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GENERALIZED HELICAL IMMERSIONS OF A RIEMANNIAN MANIFOLD ALL OF WHOSE GEODESICS ARE CLOSED INTO A EUCLIDEAN SPACE

By

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Abstract. In this paper, we investigate an isometric immersion of a compact connected Riemannian manifold M into a Euclidean space and a sphere such that every geodesic in M is closed and viewed as a helix (of general order) in the ambient space.

Introduction

Let f be an isometric immersion of a Riemannian manifold M into a Riemannian manifold $\tilde{M}$. If geodesics in M are viewed as specific curves in $\tilde{M}$, what are the shape of $f(M)$? Several geometricians studied this problem. K. Sakamoto investigated an isometric immersion f of a complete connected Riemannian manifold M into a Euclidean space and a sphere such that every geodesic in M is viewed as a helix in the ambient space and that the order and the Frenet curvatures of the helix are independent of the choice of the geodesic (cf. [13], [14]). Such a immersion is called a *helical immersion*. On the other hand, we recently investigated an isometric immersion of a compact connected Riemannian manifold M into a Euclidean space and a sphere such that every geodesic in M is viewed as a helix in the ambient space, where the order and the Frenet curvatures of the helix may depend on the choice of the geodesic. We called such a immersion a *generalized helical immersion* and the maximal order of those helices the *order of f*. It is easy to show that f is of even order if M is compact and the ambient space is a Euclidean space. In [9], we obtained the following characterizing theorem:

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Let f be a generalized helical immersion of order $2d$ of a compact connected Riemannian manifold M into a Euclidean space. Assume that the following condition holds: $(*)$ for each $p \in M$, there is at least one geodesic σ in M through p such that $f \circ \sigma$ is a generic helix of order $2d$ in the ambient space. Then the second fundamental form of f is parallel and hence f is congruent to the standard isometric embedding of a symmetric R-space of rank d.

Here a generic helix of order $2d$ is a helix in a Euclidean space whose closure is a d-dimensional Clifford torus. In case of $d \geq 2$, the above condition $(*)$ assures the existence of a non-closed geodesic in M because a generic helix of order $2d$ $(d \geq 2)$ is non-closed. In this paper, we investigate a generalized helical immersion f of a compact connected Riemannian manifold M all of whose geodesics are closed into a Euclidean space or a sphere. Concretely, we show that, if such an immersion f is an embedding, then M is a SC-manifold (see Theorem 3.2) and, under certain additional conditions, f is helical, where f may not be an embedding (see Theorem 3.6). Here a SC-manifold is a Riemannian manifold all of whose geodesics are simply closed geodesics with the same length.

In Sect. 1 and 2, we prepare basic notations, definitions and lemmas. In Sect. 3 and 4, we prove main results in terms of basic lemmas prepared in Sect. 2.

Throughout this paper, unless otherwise mentioned, we assume that all geometric objects are of class C^∞ and all manifolds are connected ones without boundary.

1. Notations and definitions

In this section, we shall state basic notations and definitions. Let $\sigma : I \to M$ be a curve in a Riemannian manifold M parametrized by the arclength s, where I is an open interval of the real line $\boldsymbol{R}$. Denote by v_0 the velocity vector field $\dot{\sigma}$ of σ. Let ∇ be the Levi-Civita connection of M. If there exist an orthonormal system field $(v_1, \dots, v_{d-1})$ along σ and positive constants $\lambda_1, \dots, \lambda_{d-1}$ satisfying the following relations

$$(1.1) \qquad \begin{cases} \nabla_{v_0} v_0 = \lambda_1 v_1 \\ \nabla_{v_0} v_1 = -\lambda_1 v_0 + \lambda_2 v_2 \\ \qquad \vdots \\ \nabla_{v_0} v_{d-2} = -\lambda_{d-2} v_{d-3} + \lambda_{d-1} v_{d-1} \\ \nabla_{v_0} v_{d-1} = -\lambda_{d-1} v_{d-2}, \end{cases}$$

then σ is called a *helix of order d*. The relation (1.1), λ_i, v_i $(1 \leq i \leq d-1)$ and $(v_0, \dots, v_{d-1})$ are called the Frenet formula, the i-th Frenet curvature, the i-th

Frenet normal vector field and the Frenet frame field of σ, respectively. In particular, a helix σ of order $2d$ in an m-dimensional Euclidean space $\boldsymbol{R}^m$ is expressed as follows:

$$\sigma(s) = c_0 + \sum_{i=1}^{d} r_i(\cos(a_i s)e_{2i-1} + \sin(a_i s)e_{2i}), \tag{1.2}$$

where c_0 is a constant vector of $\boldsymbol{R}^m$, $(e_1, \dots, e_{2d})$ is an orthonormal system of $\boldsymbol{R}^m$, r_i $(1 \le i \le d)$ are positive constants and a_i $(1 \le i \le d)$ are mutually distinct positive constants. Note that the image $\operatorname{Im}\sigma$ of σ is contained in the d-dimensional Clifford torus

$$T := \left\{ c_0 + \sum_{i=1}^{d} r_i(\cos\theta_i \cdot e_{2i-1} + \sin\theta_i \cdot e_{2i}) \,\middle|\, 0 \le \theta_i < 2\pi (i = 1, \dots, d) \right\}.$$

