TSUKUBA J. MATH.
Vol. 22 No. 3 (1998), 783–793

# NORMALITY AND COLLECTIONWISE NORMALITY OF PRODUCT SPACES, II

By

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**Abstract.** We prove that the product space $X \times Y$ is collectionwise normal if and only if $X \times Y$ is normal in the following cases; (1) $X$ is a collectionwise normal $\Sigma$-space and $Y$ is a collectionwise normal first countable $P$-space, (2) $X$ is the closed image of a normal $M$-space and $Y$ is a collectionwise normal first countable $P$-space, (3) $X$ is the closed image of a paracompact $M$-space and $Y$ is a collectionwise normal $P$-space. In particular, (2) and (3) essentially generalize K. Chiba's theorems [3].

## 1. Introduction

Throughout this paper we assume all spaces to be Hausdorff, and all maps to be continuous. For two collectionwise normal spaces $X$ and $Y$, the result which asserts normality of $X \times Y$ implies its collectionwise normality has been proved in cases $Y$ being metrizable, Lasňev, a paracompact $M$-space and $\sigma$-locally compact paracompact by Okuyama [12], Hoshina [5], Rudin-Starbird [14] and Chiba [3], respectively. In a previous paper [16, Theorem 2.2], the author proved another case that the product of a paracompact $\Sigma$-space $X$ and a collectionwise normal $P$-space $Y$ is collectionwise normal if and only if it is normal; this extends Nagami's theorem [9] with $Y$ being a paracompact $\sigma$-space as well as affirmatively answers to the problem posed by Yang [17].

In [2], K. Chiba showed the following; for a collectionwise normal $\Sigma$-space $X$ and a paracompact first countable $P$-space $Y$, $X \times Y$ is collectionwise normal. If we weaken the paracompactness of $Y$ to the collectionwise normality, even the normality of $X \times Y$ need not be implied. Being suggested by these facts, we obtain the following result.

Received December 10, 1997.
Revised January 27, 1998.

THEOREM 1.1. *Let $X$ be a collectionwise normal $\Sigma$-space and $Y$ a collectionwise normal first countable $P$-space. Then the product $X \times Y$ is collectionwise normal if and only if $X \times Y$ is normal.*

In [3], K. Chiba showed that normality of $X \times Y$ implies its collectionwise normality in the following cases;

(A) $X$ is the closed image of a normal $M$-space and $Y$ is a paracompact first countable $P$-space,

(B) $X$ is the closed image of a paracompact first countable $M$-space and $Y$ is a collectionwise normal $\Sigma$-space.

In the proof of both cases, it needed to show at first the collectionwise normality of $Z \times Y$ which is essential to the proof, where $Z$ is an $M$-space assumed in (A) or (B). In this paper, we prove further the following two theorems which generalize the Chiba's results above; it should be noted that in our results even the normality of $Z \times Y$, where $Z$ is an $M$-space in our theorems, need not be implied from our cases.

THEOREM 1.2. *Let $X$ be the closed image of a normal $M$-space and $Y$ a collectionwise normal first countable $P$-space. Then the product $X \times Y$ is collectionwise normal if and only if $X \times Y$ is normal.*

THEOREM 1.3. *Let $X$ be the closed image of a paracompact $M$-space and $Y$ a collectionwise normal $P$-space. Then the product $X \times Y$ is collectionwise normal if and only if $X \times Y$ is normal.*

## 2. Preliminaries and key lemmas

Let $N$ be the set of all positive integers.

A space $Y$ is a *$P$-space* [8] if for any index set $\Omega$ and for any collection $\{G(\alpha_1, \dots, \alpha_n) \mid \alpha_1, \dots, \alpha_n \in \Omega; n \in N\}$ of open subsets of $Y$ such that

$$G(\alpha_1, \dots, \alpha_n) \subset G(\alpha_1, \dots, \alpha_n, \alpha_{n+1}) \quad \text{for } \alpha_1, \dots, \alpha_n, \alpha_{n+1} \in \Omega,$$

there exists a collection $\{F(\alpha_1, \dots, \alpha_n) \mid \alpha_1, \dots, \alpha_n \in \Omega; n \in N\}$ of closed subsets of $Y$ such that the conditions (a), (b) below are satisfied:

(a) $F(\alpha_1, \dots, \alpha_n) \subset G(\alpha_1, \dots, \alpha_n)$ for $\alpha_1, \dots, \alpha_n \in \Omega$,

(b) $Y = \bigcup\{G(\alpha_1, \dots, \alpha_n) \mid n \in N\} \Rightarrow Y = \bigcup\{F(\alpha_1, \dots, \alpha_n) \mid n \in N\}$.

