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A NOTE ON NORMALLY GENERATED LINE BUNDLES ON COMPACT RIEMANN SURFACES

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1. Introduction

Let X denote a compact Riemann surface of genus $g(X) > 0$ and L an ample line bundle on X.

DEFINITION 1. *L is said to be normally generated if, for each $n > 0$, the natural map*

$$\mathrm{Sym}^n H^0(X, L) \to H^0(X, L^n)$$

is surjective.

There are the following two sufficient conditions for line bundles on X to be normally generated obtained by H. H. Martens and D. Mumford, respectively:

THEOREM 1 (cf. [3]). *The canononical bundle K_X on X is normally generated if and only if X is nonhyperelliptic.*

THEOREM 2 ([4]). *If $\deg L \geq 2g(X) + 1$, then L is normally generated.*

On the other hand, Homma [2] classified all the normally generated line bundles on X when the genus of X is three.

THEOREM 3 ([2]). *Supppose $g(X) = 3$.*

(i) *If X is hyperelliptic, then L is normally generated if and only if $\deg L \geq 7$.*

(ii) *If X is nonhyperelliptic, then L is normally generated if and only L satisfies one of the following conditions:*

(a) $\deg L \geq 7$.

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(b) $\deg L = 6$ *and* $H^0(X, L \otimes K_X^{-1}) = 0$.

(c) $L \cong K_X$.

Now let $\pi : X \to Y$ be a (possibly ramified) covering of compact Riemann surfaces and let $g(Y)$ (≥ 0) denote the genus of Y.

PROBLEM. Classify ample line bundles on Y such that the pull backs on X are normally generated.

In this note, we will study this problem in the cases of π being double coverings with small $g(X)$ or $g(Y)$. In §2, we will determine such line bundles on Y when $g(X) = 3$ and in §3, the cases of Y being rational or elliptic Riemann surfaces are treated.

Before closing this section, let us recall some fundamental facts on double coverings (cf. [5]):

LEMMA 1. *Let B denote the branch locus of the double covering $\pi : X \to Y$ on Y. Then there exists a line bundle F on Y with $2F \cong B$ such that the following conditions hold*:

(i) *X is embedded into F and the projection of F to Y restricted on X coincides with π.*

(ii) *The canonical bundle K_X on X is linearly equivalent to $\pi^*(K_Y \otimes F)$ where K_Y is the canonical bundle on Y.*

(iii) *For any line bundle L on Y, we have*:

$$\pi_* \mathcal{O}_X(\pi^* L) \cong \mathcal{O}_Y(L) \oplus \mathcal{O}_Y(L \otimes F^{-1}).$$

COROLLARY. *Let $\pi : X \to Y$ be a double covering of compact Riemann surfaces. Then the induced homomorphism $\pi^* : \operatorname{Pic} Y \to \operatorname{Pic} X$ is injective.*

PROOF. Let M be a line bundle on Y such that the pull back $\pi^* M$ is trivial on X. Then we have $\deg M = 0$ and $h^0(X, \pi^* M) = 1$. Hence, by Lemma 1 (iii), we have $h^0(Y, M) = h^0(X, \pi^* M) - h^0(Y, M \otimes F^{-1}) = 1$, that is, M is also trivial on Y. □

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2. The cases of $g(X) = 3$

By Lemma 1 and Theorem 3, we can determine such line bundle M on Y as in Problem when $g(X) = 3$:

Since $g(X) > g(Y)$, we have $g(Y) = 0$, 1 or 2. If $g(Y) = 0$, then X is hyperelliptic. On the other hand, we have the following result of Farkas:

LEMMA 2 ([1]). *Let $X \to Y$ be a double covering of compact Riemann surfaces with $g(X) = 3$ and $g(Y) = 2$. Then X is hyperelliptic.*

As a conclusion of Lemma 2 and Theroem 3 (i), we obtain the following result.

PROPOSITION 1. *Suppose $g(Y) = 0$ or 2. Then π^*M is normally generated if and only if* $\deg M \geq 4$.