Also, helices in an m-dimensional sphere S^m are as follows. Let σ be a helix in S^m and ι the totally umbilic embedding of S^m into $\boldsymbol{R}^{m+1}$. Then we see that $\iota \circ \sigma$ is a helix of even order in $\boldsymbol{R}^{m+1}$. Let $2d$ be the order of $\iota \circ \sigma$. It is shown that the order of σ is $2d - 1$ (resp. $2d$) if the centroid of the d-dimensional Clifford torus containing $\operatorname{Im}(\iota \circ \sigma)$ coincides (resp. does not coincide) with the center of S^m.

Let f be an isometric immersion of an n-dimensional Riemannian manifold M^n into an m-dimensional Riemannian manifold $\tilde{M}^m$. Denote by T_pM (resp. S_pM) the tangent space (resp. the unit tangent sphere) of M at p and SM the unit tangent bundle of M. We shall identify T_pM with $f_*(T_pM)$, where f_* is the differential of f. Denote by ∇ (resp. $\tilde{\nabla}$) the Levi-Civita connection on M (resp. $\tilde{M}$) and A, h and $\nabla^\perp$ the shape operator, the second fundamental form and the normal connection of f, respectively. Denote by the same symbol $\bar{\nabla}$ both $\nabla^* \otimes \cdots \otimes \nabla^* \otimes \nabla^\perp$ and $\nabla^{\perp^*} \otimes \nabla^* \otimes \cdots \otimes \nabla^* \otimes \nabla$, where ∇^* is the dual connection of ∇. Also, we shall denote the i-th order derivative of h (resp. A) with respect to $\bar{\nabla}$ by $\bar{\nabla}^i h$ (resp. $\bar{\nabla}^i A$). If, for every geodesic σ in M, $f \circ \sigma$ is a helix of order d and the Frenet curvatures of $f \circ \sigma$ do not depend on the choice of σ, then f is called a *helical immersion of order d*. Also, if, for every geodesic σ in M, $f \circ \sigma$ is a helix of order at most d and there is at least one geodesic σ_0 in M such that $f \circ \sigma_0$ is a helix of order d, then we shall call f a *generalized helical immersion of order d*.

2. Basic lemmas

In this section, we prepare basic lemmas which will be used in the next section. Let f be a generalized helical immersion of an n-dimensional complete

Riemannian manifold M^n into an m-dimensional Euclidean space $\boldsymbol{R}^m$. For each $v \in SM$, denote by σ_v the maximal geodesic in M parametrized by the arc-length s whose velocity vector at $s=0$ is equal to v. For $p \in M$, we set $V_{p,i} := \{v \in S_pM \mid f \circ \sigma_v : \text{helix of order } i\}$ $(i \geq 1)$ and define a function $\hat{\lambda}_i$ $(i \geq 1)$ on SM by

$$\hat{\lambda}_i(v) := \begin{cases} \lambda_i & \left(v \in \bigcup_{i+1 \leq j} \bigcup_{p \in M} V_{p,j}\right) \\ 0 & \left(v \in \bigcup_{1 \leq j \leq i} \bigcup_{p \in M} V_{p,j}\right), \end{cases}$$

where λ_i is the i-th Frenet curvature of $f \circ \sigma_v$. It is easy to show that $\hat{\lambda}_i$ is continuous on $\bigcup_{i \leq j} \bigcup_{p \in M} V_{p,j}$ $(i \geq 1)$. In [9], we proved the following lemma.

LEMMA 2.1. *Assume that* $V_{p,d} \neq \emptyset$ *and* $V_{p,i} = \emptyset$ $(i \geq d+1)$ *for* $p \in M$. *Then the set* $V_{p,i}$ $(1 \leq i \leq d-1)$ *are closed sets of measure zero in* S_pM *and* $V_{p,d}$ *is a dense open set in* S_pM.

In the sequel, assume that $V_{p,2d} \neq \emptyset$ and $V_{p,i} = \emptyset$ $(i \geq 2d+1)$ for some $p \in M$. For each $v \in V_{p,2d}$, $f \circ \sigma_v$ is uniquely expressed as

$$(f \circ \sigma_v)(s) = c(v) + \sum_{i=1}^{d} r_i(v)(\cos(a_i(v)s)e_{2i-1}(v) + \sin(a_i(v)s)e_{2i}(v)),$$

where $c(v)$ is a constant vector of $\boldsymbol{R}^m$, $(e_1(v), \ldots, e_{2d}(v))$ is an orthonormal system of $\boldsymbol{R}^m$, $r_i(v)$ $(1 \leq i \leq d)$ are positive constants and $a_i(v)$ $(1 \leq i \leq d)$ are positive constants with $a_1(v) < \cdots < a_d(v)$. We regard r_i and a_i $(1 \leq i \leq d)$ as functions on $V_{p,2d}$. In [9], we proved the following lemma.