A $\Sigma$-space [10] is a space $X$ having a sequence, called a *$\Sigma$-net*, $\{\mathscr{E}_n | n \in N\}$ of locally finite closed covers of $X$ which satisfies the following conditions:

(c) $\mathscr{E}_n$ is written as $\{E(\alpha_1, \dots, \alpha_n) \mid \alpha_1, \dots, \alpha_n \in \Omega\}$ with an index set $\Omega$,

(d) $E(\alpha_1, \dots, \alpha_n) = \bigcup\{E(\alpha_1, \dots, \alpha_n, \alpha_{n+1}) \mid \alpha_{n+1} \in \Omega\}$ for $\alpha_1, \dots, \alpha_n \in \Omega$,

(e) For every $x \in X$, $C(x)$ is countably compact, and there exists a sequence $\alpha_1, \alpha_2, \dots \in \Omega$ such that $C(x) \subset V$ with $V$ open implies $C(x) \subset E(\alpha_1, \dots, \alpha_n) \subset V$ for some $n$, where $C(x) = \bigcap\{E \mid y \in E \in \mathscr{E}_n,\ n \in N\}$. We call $\{E(\alpha_1, \dots, \alpha_n) \mid n \in N\}$ a *local net* of $C(x)$.

The definition of $M$-spaces is due to Morita [8].

The well-known facts are that every $M$-space is a $\Sigma$-space, every $\Sigma$-space is a $P$-space, every normal $P$-space is countably paracompact and every normal $M$-space is collectionwise normal (see Nagata [11]). We also note that not every closed image of a paracompact $M$-space is a $\Sigma$-space (see Gruenhage [4]).

The following lemma will be fundamental for the proof of our results.

LEMMA 2.1. *Let $X$ be a space. Suppose $\{K_\lambda \mid \lambda \in \Lambda\}$ a discrete collection of closed subsets of $X$ satisfies the following conditions* (i) *and* (ii) *below*:

(i) *for each $n \in N$ there exists a locally finite collection $\mathscr{O}_n = \{O_\lambda^n \mid \lambda \in \Lambda\}$ of open subsets of $X$ such that*

$$K_\lambda \subset \bigcup\{O_\lambda^n \mid n \in N\}$$

*for every $\lambda \in \Lambda$,*

(ii) *there exists an open subset $W_\lambda$ of $X$ such that*

$$K_\lambda \subset W_\lambda \quad \text{and} \quad \overline{W_\lambda} \cap \left(\bigcup\{K_\mu \mid \mu \neq \lambda, \mu \in \Lambda\}\right) = \varnothing$$

*for every $\lambda \in \Lambda$.*

*Then there exists a disjoint collection $\{Q_\lambda \mid \lambda \in \Lambda\}$ of open subsets of $X$ such that $K_\lambda \subset Q_\lambda$ for every $\lambda \in \Lambda$.*

PROOF. Suppose $\{K_\lambda \mid \lambda \in \Lambda\}$ satisfies the conditions (i) and (ii). Let $\{\mathscr{O}_n \mid n \in N\}$ and $\{W_\lambda \mid \lambda \in \Lambda\}$ be the collections described in the conditions (i) and (ii). Here we put

$$R_\lambda^n = O_\lambda^n \cap W_\lambda$$

for each $\lambda \in \Lambda$ and $n \in N$. Then $\{R_\lambda^n \mid \lambda \in \Lambda\}$ is a locally finite collection of open subsets of $X$ for every $n \in N$, which satisfies

$$\overline{R_\lambda^n} \cap \left(\bigcup\{K_\mu \mid \mu \neq \lambda, \mu \in \Lambda\}\right) = \varnothing$$

