And, by $R-i$) $(i\ne n)$, $\tilde{\gamma}_n^{\Gamma_{in}}(\gamma)>0$ is equivalent to $\gamma\in G_i^{\Gamma_{in}}$. Thus $\tilde{G}_i^{\Gamma_{in}}=G_i^{\Gamma_{in}}(i\ne n)$. By $R-i$) $(i\ne n)$, we also have $\Delta_n(\Gamma_{in},\gamma)=\{\alpha\,|\,\tilde{\gamma}_n^{\Gamma_{in}}(\gamma)\le\alpha\}$. Then $\tilde{\gamma}_n^{\Gamma_{in}}(*)=\gamma_n^{\Gamma_{in}}(*)$ and $\tilde{\gamma}_n^{\Gamma_{in}}(*)=\gamma_i^{\Gamma_{in}}(*)$.

Next we will prove $\tilde{\gamma}_n(\tilde{G}_i^{\Gamma_{in}})=G_n^{\Gamma_{in}}$.
Take $\gamma\in\tilde{\gamma}_n^{\Gamma_{in}}(\tilde{G}_i^{\Gamma_{in}})=\gamma_n^{\Gamma_{in}}(G_i^{\Gamma_{in}})$. Then

$$\tilde{G}_i^{\Gamma_{in}}\ni(\tilde{\gamma}_n^{\Gamma_{in}})^{-1}(\gamma)=\tilde{\gamma}_i^{\Gamma_{in}}(\gamma)>0.$$

Thus, by $R-n$),

$$[\{\Gamma_{in}+\gamma\mathbf{e}_n\}-\mathbf{e}_n,\{\Gamma_{in}+\gamma\mathbf{e}_n\}]=0$$

and $\gamma\in G_n^{\Gamma_{in}}$.

Conversely, if $\gamma\in G_n^{\Gamma_{in}}$, then $\tilde{\gamma}_i^{\Gamma_{in}}(\gamma)>0$ by $R-n$). And we have $\gamma\in\tilde{\gamma}_n^{\Gamma_{in}}(\tilde{G}_i^{\Gamma_{in}})$ by γ) in Definition A-5.

(2) Assume $\gamma_k^{\Gamma_{ik}}(\gamma_i))\le\gamma_k$, and put $\Gamma'=\Gamma_{ik}+\gamma_i\mathbf{e}_i+\gamma_k^{\Gamma_{ik}}(\gamma_i)\mathbf{e}_k$. Then $[\Gamma'-\mathbf{e}_i,\Gamma']=1$. Let Γ'_{in} be the $(n-2)$-tuple satisfying $\Gamma'=\Gamma'_{in}+\gamma_i\mathbf{e}_i+\gamma_n\mathbf{e}_n$. Then, by $R-i$),

$$\gamma_n^{\Gamma'_{in}}(\gamma_i)\le\gamma_n.$$

Since $\Gamma'_{in}\le\Gamma_{in}$,

$$\gamma_n^{\Gamma_{in}}(\gamma_i)\le\gamma_n^{\Gamma'_{in}}(\gamma_i)\qquad\text{by C-2)}.$$

Hence $\gamma_n^{\Gamma_{in}}(\gamma_i)\le\gamma_n$. We proved that $[\Gamma-\mathbf{e}_i,\Gamma]=1$ if $\gamma_k^{\Gamma_{ik}}(\gamma_i))\le\gamma_k$.

Conversely if $[\Gamma-\mathbf{e}_i,\Gamma]=1$, then $\gamma_k\in\Delta_k(\Gamma_{ik},\gamma_i)\ne\varnothing$ and $\gamma_k^{\Gamma_{ik}}(\gamma_i)$ is equal to $\min\{\alpha\,|\,\alpha\in\Delta_k(\Gamma_{ik},\gamma_i)\}$. Thus $\gamma_k^{\Gamma_{ik}}(\gamma_i))\le\gamma_k$. □

By Lemma A-7(2) and R), we can see easily that the graph $(D^{(n)}, R)$ satisfies the condition $*_n - 1)$.

Now we add the following assumption so that the graph $(D^{(n)}, R)$ satisfies $*_n - 2)$.