LEMMA 2.2. *The functions* a_i $(1 \leq i \leq d)$ *are analytic.*

Also, we prepare the following lemma.

LEMMA 2.3 *On* $V_{p,2d}$, *the following relation holds*:

$$\begin{pmatrix} a_1^2 & \cdots & a_d^2 \\ a_1^4 & \cdots & a_d^4 \\ \vdots & & \vdots \\ a_1^{2d} & \cdots & a_d^{2d} \end{pmatrix} \begin{pmatrix} r_1^2 \\ \vdots \\ r_d^2 \end{pmatrix} = \begin{pmatrix} 1 \\ F_1(\hat{\lambda}_1) \\ \vdots \\ F_{d-1}(\hat{\lambda}_1, \ldots, \hat{\lambda}_{d-1}) \end{pmatrix},$$

where F_i *is a polynomial of i-variables* $(1 \leq i \leq d-1)$.

PROOF. Fix $v \in V_{p,2d}$. Let $(v_0, v_1, \dots, v_{2d-1})$ be the Frenet frame field of $f \circ \sigma_v$. Then we have

$$v_0 = \sum_{i=1}^{d} r_i(v)a_i(v)(-\sin(a_i(v)s)e_{2i-1}(v) + \cos(a_i(v)s)e_{2i}(v)) \tag{2.1}$$

and hence $\sum_{i=1}^{d} r_i(v)^2 a_i(v)^2 = 1$. Thus, if $d = 1$, then the proof is completed. In the sequel, assume $d \geq 2$. By operating $\tilde{\nabla}_{v_0}$ to (2.1), we have

$$\hat{\lambda}_1(v)v_1 = -\sum_{i=1}^{d} r_i(v)a_i(v)^2(\cos(a_i(v)s)e_{2i-1}(v) + \sin(a_i(v)s)e_{2i}(v)) \tag{2.2}$$

and hence $\sum_{i=1}^{d} r_i(v)^2 a_i(v)^4 = \hat{\lambda}_1(v)^2$. Thus, if $d = 2$, then the proof is completed. In the sequel, assume $d \geq 3$. Furthermore, by operating $\tilde{\nabla}_{v_0}$ to (2.2), we have

$$\begin{aligned}&\hat{\lambda}_1(v)(-\hat{\lambda}_1(v)v_0 + \hat{\lambda}_2(v)v_2)\\&= \sum_{i=1}^{d} r_i(v)a_i(v)^3(\sin(a_i(v)s)e_{2i-1}(v) - \cos(a_i(v)s)e_{2i}(v))\end{aligned}$$

and hence $\sum_{i=1}^{d} r_i(v)^2 a_i(v)^6 = \hat{\lambda}_1(v)^4 + \hat{\lambda}_1(v)^2\hat{\lambda}_2(v)^2$. Thus, if $d = 3$, then the proof is completed. In case of $d \geq 4$, by repeating the same process, we can obtain

$$\sum_{i=1}^{d} r_i(v)^2 a_i(v)^{2j} = F_{j-1}(\hat{\lambda}_1(v), \dots, \hat{\lambda}_{j-1}(v)) \quad (4 \leq j \leq d),$$

where F_{j-1} is a polynomial of $(j-1)$-variables $(4 \leq j \leq d)$. This completes the proof. □

3. Generalized helical immersions into a Euclidean space

In this section, we shall investigate a generalized helical immersion f of an n-dimensional compact Riemannian manifold M all of whose geodesics are closed into an m-dimensional Euclidean space $\boldsymbol{R}^m$. Since M is compact, f is of even order. Let $2d$ be the order of f. Take $p \in M$ with $V_{p,2d} \neq \emptyset$. Since all of geodesics in M are closed, they admit a common period by Lemma 7.11 of [1, P182]. Let $\mu := \max_{v \in S_pM} l(\sigma_v)$, where $l(\sigma_v)$ is the length of σ_v (i.e., the minimal period of σ_v). Let $W := \{v \in V_{p,2d} \mid l(\sigma_v) = \mu\}$. Since $l(\sigma_v)$ $(v \in V_{p,2d})$ are divisors of the common period, $\{l(\sigma_v) \mid v \in V_{p,2d}\}$ is a discrete set. The function ϕ on $V_{p,2d}$ defined by $\phi(v) = l(\sigma_v)$ is lower semi-continuous. These facts deduce that $W = \phi^{-1}(\mu)$ is an open set in $V_{p,2d}$. Let a_i $(1 \leq i \leq d)$ be functions on $V_{p,2d}$ stated in Sect. 2. First we shall show the following lemma.