for each $\lambda \in \Lambda$, $n \in N$ and

$$K_\lambda \subset \bigcup\{R_\lambda^n \mid n \in N\}$$

for every $\lambda \in \Lambda$. We define

$$Q_\lambda = \bigcup\left\{R_\lambda^n - \bigcup\{\overline{R_\mu^i} \mid \mu \neq \lambda, \mu \in \Lambda; i \leq n\} \mid n \in N\right\}$$

for each $\lambda \in \Lambda$. Then we can easily show that $\{Q_\lambda \mid \lambda \in \Lambda\}$ is a disjoint collection of open subsets of $X$ such that $K_\lambda \subset Q_\lambda$ for every $\lambda \in \Lambda$. It completes the proof of Lemma 2.1. □

REMARK. $X$ is collectionwise normal if and only if $X$ is normal and every discrete collection $\{K_\lambda \mid \lambda \in \Lambda\}$ of closed subsets of $X$ satisfies the condition (i) of Lemma 2.1.

We need the following lemmas for our results. Our proofs of the lemmas are based on the proof of [16, Theorem 2.2].

LEMMA 2.2. *Let $X$ be a collectionwise normal $\Sigma$-space and $Y$ a collectionwise normal first countable $P$-space. Let $\{K_\lambda \mid \lambda \in \Lambda\}$ be a locally finite collection of closed subsets of $X \times Y$. Then $\{K_\lambda \mid \lambda \in \Lambda\}$ satisfies the condition* (i) *of Lemma* 2.1.

PROOF. Let $\{\mathscr{E}_n \mid n \in N\}$ be a $\Sigma$-net of $X$, where we express

$$\mathscr{E}_n = \{E(\alpha_1, \ldots, \alpha_n) \mid \alpha_1, \ldots, \alpha_n \in \Omega\}.$$

Since $X$ is collectionwise normal and countably paracompact, for each $n \in N$ there exists a locally finite open cover $\{L(\alpha_1, \ldots, \alpha_n) \mid \alpha_1, \ldots, \alpha_n \in \Omega\}$ of $X$ such that

$$E(\alpha_1, \ldots, \alpha_n) \subset L(\alpha_1, \ldots, \alpha_n) \tag{1}$$

for each $\alpha_1, \ldots, \alpha_n \in \Omega$.

Define $\Delta = \{\Gamma \subset \Lambda \mid \text{Card}\,\Gamma \text{ is finite}\}$. For each $\alpha_1, \ldots, \alpha_n \in \Omega$, $n \in N$ and $\Gamma \in \Delta$, let us put

$$G_\Gamma(\alpha_1, \ldots, \alpha_n) = \bigcup\Big\{O \mid O \text{ is open in } Y \text{ and}$$
$$(E(\alpha_1, \ldots, \alpha_n) \times O) \cap \left(\bigcup\{K_\lambda | \lambda \notin \Gamma\}\right) = \varnothing\Big\}.$$

Then $G_\Gamma(\alpha_1, \ldots, \alpha_n)$ is open in $Y$ and we have

$$G_\Gamma(\alpha_1, \ldots, \alpha_n) \subset G_\Gamma(\alpha_1, \ldots, \alpha_n, \alpha_{n+1})$$

for each $\alpha_1, \dots, \alpha_n, \alpha_{n+1} \in \Omega$ since $E(\alpha_1, \dots, \alpha_n) \supset E(\alpha_1, \dots, \alpha_n, \alpha_{n+1})$. Hence if we put

$$G(\alpha_1, \dots, \alpha_n) = \bigcup\{G_\Gamma(\alpha_1, \dots, \alpha_n) \mid \Gamma \in \Delta\},$$

then $G(\alpha_1, \dots, \alpha_n)$ is an open subset of $Y$ and we have

$$G(\alpha_1, \dots, \alpha_n) \subset G(\alpha_1, \dots, \alpha_n, \alpha_{n+1})$$

for each $\alpha_1, \dots, \alpha_n, \alpha_{n+1} \in \Omega$. Since $Y$ is a $P$-space, there exists a collection

$$\{M(\alpha_1, \dots, \alpha_n) \mid \alpha_1, \dots, \alpha_n \in \Omega;\ n \in N\}$$

of closed subsets of $Y$ such that

$$M(\alpha_1, \dots, \alpha_n) \subset G(\alpha_1, \dots, \alpha_n) \tag{2}$$

for each $\alpha_1, \dots, \alpha_n \in \Omega$; $n \in N$, and

$$Y = \bigcup\{G(\alpha_1, \dots, \alpha_n) \mid n \in N\} \Longrightarrow Y = \bigcup\{M(\alpha_1, \dots, \alpha_n) \mid n \in N\}. \tag{3}$$