Now suppose X is nonhyperelliptic. Then we have $g(Y) = 1$ and hence, by Lemma 1 (ii), $K_X \cong \pi^*F$ and $\deg F = 2$.

By Theorem 3 (ii), π^*M is normally generated if $\deg M \geq 4$ and not normally generated if $\deg M = 1$.

Suppose $\deg M = 2$. Then π^*M is normally generated if and only if $\pi^*M \cong \pi^*F$, that is, by the corollary to Lemma 1, $M \cong F$.

Suppose $\deg M = 3$. Then, since $g(Y) = 1$ and $\deg M \otimes F^{-1} = 1$, we have

$$h^0(X, \pi^*M \otimes K_X^{-1}) = h^0(Y, M \otimes F^{-1}) + h^0(Y, M \otimes F^{-2}) > 0.$$

Consequently we get the following:

PROPOSITION 2. *Suppose $g(Y) = 1$.*

(i) *If X is hyperelliptic then π^*M is normally generated if and only if* $\deg M \geq 4$.

(ii) *If X is nonhyperelliptic then π^*M is normally generated if and only if* $\deg M \geq 4$ *or* $M \cong F$.

3. The cases of $g(X) \geq 4$ and $g(Y) \leq 1$

3.1. The cases of $g(Y) = 0$

If $g(Y) = 0$, then $\deg F = g(X) + 1$ and hence, if $\deg M < g(X) + 1$ for a line bundle M on Y, we have

$$H^0(X, \pi^*M) \cong H^0(Y, M)$$

by Lemma 1 (iii), that is, each section in $H^0(X, \pi^* M)$ is the pull back of a section in $H^0(Y, M)$. But, for a sufficiently large n,

$$H^0(X, \pi^* M^n) \neq H^0(Y, M^n)$$

by Lemma 1 (iii) again. We therefore conclude that $\pi^* M$ is not normally generated in this case.

On the other hand, by Theorem 2, $\pi^* M$ is normally generated if $\deg M \geq g(X) + 1$.

Consequently we have:

PROPOSITION 3. *Suppose $g(Y) = 0$. Then $\pi^* M$ is normally generated if and only if $\deg M \geq g(X) + 1$.*

3.2. The cases of $g(Y) = 1$

If $g(Y) = 1$, then we have $\deg F = g(X) - 1$ and $K_X \cong \pi^* F$. If moreover $g(X) \geq 4$, then X is always nonhyperelliptic.

Now by the same arguments as in §3.1, we can conclude that, for a line bundle M on Y, $\pi^* M$ is not normally generated if $H^0(X, M \otimes F^{-1}) = 0$. Therefore if $\pi^* M$ is normally generated, then $\deg M \geq g(X)$ or $M \cong F$. On the other hand, by Theorems 1 and 2, $\pi^* M$ is normally generated if $\deg M \geq g(X) + 1$ or $M \cong F$.

Now suppose $\deg M = g(X)$. By Lemma 1 (iii), we have

$$H^0(X, \pi^* M) \cong H^0(Y, M) \oplus H^0(Y, M \otimes F^{-1})$$

and

$$H^0(X, \pi^* M^2) \cong H^0(Y, M^2) \oplus H^0(Y, M^2 \otimes F^{-1}).$$

But by the Riemann-Roch theorem, we have $h^0(Y, M) = g$, $h^0(Y, M \otimes F^{-1}) = 1$ and $h^0(Y, M^2 \otimes F^{-1}) = g + 1$. Hence the natural map

$$H^0(Y, M) \otimes H^0(Y, M \otimes F^{-1}) \to H^0(Y, M^2 \otimes F^{-1})$$

is not surjective, and neither is

$$H^0(X, \pi^* M) \otimes H^0(X, \pi^* M) \to H^0(X, \pi^* M^2).$$

Therefore we conclude that, in this case, $\pi^* M$ is not normally generated.

Consequently we have:

PROPOSITION 4. Suppose $g(X) \geq 4$ and $g(Y) = 1$. Then, for a line bundle M on Y, $\pi^* M$ is normally generated if and only if $\deg M \geq g(X) + 1$ or $M \cong F$.

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