Lemma 3.1. *The set $V_{p,2d}$ coincides with S_pM and a_i $(1 \le i \le d)$ are constant on S_pM.*

Proof. (Step I) First we shall show that the functions a_i $(1 \le i \le d)$ are constant on each component of $V_{p,2d}$ which intersects with W. Let W_0 be a component of W. For each $v \in W_0$, set $s_v := \min\{s \mid a_i(v)s \in \boldsymbol{N}\ (1 \le i \le d)\}$. Clearly $l(f \circ \sigma_v) = 2\pi s_v$ holds. Also, we can show $l(\sigma_v)/l(f \circ \sigma_v) \in \boldsymbol{N}$. Hence we have $\mu/2\pi s_v \in \boldsymbol{N}$, which together with $a_i(v)s_v \in \boldsymbol{N}$ implies $a_i(v)\mu/2\pi \in \boldsymbol{N}$. Therefore, it follows from the continuity of a_i that a_i is constant on W_0. Thus a_i is constant on each component of W. This together with the analyticity of a_i (by Lemma 2.2) implies that a_i is constant on each component of $V_{p,2d}$ which intersects with W.

(Step II) Next we shall show $V_{p,2d} = S_pM$. Let V_0 be a component of $V_{p,2d}$ which intersects with W. We showed that a_i $(1 \le i \le d)$ are constant on V_0. Denote by $\overline{V}_0$ the closure of V_0 in S_pM. Take $v \in \overline{V}_0$ and a sequence $\{w_k\}_{k=1}^{\infty}$ in V_0 with $\lim_{k\to\infty} w_k = v$. The helix $f \circ \sigma_{w_k}$ is uniquely expressed as

(3.1)
$$(f \circ \sigma_{w_k})(s) = c(w_k) + \sum_{i=1}^{d} r_i(w_k)(\cos(a_i(w_k)s)e_{2i-1}(w_k) + \sin(a_i(w_k)s)e_{2i}(w_k)).$$

Since helices $f \circ \sigma_{w_k}$ $(k \in \boldsymbol{N})$ are contained in a compact set $f(M)$, we have $\sup_k \|c(w_k)\| < \infty$ and $\sup_k r_i(w_k) < \infty$ $(1 \le i \le d)$. Set $C := \sup_k \|c(w_k)\|$ and $R_i := \sup_k r_i(w_k) (1 \le i \le d)$. Since $\{(e_1(w_k), \dots, e_{2d}(w_k), c(w_k), r_1(w_k), \dots, r_d(w_k))\}_{k=1}^{\infty}$ is a sequence in a compact set $S_{m,2d} \times B^m(C) \times [0, R_1] \times \cdots \times [0, R_d]$, its convergent subsequence $\{(e_1(w_{\alpha(k)}), \dots, e_{2d}(w_{\alpha(k)}), c(w_{\alpha(k)}), r_1(w_{\alpha(k)}), \dots, r_d(w_{\alpha(k)}))\}_{k=1}^{\infty}$ exists, where $S_{m,2d}$ is the Stiefel manifold of all orthonormal $2d$-frames in $\boldsymbol{R}^m$, $B^m(C)$ is the m-dimensional ball of center O and radius C in $\boldsymbol{R}^m$ and $[0, R_i]$ $(1 \le i \le d)$ are closed intervals. Let $(e_1^0, \dots, e_{2d}^0, c^0, r_1^0, \dots, r_d^0) := \lim_{k\to\infty} (e_1(w_{\alpha(k)}), \dots, e_{2d}(w_{\alpha(k)}), c(w_{\alpha(k)}), r_1(w_{\alpha(k)}), \dots, r_d(w_{\alpha(k)}))$. From (3.1) and the constancy of a_i on V_0, we have

$$\begin{aligned}\lim_{k\to\infty} (f \circ \sigma_{w_{\alpha(k)}})(s) &= \lim_{k\to\infty} \left\{ c(w_{\alpha(k)}) + \sum_{i=1}^{d} r_i(w_{\alpha(k)})(\cos(a_i(w_{\alpha(k)})s)e_{2i-1}(w_{\alpha(k)}) \right. \\ &\qquad \left. + \sin(a_i(w_{\alpha(k)})s)e_{2i}(w_{\alpha(k)}) \right\} \\ &= c_0 + \sum_{i=1}^{d} r_i^0(\cos(a_i(w_1)s)e_{2i-1}^0 + \sin(a_i(w_1)s)e_{2i}^0).\end{aligned}$$

On the other hand, we have

$$\lim_{k\to\infty}(f\circ\sigma_{w_{\alpha(k)}})(s)=\lim_{k\to\infty}(f\circ\exp_p)(sw_{\alpha(k)})=(f\circ\exp_p)(sv)$$
$$=(f\circ\sigma_v)(s),$$

where $\exp_p$ is the exponential map of M at p. Thus we can obtain

$$(f\circ\sigma_v)(s)=c_0+\sum_{i=1}^{d}r_i^0(\cos(a_i(w_1)s)e_{2i-1}^0+\sin(a_i(w_1)s)e_{2i}^0),$$

which implies that $f\circ\sigma_v$ is a helix of order $2d$, that is, $v\in V_{p,2d}$. Clearly, this implies $v\in V_0$. Therefore, we have $\bar{V}_0=V_0$, that is, V_0 is closed in S_pM. On the other hand, since $V_{p,2d}$ is open in S_pM by Lemma 2.1, so is also V_0. Hence, it follows from the connectedness of S_pM that $V_0=S_pM$, that is, $V_{p,2d}=S_pM$. This completes the proof. □

From this lemma, we can prove the following result.