Here we may assume

$$M(\alpha_1, \dots, \alpha_n) \subset M(\alpha_1, \dots, \alpha_n, \alpha_{n+1}) \tag{4}$$

for each $\alpha_1, \dots, \alpha_n, \alpha_{n+1} \in \Omega$.

Fix $n \in N$ and $\alpha_1, \dots, \alpha_n \in \Omega$ arbitrarily. Define

$$P_\Gamma(\alpha_1, \dots, \alpha_n) = \{y \in Y \mid (E(\alpha_1, \dots, \alpha_n) \times \{y\}) \cap K_\lambda \neq \varnothing \text{ if and only if } \lambda \in \Gamma\}$$

for each $\Gamma \in \Delta$. We show that the collection

$$\{M(\alpha_1, \dots, \alpha_n) \cap P_\Gamma(\alpha_1, \dots, \alpha_n) \mid \Gamma \in \Delta\} \tag{5}$$

is locally finite in $Y$. To prove this, let $y \in Y$ and we show that the above collection is locally finite at $y$. Since $M(\alpha_1, \dots, \alpha_n)$ is closed in $Y$, we may assume that $y \in M(\alpha_1, \dots, \alpha_n)$. Then by (2) we have $y \in G(\alpha_1, \dots, \alpha_n)$, and hence there exists a $\Gamma_y \in \Delta$ such that $y \in G_{\Gamma_y}(\alpha_1, \dots, \alpha_n)$. Suppose

$$G_{\Gamma_y}(\alpha_1, \dots, \alpha_n) \cap P_\Gamma(\alpha_1, \dots, \alpha_n) \neq \varnothing,$$

then we show $\Gamma \subset \Gamma_y$. To show this, let a $\lambda \in \Gamma$. Select a point

$$z \in G_{\Gamma_y}(\alpha_1, \dots, \alpha_n) \cap P_\Gamma(\alpha_1, \dots, \alpha_n).$$

Since $z \in P_\Gamma(\alpha_1, \dots, \alpha_n)$, we have

$$(E(\alpha_1, \dots, \alpha_n) \times \{z\}) \cap K_\lambda \neq \varnothing,$$

so

$$(E(\alpha_1,\dots,\alpha_n)\times G_{\Gamma_y}(\alpha_1,\dots,\alpha_n))\cap K_\lambda\neq\varnothing. \tag{6}$$

By the definition of $G_{\Gamma_y}(\alpha_1,\dots,\alpha_n)$, we have

$$(E(\alpha_1,\dots,\alpha_n)\times G_{\Gamma_y}(\alpha_1,\dots,\alpha_n))\cap\left(\bigcup\{K_\mu\mid\mu\notin\Gamma_y\}\right)=\varnothing. \tag{7}$$

The formulations (6) and (7) show $\lambda\in\Gamma_y$, it follows that $\Gamma\subset\Gamma_y$. And since $G_{\Gamma_y}(\alpha_1,\dots,\alpha_n)$ is an open neighborhood of $y$ and Card $\Gamma_y$ is finite, we have shown that the collection (5) above is localy finite at $y$.

Since $Y$ is collectionwise normal and countably paracompact, there exists a locally finite collection $\{H_\Gamma(\alpha_1,\dots,\alpha_n)\mid\Gamma\in\Delta\}$ of open subsets of $Y$ such that

$$M(\alpha_1,\dots,\alpha_n)\cap P_\Gamma(\alpha_1,\dots,\alpha_n)\subset H_\Gamma(\alpha_1,\dots,\alpha_n)$$

for each $\Gamma\in\Delta$.