C-4) Let $1 \le i, k \le n-1$, $\gamma_k^{\Gamma_{ik}}$ is a bijection from $G_i^{\Gamma_{ik}}$ to $G_k^{\Gamma_{ik}}$ so that

$$(\gamma_k^{\Gamma_{ik}})^{-1}(*) = \gamma_i^{\Gamma_{ik}}(*) \quad \text{on } G_k^{\Gamma_{ik}}.$$

THEOREM A-8. *Assume that*

$$(V^{(n-1)}, *_{n-1}) \quad and \quad \{\tilde{\gamma}_n^{\Gamma_{in}} \mid 1 \le i \le n-1, \Gamma_{in} \in V^{(n-2)}\}$$

satisfy the conditions C-1) ~ C-4). Then the graph $(D^{(n)}, R)$ is equipped with $*_n$ which induces the given $*_{n-1}$.

PROOF. We only have to show that $*_n - 2)$ is satisfied.

Let $\Gamma = \sum_{k=1}^{n} \gamma_k \mathbf{e}_k \in V^{(n)}$ satisfying $\gamma_i > 0$ and $\gamma_j > 0$ for some i and j $(1 \le i, j \le n, i \ne j)$. Let Γ_{ij} be the $(n-2)$-tuple satisfying $\Gamma = \Gamma_{ij} + \gamma_i \mathbf{e}_i + \gamma_j \mathbf{e}_j$. By $*_n - 1)$ and $(\gamma_j^{\Gamma_{ij}})^{-1}(*) = \gamma_i^{\Gamma_{ij}}(*)(C\text{-}4)$ and γ)), the following two cases can be happened.

$$\dagger)\begin{cases} \text{i)} \quad [\{\Gamma - \mathbf{e}_j\} - \mathbf{e}_i, \{\Gamma - \mathbf{e}_j\}] = [\Gamma - \mathbf{e}_i, \Gamma] \quad \text{and} \quad [\{\Gamma - \mathbf{e}_i\} - \mathbf{e}_j, \{\Gamma - \mathbf{e}_i\}] \\ \qquad = [\Gamma - \mathbf{e}_j, \Gamma]. \\ \text{ii)} \quad [\{\Gamma - \mathbf{e}_j\} - \mathbf{e}_i, \{\Gamma - \mathbf{e}_j\}] \\ \qquad = [\{\Gamma - \mathbf{e}_i\} - \mathbf{e}_j, \{\Gamma - \mathbf{e}_i\}] = 0 \quad \text{and} \quad [\Gamma - \mathbf{e}_i, \Gamma] = [\Gamma - \mathbf{e}_j, \Gamma] = 1. \end{cases}$$

(In fact, for example, if $[\Gamma - \mathbf{e}_i, \Gamma] = 1$ and $[\{\Gamma - \mathbf{e}_j\} - \mathbf{e}_i, \{\Gamma - \mathbf{e}_j\}] = 0$, then $\gamma_j^{\Gamma_{ij}}(\gamma_i) = \gamma_j$. As $\gamma_i = (\gamma_j^{\Gamma_{ij}})^{-1}(\gamma_j) = \gamma_i^{\Gamma_{ij}}(\gamma_j)$, $[\Gamma - \mathbf{e}_j, \Gamma] = 1$ and $[\{\Gamma - \mathbf{e}_i\} - \mathbf{e}_j, \{\Gamma - \mathbf{e}_i\}] = 0$. This is the case †) ii).)

†) implies the condition A) of Lemma 1-1.

Let $\Gamma = \sum_{k=1}^{n} \gamma_k \mathbf{e}_k \in V^{(n)}$ with $deg\,\Gamma = 2g-2$. Then we have

$$\dagger\dagger) \qquad [\Gamma, \Gamma + \mathbf{e}_i] = [\Gamma, \Gamma + \mathbf{e}_n] \quad \text{for } 1 \le i \le n-1.$$

(In fact, $[\Gamma, \Gamma + \mathbf{e}_i] = 0$ is equivalent to $\gamma_n^{\Gamma_{in}}(\gamma_i + 1) = \gamma_n + 1$ by $R - i$). As $(\gamma_n^{\Gamma_{in}})^{-1} = \gamma_i^{\Gamma_{in}}$, we have $\gamma_i^{\Gamma_{in}}(\gamma_n + 1) = \gamma_i + 1$. This is equivalent to $[\Gamma, \Gamma + \mathbf{e}_n] = 0$ by $R - n$).)