THEOREM 3.2. *Let f be a generalized helical immersion of an n-dimensional compact Riemannian manifold M all of whose geodesics are closed into an m-dimensional Euclidean space $\boldsymbol{R}^m$. Then the following statements* (i) *and* (ii) *hold:*

(i) *all geodesics in M are viewed as closed helices of the same order with the same length in $\boldsymbol{R}^m$,*

(ii) *if f is an embedding, then M is a SC-manifold.*

PROOF. Let $2d$ be the order of f. Take $p\in M$ with $V_{p,2d}\neq\emptyset$. From Lemma 3.1, it follows that $V_{p,2d}=S_pM$ and that a_i $(1\leq i\leq d)$ are constant on S_pM. This implies that all geodesics in M through p are viewed as closed helices of order $2d$ with the same length in $\boldsymbol{R}^m$. Take an arbitrary $q\in M$. Since M is compact and hence complete, there is a geodesic in M through p and q. This implies $V_{q,2d}\neq\emptyset$. Hence, we see that all geodesics in M through p or q are viewed as closed helices of order $2d$ with the same length in $\boldsymbol{R}^m$. Thus the statement (i) is deduced from the arbitrarity of q. Assume that f is an embedding. Since a closed helix in $\boldsymbol{R}^m$ are simply closed, all geodesics in M are simply closed geodesics. Also, $l(\sigma)=l(f\circ\sigma)$ holds for each geodesic σ in M. From the statement (i), $l(f\circ\sigma)$ is independent of the choice of σ. Therefore, so is also $l(\sigma)$, that is, all geodesics in M have the same length. Thus M is a SC-manifold. □

From the statement (ii) of this theorem, we can obtain the following corollary.

Corollary 3.3. *Let $f : M \hookrightarrow \boldsymbol{R}^m$ be an immersion as in Theorem 3.2. If f is an embedding and M is a Riemannian homogeneous space, then M is isometric to a compact symmetric space of rank one.*

Proof. By the statement (ii) of Theorem 3.2, M is a SC-manifold. Hence, since M is a Riemannian homogeneous space, M is isometric to a compact symmetric space of rank one by Theorem 7.55 of [1, P196]. □

Next we shall investigate in what case an immersion as in Theorem 3.2 is helical. First we shall show the following lemma.

Lemma 3.4. *Let f be an immersion as in Theorem 3.2 and $2d$ the order of f. In case of $d \geq 2$, assume that $\hat{\lambda}_i$ $(1 \leq i \leq d-1)$ are constant on SM, where $\hat{\lambda}_i$ $(1 \leq i \leq d-1)$ are functions defined in Sect. 2. Then f is helical.*

Proof. Fix $p \in M$. By Theorem 3.2, $V_{p,2d} = S_pM$ holds. For each $v \in S_pM$, $f \circ \sigma_v$ is uniquely expressed as

$$(f \circ \sigma_v)(s) = c(v) + \sum_{i=1}^{d} r_i(v)(\cos(a_i(v)s)e_{2i-1}(v) + \sin(a_i(v)s)e_{2i}(v)). \tag{3.2}$$

It follows from Lemma 3.1 that a_i $(1 \leq i \leq d)$ are constant on S_pM. Hence, since $\hat{\lambda}_i$ $(1 \leq i \leq d-1)$ are constant on S_pM by the assumption, so are also r_i $(1 \leq i \leq d)$ by Lemma 2.3. Therefore, by (3.2), $f \circ \sigma_v$ $(v \in S_pM)$ are mutually congruent, that is, they have the same Frenet curvatures. This together with the arbitrarity of p and the completeness of M implies that f is helical. □

Define functions F_{ij} $(i \geq 0, j \geq 0)$ on SM by

$$F_{ij}(v) := \langle (\bar{\nabla}^i h)(v, \dots, v), (\bar{\nabla}^j h)(v, \dots, v) \rangle \quad (v \in SM)$$

and functions G_{ij} on the Stiefel bundle $V_2(M)$ of M of all orthonormal 2-frames of M by

$$G_{ij}(v, w) := \langle (\bar{\nabla}^i h)(v, \dots, v), (\bar{\nabla}^j h)(v, \dots, v, w) \rangle \quad ((v, w) \in V_2(M)).$$

Now we shall prepare another lemma.