We define

$$O_\lambda^n=\bigcup\{L(\alpha_1,\dots,\alpha_n)\times H_\Gamma(\alpha_1,\dots,\alpha_n)\mid\Gamma\in\Delta\text{ and }\lambda\in\Gamma;\ \alpha_1,\dots,\alpha_n\in\Omega\},$$

for each $n\in N$ and $\lambda\in\Lambda$. Then we can see that $\{O_\lambda^n\mid\lambda\in\Lambda\}$ is a locally finite collection of open subsets of $X\times Y$ for each $n\in N$.

We shall show that $K_\lambda\subset\bigcup\{O_\lambda^n\mid n\in N\}$ for every $\lambda\in\Lambda$. To see this, let $(x,y)\in K_\lambda$. We choose $\alpha_1,\alpha_2,\ldots\in\Omega$ so that

$$\{E(\alpha_1,\dots,\alpha_n)\mid n\in N\}\text{ is a local net of }C(x).$$

Before everything, we show that $Y=\bigcup\{G(\alpha_1,\dots,\alpha_n)\mid n\in N\}$. Let any $z\in Y$. Put

$$\Gamma_{xz}=\{\mu\in\Lambda\mid(C(x)\times\{z\})\cap K_\mu\neq\varnothing\}.$$

Since $C(x)\times\{z\}$ is countably compact and $\{K_\mu\mid\mu\in\Lambda\}$ is locally finite, $\Gamma_{xz}$ is finite, that is, $\Gamma_{xz}\in\Delta$. Moreover, since $C(x)\times\{z\}$ is countably compact and $Y$ is first countable, there exist open subsets $O$ and $O'$ of $X$ and $Y$, respectively, such that

$$C(x)\times\{z\}\subset O\times O'\subset X\times Y-\bigcup\{K_\mu\mid\mu\notin\Gamma_{xz}\}.$$

From the property of the local net, there exists an $n\in N$ such that

$$C(x)\subset E(\alpha_1,\dots,\alpha_n)\subset O.$$

Therefore

$$(E(\alpha_1,\dots,\alpha_n)\times O')\cap\left(\bigcup\{K_\mu\mid\mu\notin\Gamma_{xz}\}\right)=\varnothing.$$

Thus we can verify $z \in O' \subset G_{\Gamma_{xz}}(\alpha_1, \dots, \alpha_n) \subset G(\alpha_1, \dots, \alpha_n)$. Hence we have $Y = \bigcup\{G(\alpha_1, \dots, \alpha_n) \mid n \in N\}$.

It follows from (3) that

$$Y = \bigcup\{M(\alpha_1, \dots, \alpha_n) \mid n \in N\}.$$

So there exists an $n \in N$ such that $y \in M(\alpha_1, \dots, \alpha_n)$. Let

$$\Gamma_{xy} = \{\mu \in \Lambda \mid (C(x) \times \{y\}) \cap K_\mu \neq \varnothing\}.$$

Likewise the matter shown above, we have $\Gamma_{xy} \in \Delta$, and there exist open subsets $O_x$ and $O_y$ of $X$ and $Y$, respectively, such that

$$C(x) \times \{y\} \subset O_x \times O_y \subset X \times Y - \bigcup\{K_\mu \mid \mu \notin \Gamma_{xy}\}.$$

From the property of the local net, there exists an $m \in N$ such that

$$C(x) \times \{y\} \subset E(\alpha_1, \dots, \alpha_m) \times \{y\} \subset O_x \times O_y,$$

where we can select $m \geq n$. So we have

$$(8) \qquad (E(\alpha_1, \dots, \alpha_m) \times \{y\}) \cap (\bigcup\{K_\mu \mid \mu \notin \Gamma_{xy}\}) = \varnothing.$$

Moreover, by the definition of $\Gamma_{xy}$ and the fact $C(x) \subset E(\alpha_1, \dots, a_m)$, we have

$$(9) \qquad (E(\alpha_1, \dots, \alpha_m) \times \{y\}) \cap K_\mu \neq \varnothing \quad \text{for every } \mu \in \Gamma_{xy}.$$