†) and ††) imply The condition B) in Lemma 1-1.

When $\Gamma_{kn} = (0, \dots, 0)$ (write O_{kn}) for $k \ne n$, the subset $\tilde{\gamma}_n(G_k^{O_{kn}}) = G_n^{O_{kn}}$ (Lemma A-7 (1)) of $\{\gamma \mid 1 \le \gamma \le 2g-1\}$ is uniquely determined whichever k

we may take (by C-3 and $R-n$)). We denote this set by $\tilde{G}$. Then $\tilde{G} = \{\gamma \mid \gamma \mathbf{e}_n \in V^{(n)}, [(\gamma-1)\mathbf{e}_n, \gamma\mathbf{e}_n] = 0\}$ and $\#\tilde{G} = \#(G_k^{O_{kn}}) = g$. This means that the condition C) in Lemma 1-1 is satisfied. □

Remark A-9. The non-negative integer δ^{Γ_n} defined in §.1 can be re-defined by

$$\delta^{\Gamma_n} := max\{\gamma_n^{\Gamma_{in}}(\gamma_i) \mid 1 \le i \le n-1\},$$

where $\Gamma = (\gamma_1, \ldots, \gamma_n) = (\Gamma_n, \gamma_n) = \Gamma_{in} + \gamma_i \mathbf{e}_i + \gamma_n \mathbf{e}_n$ for $1 \le i \le n-1$.

Example A-10. Let $(V^{(3)}, *_3)$ be the graph in Theorem 3-3. Let $\Gamma = (\gamma_1, \gamma_2, \gamma_3) \in V^{(3)}$. Then $\Gamma_{23} = \gamma_1$ and $\Gamma_3 = (\gamma_1, \gamma_2)$. If $\gamma_1 = 2k-1$ or $2k$ with $1 \le k \le g-1$, then

$$G_2^{\Gamma_{23}} = G_3^{\Gamma_{23}} = \{1, 3, 5, \ldots, 2(g-k)-1\}$$

and

$$\gamma_3^{\Gamma_{23}}(\gamma_2) = \begin{cases} 0 & \text{if } \gamma_2 \text{ is even} \\ 2(g-k)-\gamma_2 & \text{if } \gamma_2 \text{ is odd.} \end{cases}$$

Then, for $(\gamma_1, \gamma_2) \in V^{(2)}$,

$$\delta^{\Gamma_3} = \begin{cases} 2g-1-\gamma_1-\gamma_2 & \text{if } \gamma_1 \text{ and } \gamma_2 \text{ are odd} \\ 2g-\gamma_1-\gamma_2 & \text{if } \gamma_1 \text{ is odd(resp. even) and } \gamma_2 \text{ is even(resp. odd)} \\ 0 & \text{if } \gamma_1 \text{ and } \gamma_2 \text{ are even.} \end{cases}$$

(In fact, if $\gamma_1 = 2k-1$ and $\gamma_2 = 2l-1$ $(0 \le k, l \le g-1,\ k+l \le g)$ then

$$\begin{aligned} \delta^{\Gamma_3} &= max\{\gamma_3^{\Gamma_{13}}(\gamma_1) = 2(g-l)-\gamma_1, \gamma_3^{\Gamma_{23}}(\gamma_2) = 2(g-k)-\gamma_2\} \\ &= 2g-1-\gamma_1-\gamma_2. \end{aligned}$$

If $\gamma_1 = 2k-1$ and $\gamma_2 = 2l$ $(0 \le k, l \le g-1,\ k+l \le g)$, then

$$\begin{aligned} \delta^{\Gamma_3} &= max\{\gamma_3^{\Gamma_{13}}(\gamma_1) = 2(g-l)-\gamma_1, \gamma_3^{\Gamma_{23}}(\gamma_2) = 0\} \\ &= 2g-\gamma_1-\gamma_2. \end{aligned}$$

(See Proposition 3-7).)

References

[1] E. Albarello, M. Cornalba, P. Griffiths and J. Harris, Geometry of Algebraic curves I, Springer-Verlag 1985.

[2] M. Homma, The Weierstrass semigroup of points on a curve, Archv. der Mathematik **67**, 337–348, 1996.

[3] S. J. Kim, On the index of the Weierstrass semigroup of a pair of points on a curve, Archv. der Mathematik **62**, 73–82, 1994.

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