LEMMA 3.5. *Let f be an immersion as in Theorem 3.2 and $2d$ the order of f. Assume that F_{kk} is constant on SM and $\bar{\nabla}^i h$ $(i \geq 2)$ are symmetric, where k is a fixed non-negative integer. Then the following relations hold*:

$$G_{kk} = 0, \quad \sum_{i=0}^{j} \binom{j}{i} F_{k+j-i,k+i+1} = 0,$$

$$\sum_{i=0}^{j} \binom{j}{i} G_{k+i+1,k+j-i} = 0, \quad \sum_{i=0}^{j} \binom{j}{i} G_{k+i,k+j-i+1} = 0 \quad (j \geq 0).$$

PROOF. Take an arbitrary point p of M and furthermore, take an arbitrary orthonormal 2-frame (v, w) of M at p. Let $\tilde{v}$ be the velocity vector field of the geodesic σ_v. By operating d/ds to the constant function $F_{kk}(\tilde{v})$, we have $F_{k,k+1}(\tilde{v}) = 0$, where s is the arclength of σ_v. Furthermore, by operating $(d/ds)^j$ to $F_{k,k+1}(\tilde{v}) = 0$ and substituting $s = 0$, we can obtain $\sum_{i=0}^{j} \binom{j}{i} F_{k+j-i,k+i+1}(v) = 0$. Hence, by the arbitrarity of v and p, $\sum_{i=0}^{j} \binom{j}{i} F_{k+j-i,k+i+1} = 0$ holds on SM. By differentiating $F_{kk}|_{S_pM}$ in the direction $w(\in T_v(S_pM))$, we have $G_{kk}(v, w) = 0$. By the arbitrarity of (v, w) and p, $G_{kk} = 0$ holds on $V_2(M)$. Let $\tilde{w}$ be the parallel vector field along σ_v with $\tilde{w}(0) = w$. By operating d/ds to $G_{kk}(\tilde{v}, \tilde{w})$ and substituting $s = 0$, we have $G_{k+1,k}(v, w) + G_{k,k+1}(v, w) = 0$. Also, by differentiating $F_{k,k+1} \mid_{S_pM} = 0$ in the direction w $(\in T_v(S_pM))$, we have

$$(k+2)G_{k+1,k}(v, w) + (k+3)G_{k,k+1}(v, w) = 0.$$

Therefore, we can obtain $G_{k+1,k}(v, w) = G_{k,k+1}(v, w) = 0$ and hence, by the arbitrarity of (v, w) and p, $G_{k+1,k} = G_{k,k+1} = 0$ holds on $V_2(M)$. Furthermore, by operating $(d/ds)^j$ to $G_{k+1,k}(\tilde{v}, \tilde{w}) = G_{k,k+1}(\tilde{v}, \tilde{w}) = 0$ and substituting $s = 0$, we can obtain

$$\sum_{i=0}^{j} \binom{j}{i} G_{k+i+1,k+j-i}(v, w) = \sum_{i=0}^{j} \binom{j}{i} G_{k+i,k+j-i+1}(v, w) = 0.$$

Hence, by the arbitrarity of (v, w) and p,

$$\sum_{i=0}^{j} \binom{j}{i} G_{k+i+1,k+j-i} = \sum_{i=0}^{j} \binom{j}{i} G_{k+i,k+j-i+1} = 0$$

holds on $V_2(M)$. □

From these lemmas, we can show the following result.

THEOREM 3.6. *Let f be a generalized helical immersion of order $2d$ of an n-dimensional compact Riemannian manifold M all of whose geodesics are closed into an m-dimensional Euclidean space $\boldsymbol{R}^m$. In case of $d \geq 2$, assume that, for each $p \in M$, $\|(\bar{\nabla}^i h)(v, \ldots, v)\|$ is independent of the choice of $v \in S_pM$ $(0 \leq i \leq d-2)$ and furthermore, in case of $d \geq 6$, assume that $\bar{\nabla}^i h$ is symmetric $(2 \leq i \leq [d/2]-1)$, where [ ] is the Gauss's symbol. Then f is helical.*

PROOF. If $d = 1$, then f is a planar geodesic immersion and hence a helical immersion of order 2. In the sequel, assume that $d \geq 2$. Take an arbitrary point p_0 of M and furthermore take an arbitrary orthonormal 2-frame (v, w) of M at p_0. Let $(v_0, \ldots, v_{2d-1})$ (resp. λ_i $(1 \leq i \leq 2d-1)$) be the Frenet frame (resp. i-th Frenet curvature) of $f \circ \sigma_v$, where we note that $f \circ \sigma_v$ is of order $2d$ by Theorem 3.2. From the Gauss formula and the Frenet formula, we have

$$\lambda_1 v_1 = h(v_0, v_0) \tag{3.3}$$

and hence $\hat{\lambda}_1(v)^2 = F_{00}(v)$. By the arbitrarity of v and p_0, we see that $\hat{\lambda}_1^2 = F_{00}$ holds on SM. By the assumption, $\hat{\lambda}_1$ is constant on S_pM for each $p \in M$. Furthermore, since f is generalized helical and there exists a geodesic through arbitrary two points of M by the compactness of M, $\hat{\lambda}_1(= F_{00})$ is constant on SM. Thus, if $d = 2$, then f is helical by Lemma 3.4. In the sequel, assume $d \geq 3$. By operating $\tilde{\nabla}_{v_0}$ to (3.3), we have

$$-\lambda_1^2 v_0 + \lambda_1 \lambda_2 v_2 = -A_{h(v_0, v_0)} v_0 + (\bar{\nabla} h)(v_0, v_0, v_0).$$