It follows from the formulations (8) and (9) that

$$y \in P_{\Gamma_{xy}}(\alpha_1, \dots, \alpha_m).$$

By (4), $y \in M(\alpha_1, \dots, \alpha_n) \subset M(\alpha_1, \dots, \alpha_m)$. So we have

$$(10) \qquad y \in M(\alpha_1, \dots, \alpha_m) \cap P_{\Gamma_{xy}}(\alpha_1, \dots, \alpha_m) \subset H_{\Gamma_{xy}}(\alpha_1, \dots, \alpha_m).$$

It follows from (1) and (10) that

$$(x, y) \in L(\alpha_1, \dots, \alpha_m) \times H_{\Gamma_{xy}}(\alpha_1, \dots, \alpha_m).$$

Since $(x, y) \in K_\lambda$, it is clear that $\lambda \in \Gamma_{xy}$. Thus we have $(x, y) \in O_\lambda^m$, which proves that $K_\lambda \subset \bigcup\{O_\lambda^n \mid n \in N\}$. It follows that $\{K_\lambda \mid \lambda \in \Lambda\}$ satisfies the condition (i) of Lemma 2.1. This completes the proof of the lemma. $\square$

If $C(x)$ in the proof of Lemma 2.2 is compact, then without using the first countability of $Y$ the similar proof to the above yields the following lemma.

LEMMA 2.3. *Let $X$ be a paracompact $\Sigma$-space and $Y$ a collectionwise normal $P$-space. Let $\{K_\lambda \mid \lambda \in \Lambda\}$ be a locally finite collection of closed subsets of $X \times Y$. Then $\{K_\lambda \mid \lambda \in \Lambda\}$ satisfies the condition* (i) *of Lemma* 2.1.

## 3. The proofs of Theorems

Let us proceed to the proofs of Theorems.

PROOF OF THEOREM 1.1. It is shown by Remark of Lemma 2.1 and Lemma 2.2. □

PROOF OF THEOREM 1.2. Let $Z$ be a normal $M$-space and $f$ a closed continuous map from $Z$ onto $X$. Then we can express that $X = \bigcup\{X_i \mid i \geq 0\}$, where $X_i$ is closed discrete for every $i \geq 1$ and $f^{-1}(x)$ is countably compact for each $x \in X_0$ (see Nagata [11]).

First we observe that for a subset $A$ of $Z \times Y$ the following equality holds:

$$(f \times 1_Y)(\bar{A}) \cap (X_0 \times Y) = \overline{(f \times 1_Y)(A)} \cap (X_0 \times Y). \tag{11}$$

Let $\{D_\lambda \mid \lambda \in \Lambda\}$ be a discrete collection of closed subsets of $X \times Y$. For each $i \geq 1$, we can take a discrete collection $\{U_\lambda^i \mid \lambda \in \Lambda\}$ of open subsets of $X \times Y$ such that

$$D_\lambda \cap (X_i \times Y) \subset U_\lambda^i$$

for every $\lambda \in \Lambda$, because $X$ and $Y$ are collectionwise normal.

Let

$$F_\lambda = D_\lambda - \bigcup\{U_\lambda^i \mid i \geq 1\}$$

for each $\lambda \in \Lambda$. Then $F_\lambda$ is closed in $X \times Y$ and we have $F_\lambda \subset X_0 \times Y$.

Here $\{(f \times 1_Y)^{-1}(F_\lambda) \mid \lambda \in \Lambda\}$ is a discrete collection of closed subsets of $Z \times Y$.

Since $Z$ is a normal $M$-space (therefore a collectionwise normal $\Sigma$-space), it follows that $\{(f \times 1_Y)^{-1}(F_\lambda) \mid \lambda \in \Lambda\}$ satisfies the condition (i) of Lemma 2.1 because of Lemma 2.2.

Since $X \times Y$ is normal, for each $\lambda \in \Lambda$, there exists an open subset $W_\lambda$ of $X \times Y$ such that

$$F_\lambda \subset W_\lambda \subset \overline{W_\lambda} \subset X \times Y - \bigcup\{F_\mu \mid \mu \neq \lambda, \mu \in \Lambda\}.$$

Then $\{(f \times 1_Y)^{-1}(W_\lambda) \mid \lambda \in \Lambda\}$ is the required collection for $\{(f \times 1_Y)^{-1}(F_\lambda) \mid \lambda \in \Lambda\}$ to satisfy the condition (ii) of Lemma 2.1. Hence $\{(f \times 1_Y)^{-1}(F_\lambda) \mid \lambda \in \Lambda\}$ satisfies the condition (ii) of Lemma 2.1.