Also, since F_{00} is constant on SM, it follows from Lemma 3.5 that $G_{00} = 0$ and hence $A_{h(v_0, v_0)} v_0 = F_{00}(v_0) v_0$, where we note that the symmetricness of h is used. So we can obtain

$$-\lambda_1^2 v_0 + \lambda_1 \lambda_2 v_2 = -F_{00}(v_0) v_0 + (\bar{\nabla} h)(v_0, v_0, v_0) \tag{3.4}$$

and hence

$$\hat{\lambda}_1(v)^4 + \hat{\lambda}_1(v)^2 \hat{\lambda}_2(v)^2 = F_{00}(v)^2 + F_{11}(v).$$

By the arbitrarity of v and p_0, we see that

$$\hat{\lambda}_1^4 + \hat{\lambda}_1^2 \hat{\lambda}_2^2 = F_{00}^2 + F_{11} \tag{3.5}$$

holds on SM. Since $\hat{\lambda}_1$ and F_{00} are constant on SM and F_{11} is constant on S_pM for each $p \in M$, it follows from (3.5) that $\hat{\lambda}_2$ is constant on S_pM for each $p \in M$. Furthermore, since f is a generalized helical and there exists a geodesic through arbitrary two points of M, $\hat{\lambda}_2$ is constant on SM. Thus, if $d = 3$, then f is helical by Lemma 3.4. In the sequel, assume $d \geq 4$. By operating $\tilde{\nabla}_{v_0}$ to (3.4), we have

$$\begin{aligned} &- \lambda_1(\lambda_1^2 + \lambda_2^2)v_1 + \lambda_1\lambda_2\lambda_3 v_3 \\ &\qquad = -2F_{10}(v_0)v_0 + F_{00}(v_0)h(v_0, v_0) \\ &\qquad\quad - A_{(\bar{\nabla}h)(v_0,,v_0,v_0)}v_0 + (\bar{\nabla}^2 h)(v_0, \ldots, v_0). \end{aligned}$$

Also, since F_{00} is constant on SM, we have $F_{10} = 0$ and $G_{10} = 0$ by Lemma 3.5, where we note that the symmetricness of $\bar{\nabla}h$ is used. So we have

$$-\lambda_1(\lambda_1^2 + \lambda_2^2)v_1 + \lambda_1\lambda_2\lambda_3 v_3 = F_{00}(v_0)h(v_0, v_0) + (\bar{\nabla}^2 h)(v_0, \ldots, v_0). \tag{3.6}$$

and hence

$$\begin{aligned} &\hat{\lambda}_1(v)^2(\hat{\lambda}_1(v)^2 + \hat{\lambda}_2(v)^2)^2 + \hat{\lambda}_1(v)^2\hat{\lambda}_2(v)^2\hat{\lambda}_3(v)^2 \\ &\qquad = F_{00}(v)^3 + 2F_{00}(v)F_{20}(v) + F_{22}(v). \end{aligned}$$

By the arbitrarity of v and p_0, we see that

$$\hat{\lambda}_1^2(\hat{\lambda}_1^2 + \hat{\lambda}_2^2)^2 + \hat{\lambda}_1^2\hat{\lambda}_2^2\hat{\lambda}_3^2 = F_{00}^3 + 2F_{00}F_{20} + F_{22} \tag{3.7}$$

holds on SM. Since $\hat{\lambda}_1$, $\hat{\lambda}_2$ and F_{00} are constant on SM, so is also F_{11} by (3.5). Furthermore, since F_{00} and F_{11} are constant on SM, so is also F_{20} by Lemma 3.5. Therefore, since $\hat{\lambda}_1$, $\hat{\lambda}_2$, F_{00} and F_{20} are constant on SM and F_{22} is constant on S_pM for each $p \in M$, it follows from (3.7) that $\hat{\lambda}_3$ is constant on S_pM for each $p \in M$. Moreover, since f is generalized helical and there exists a geodesic through arbitrary two points of M, $\hat{\lambda}_3$ is constant on SM. Thus, if $d = 4$, then f is helical by Lemma 3.4. In case of $d \geq 5$, by repeating the same process, we can show that $\hat{\lambda}_i$ $(4 \leq i \leq d-1)$ are constant on SM. Hence f is helical by Lemma 3.4. □

Now we shall recall examples of a helical immersion into a sphere (or a Euclidean space) given by K. Tsukada in [18]. Let M be an n-dimensional compact symmetric space of rank one. Let V_k be the eigenspace for k-th eigenvalue λ_k of the Laplace operator on M and let $\dim V_k = m(k) + 1$. We define an inner product $\langle\,,\rangle$ on V_k by $\langle\phi, \psi\rangle := \int_M \phi\psi\, dV$, where dV is the volume element of M. We define a map $\Phi_k : M \to \boldsymbol{R}^{m(k)+1}$ by $\Phi_k(p) := \sqrt{n/\lambda_k}(\phi_0(p), \ldots, \phi_{m(k)}(p))$, where $(\phi_0, \ldots, \phi_{m(k)})$ is an orthonormal base of V_k.