By Lemma 2.1, we can take a disjoint collection $\{Q_\lambda \mid \lambda \in \Lambda\}$ of open subsets of $Z \times Y$ such that

$$(f \times 1_Y)^{-1}(F_\lambda) \subset Q_\lambda$$

for each $\lambda \in \Lambda$.

Define

$$V_\lambda = X \times Y - \overline{(f \times 1_Y)(Z \times Y - Q_\lambda)}$$

for each $\lambda \in \Lambda$. It is clear that $\{V_\lambda \mid \lambda \in \Lambda\}$ is a disjoint collection of open subsets of $X \times Y$. Since $F_\lambda \subset X_0 \times Y$, we can show that

$$F_\lambda \subset V_\lambda$$

for each $\lambda \in \Lambda$ by (11). By the normality of $X \times Y$, there exists a discrete collection $\{U_\lambda^0 \mid \lambda \in \Lambda\}$ of open subsets of $X \times Y$ such that

$$F_\lambda \subset U_\lambda^0 \subset V_\lambda$$

for each $\lambda \in \Lambda$.

The collection $\{U_\lambda^i \mid \lambda \in \Lambda, i \geq 0\}$ has the properties that

$$D_\lambda \subset \bigcup\{U_\lambda^i \mid i \geq 0\}$$

for each $\lambda \in \Lambda$ and that $\{U_\lambda^i \mid \lambda \in \Lambda\}$ is discrete for each $i \geq 0$. Namely $\{D_\lambda \mid \lambda \in \Lambda\}$ satisfies the condition (i) of Lemma 2.1, and also satisfies the condition (ii) of Lemma 2.1 because of the normality of $X \times Y$. Therefore we can get a disjoint collection $\{B_\lambda \mid \lambda \in \Lambda\}$ of open subsets of $X \times Y$ such that $D_\lambda \subset B_\lambda$ for each $\lambda \in \Lambda$ by Lemma 2.1. Hence $X \times Y$ is collectionwise normal, which completes the proof of Theorem 1.2. □

Proof of Theorem 1.3. Let $Z$ be a paracompact $M$-space and $f$ a closed continuous map from $Z$ onto $X$. We can express $X = \bigcup\{X_i \mid i \geq 0\}$ that has the properties of the proof of Theorem 1.2. Let $\{D_\lambda \mid \lambda \in \Lambda\}$ be a discrete collection of closed subsets of $X \times Y$. The proof of this theorem is similar to that of Theorem 1.2, because we can use Lemma 2.3 instead of Lemma 2.2. So we can take a discrete collection $\{U_\lambda^i \mid \lambda \in \Lambda\}$ of open subsets of $X \times Y$ for each $i \geq 0$ such that

$$D_\lambda \subset \bigcup\{U_\lambda^i \mid i \geq 0\}$$

for each $\lambda \in \Lambda$ as the proof of Theorem 1.2. Hence $\{D_\lambda \mid \lambda \in \Lambda\}$ satisfies the condition (i) of Lemma 2.1. Since $X \times Y$ is normal, $\{D_\lambda \mid \lambda \in \Lambda\}$ satisfies the condition (ii) of Lemma 2.1. Therefore we can get a disjoint collection $\{B_\lambda \mid \lambda \in \Lambda\}$ of open subsets of $X \times Y$ such that $D_\lambda \subset B_\lambda$ for each $\lambda \in \Lambda$. Hence $X \times Y$ is collectionwise normal, which completes the proof of Theorem 1.3. □

REMARK. Since $\omega_1$ is a collectionwise normal first countable $M$-space and $\omega_1 + 1$ is a compact space, the assumption of the normality of $X \times Y$ of Theorems 1.1, 1.2 and 1.3 can not be dropped.