Then Φ_k becomes a helical immersion. Furthermore, it is shown that $\Phi_k(M)$ is contained in a hypersphere $S^{m(k)}$ of $\boldsymbol{R}^{m(k)+1}$ and that $\Phi_k : M \hookrightarrow S^{m(k)}$ is minimal and helical. The isometric immersion Φ_k is called the k-th *standard minimal immersion* into $S^{m(k)}$. K. Tsukada defined an isometric immersion $\Phi_{k_1\cdots k_r}$ of M into $\boldsymbol{R}^{m(k_1)+\cdots m(k_r)+r}$ by

$$\Phi_{k_1\cdots k_r}(p) := (c_1\Phi_{k_1}(p), \dots, c_r\Phi_{k_r}(p)),$$

where $k_1, \dots, k_r$ are positive integers and $c_1, \dots, c_r$ are positive numbers with $c_1^2 + \cdots + c_r^2 = 1$. He showed that it is a helical immersion into a hypersphere $S^{m(k_1)+\cdots m(k_r)+r-1}$ of $\boldsymbol{R}^{m(k_1)+\cdots m(k_r)+r}$ (cf. [18]). Now we can obtain the following result in terms of Theorem 3.6 and Theorem 4.7 of [14].

COROLLARY 3.7. *Under the hypothesis in Theorem* 3.6, *assume that f is a full embedding and* $\dim M = 2$ *or odd integer. Then M is isometric to a sphere or a real projective space and f is congruent to the above immersion $\Phi_{k_1\cdots k_r}$.*

4. Generalized helical immersions into a sphere

In this section, we shall deduce some results for a generalized helical immersion into a sphere in terms of results in the previous section. First we can deduce the following result from Theorem 3.2.

THEOREM 4.1. *Let f be a generalized helical immersion of an n-dimensional compact Riemannian manifold M all of whose geodesics are closed into an m-dimensional sphere S^m. Then the following statements* (i) *and* (ii) *hold*:

(i) *if f is of odd* (*resp. even*) *order d, then all geodesics in M are viewed as closed helices of order d* (*resp. d or $d-1$*) *with the same length in S^m*,

(ii) *if f is an embedding, then M is a SC-manifold.*

PROOF. Let ι be the totally umbilical embedding of S^m into $\boldsymbol{R}^{m+1}$ and set $\tilde{f} := \iota \circ f$. It is clear that $\tilde{f}$ is generalized helical. Hence, it follows from Theorem 3.2 that all geodesics in M are viewed as closed helices of the same order with the same length in $\boldsymbol{R}^{m+1}$. This deduces the statement (i) because a helix of order d in S^m is viewed as a helix of order $2[(d+1)/2]$ in $\boldsymbol{R}^{m+1}$. If f is an embedding, then so is also $\tilde{f}$. Hence, the statement (ii) is deduced from Theorem 3.2. □

From the statement (ii) of this theorem, we can obtain the following corollary.

COROLLARY 4.2. *Let $f : M \hookrightarrow S^m$ be an immersion as in Theorem* 4.1. *If f is an embedding and M is a Riemannian homogeneous space, then M is isometric to a compact symmetric space of rank one.*

Also, we can deduce the following result from Theorem 3.6.

THEOREM 4.3. *Let f be a generalized helical immersion of order $2d - 1$ or $2d$ of an n-dimensional compact Riemannian manifold M all of whose geodesics are closed into an m-dimensional sphere S^m. In case of $d \geq 2$, assume that, for each $p \in M$, $\|(\bar{\nabla}^i h)(v, \ldots, v)\|$ is independent of $v \in S_pM$ $(0 \leq i \leq d-2)$ and furthermore, in case of $d \geq 6$, assume that $\bar{\nabla}^i h$ is symmetric $(2 \leq i \leq [d/2] - 1)$. Then f is helical.*

PROOF. Let ι be the totally umbilical embedding of S^m into $\boldsymbol{R}^{m+1}$ and set $\tilde{f} := \iota \circ f$. From the assumptions, we can show that $\tilde{f}$ satisfies the conditions of Theorem 3.6. Hence $\tilde{f}$ is helical by Theorem 3.6. This implies that so is also f. □

Also, we can obtain the following result from Corollary 3.7.

COROLLARY 4.4. *Under the hypothesis in Theorem* 4.3, *assume that f is a full embedding and $\dim M = 2$ or odd integer. Then M is isometric to a sphere or a real projective space and f is congruent to the immersion $\Phi_{k_1 \cdots k_r}$ stated in Sect.* 3.

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