## 4. Some problems

It is well-known by Nagami [10, Corollary 4.2] that for a paracompact $\Sigma$-space $X$ and a paracompact $P$-space $Y$ the product $X \times Y$ is paracompact. If we replace the paracompactness of $X$ and $Y$ by the collectionwise normality, then even the normality of $X \times Y$ need not implied in general. Thus the following problem naturally arises. [16, Theorem 2.2] or Theorem 1.1 can be regarded as a partial answer to this problem.

PROBLEM 4.1. *Let $X$ be a collectionwise normal $\Sigma$-space and $Y$ a collectionwise normal (or a paracompact) $P$-space. Is it true that $X \times Y$ is collectionwise normal if and only if $X \times Y$ is normal?*

Corresponding to Nagami's result above, we have the following theorem.

THEOREM 4.2. *Let $X$ be the closed image of a paracompact $M$-space and $Y$ a paracompact $P$-space. Then $X \times Y$ is paracompact if and only if $X \times Y$ normal.*

PROOF. First we note the fact that for spaces $X$ and $Y$ given in the theorem $X \times Y$ is normal iff $X \times Y$ is countably paracompact; the proof is similar to Bešlagić and Chiba [1, Section 5]. Assume $X \times Y$ is normal, therefore it is countably paracompact. Let $K$ be a compact space. Then $Y \times K$ is a paracompact $P$-space, and by the fact above the countably paracompact space $(X \times Y) \times K = X \times (Y \times K)$ is normal. Hence $X \times Y$ is paracompact from Tamano's theorem [15, Theorem 2]. □

In view of this result, under the similar consideration to Problem 4.1, the following problem also arises. A partial answer to this problem is Theorem 1.2 or Theorem 1.3.

PROBLEM 4.3. *Let $X$ be the closed image of a normal $M$-space and $Y$ a collectionwise normal* (*or a paracompact*) *$P$-space. Is it true that $X \times Y$ is collectionwise normal if and only if $X \times Y$ is normal*?

## References

[1] A. Bešlagić and K. Chiba, Normality of product spaces, Fund. Math. **128** (1987), 99–112.
[2] K. Chiba, On collectionwise normality of product spaces, I, Proc. Amer. Math. Soc. **91** (1984), 649–652.
[3] ———, On collectionwise normality of product spaces, II, Proc. Amer. Math. Soc. **91** (1984), 653–657.
[4] G. Gruenhage, Generalized metric spaces, in: Handbook of Set-Theoretic Topology, ed. by K. Kunen and J. E. Vaughan, North Holland (1984), 423–501.
[5] T. Hoshina, Products of normal spaces with Lasňev spaces, Fund. Math. **124** (1984), 143–153.
[6] ———, Normality of product spaces, II, in: Topics in Gen. Top., K. Morita and J. Nagata, eds., North-Holland (1989), 121–160.
[7] ——— and K. Yamazaki, Extensions of functions on product spaces, Topology Appl. **82** (1998), 195–204.
[8] K. Morita, Products of normal spaces with metric spaces, Math. Ann. **154** (1964), 365–382.
[9] K. Nagami, $\sigma$-spaces and product spaces, Math. Ann. **181** (1969), 109–118.
[10] ———, $\Sigma$-spaces, Fund. Math. **65** (1969), 169–192.
[11] J. Nagata, Modern General Topology, North-Holland, 1985.
[12] A. Okuyama, Some generalizations of metric spaces, their metrization theorems and product spaces, Sci. Rep. Tokyo Kyoiku Daigaku Sect. A **8** (1967), 236–254.
[13] T. C. Przymusiński, Products of normal spaces, in: Handbook of Set-Theoretic Topology, ed. by K. Kunen and J. E. Vaughan, North Holland (1984), 781–826.
[14] M. E. Rudin and M. Starbird, Products with a metric factor, Gen. Top. Appl. **5** (1975), 235–248.
[15] H. Tamano, On paracompactness, Pacific. J. Math. **10** (1960), 1043–1047.
[16] K. Yamazaki, Normality and collectionwise normality of product spaces, Topology Proc. **22** (to appear).
[17] L. Yang, Normality in product spaces and covering properties, Doctor Thesis, Univ. Tsukuba (1995).

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