TSUKUBA J. MATH.
Vol. 22 No. 3 (1998), 747–768

ON GLOBAL SMALL SOLUTIONS OF NONLINEAR TIMOSHENKO'S TYPE EQUATIONS

By

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1. Introduction

This paper is concerned with the global existence of small, regular solutions to the Timoshenko's type equation in $\boldsymbol{R}_t \times \boldsymbol{R}_x^n$

$$u_{tt} + a(\|D^{\beta_1}u\|_{L^2}^2, \dots, \|D^{\beta_N}u\|_{L^2}^2)\Delta^2 u = m(\|D^{\beta_1}u\|_{L^2}^2, \dots, \|D^{\beta_N}u\|_{L^2}^2)\Delta u, \tag{1.1}$$

which appears in various models in the study of nonlinear vibrations of beams and plates.

We assume that the functions $a(s)$, $m(s) : \boldsymbol{R}^N \to \boldsymbol{R}$ (with $s = (s_1, \dots, s_N)$) satisfy a *strict hyperbolicity* condition

$$\begin{cases} a(s), m(s) \in \mathbf{C}^1([-\delta,\delta]^N) \\ a(s), m(s) \geq \eta > 0, \qquad \forall s \in [-\delta,\delta]^N. \end{cases} \tag{1.2}$$

Moreover, for the multi-indices $\beta_\nu = (\beta_{\nu,t}, \beta_{\nu,x}) \in \boldsymbol{N} \times \boldsymbol{N}^n$ we require that

$$|\beta_{\nu,t}| \leq 1, \quad |\beta_\nu| \geq 1 \quad \text{for } 1 \leq \nu \leq N. \tag{1.3}$$

Then, we shall prove the following:

THEOREM 1.1. *Let $n \geq 2$. Under the above hypotheses, the Cauchy problem*

$$\begin{aligned} &u_{tt} + a(\|D^{\beta_1}u\|_{L^2}^2, \dots, \|D^{\beta_N}u\|_{L^2}^2)\Delta^2 u = m(\|D^{\beta_1}u\|_{L^2}^2, \dots, \|D^{\beta_N}u\|_{L^2}^2)\Delta u, \\ &u(0,x) = \varepsilon \cdot u_0(x), \quad u_t(0,x) = \varepsilon \cdot u_1(x), \end{aligned} \tag{1.4}$$

where $u_0(x), u_1(x) \in \mathbf{C}_0^\infty(\boldsymbol{R}_x^n)$, has a unique classical solution $u(t,x) \in \mathbf{C}^2(\boldsymbol{R}_t; H^\infty(\boldsymbol{R}_x^n))$ provided the parameter ε is small enough, namely $|\varepsilon| \leq \varepsilon_0$ for a suitable $\varepsilon_0 = \varepsilon_0(u_0, u_1) > 0$.

Received November 14, 1997.

Results on the existence of classical global solutions for equations of type (1.1) are proved by G. Perla Menzala [15] for the evolution equation

$$u_{tt} + \Delta^2 u - M(\|\nabla u(t,\cdot)\|_{L^2}^2)\Delta u = 0 \quad \text{in } \boldsymbol{R}_t \times \boldsymbol{R}_x^n, \tag{1.5}$$

assuming $M(\lambda) \in \mathbf{C}^1(\boldsymbol{R}^+)$, $M(\lambda) \geq \lambda_0 > 0$, but without restrictions on the size of the initial date $u(0,x)$, $u_t(0,x)$.

In fact, thanks to the special form of the nonlinearity, setting $F(\lambda) = \int_0^\lambda M(z)\,dz$ it is easy to verify that the quantity

$$\mathscr{E}(t) = \int |u_t|^2\,dx + \int |\Delta u|^2\,dx + F(\|\nabla u\|_{L^2}^2) \tag{1.6}$$

is constant, that is $\mathscr{E}(t) = \mathscr{E}(0)$ $\forall t \in \boldsymbol{R}_t$, for any *sufficiently regular* solution $u(t,x)$ of equation (1.5). This immediately gives the inequality

$$\left|\frac{d}{dt}M(\|\nabla u\|_{L^2}^2)\right| \leq \mathscr{E}(0) \sup_{0 \leq \lambda \leq \mathscr{E}(0)/\lambda_0} |M'(\lambda)| = C. \tag{1.7}$$

Defining for $\alpha \in \boldsymbol{N}^n$ the energies

$$E_\alpha(t) = \int |\partial_x^\alpha u_t|^2\,dx + \int |\Delta\partial_x^\alpha u|^2\,dx + M(\|\nabla u\|_{L^2}^2)\int |\nabla\partial_x^\alpha u|^2\,dx, \tag{1.8}$$

estimate (1.7) implies that

$$E_\alpha(t) \leq E_\alpha(0)e^{C|t|} \quad \forall t \in \boldsymbol{R}_t \quad \text{and} \quad \forall \alpha \in \boldsymbol{N}^n, \tag{1.9}$$

where C is the constant obtained in (1.7). Then, using the *a priori* estimates (1.9) it is not difficult to prove that equation (1.5) is globally solvable, provided the initial data $u(0,x)$, $u_t(0,x)$ belong to some $H^k(\boldsymbol{R}_x^n)$ with $k \geq 0$ large enough.

Clearly, in the case of more general nonlinearity, like in (1.1), this argument does not work because we are not able to prove an *a priori* estimate like (1.7).

REMARK 1.1. For $n = 1$ we are able to prove Theorem 1.1 only in the case

$$m(s_1,\dots,s_N) = \mu a(s_1,\dots,s_N)$$

for some constant $\mu > 0$. This is due to a technical difficulty. We suspect that the result should be valid without this restriction. □

We shall treat problem (1.4) by the Fourier transform and the energy estimates. As a by-product of these methods of proof we have:

Corollary 1.1. *Let $n \geq 2$. Then the solution $u(t,x)$ of the Cauchy problem* (1.4) *satisfies the estimate*

$$\left| \frac{d}{dt} \int_{\boldsymbol{R}_x^n} |D^\beta u(t,x)|^2 \, dx \right| \leq \frac{C\varepsilon^2}{1+|t|^{n+1}} \tag{1.10}$$

as $|t| \to \infty$, *for any* $\beta = (\beta_t, \beta_x)$ *with* $|\beta_t| \leq 1$, $|\beta| \geq 1$.

Let us briefly recall that the problem of the global solvability for nonlinear evolution equations with *non local nonlinearities* has been extensively studied in the last years, starting from the paper of Greenberg and Hu [10] where the case of the classical Kirchhoff equation [11] in *one* space dimension,

$$u_{tt} - \left(1 + \int_{\boldsymbol{R}_x} u_x(t,x)^2 \, dx\right) u_{xx} = 0,$$

was considered. More recently, the result of [10] was generalized by D'Ancona and Spagnolo [5], [6], [7]. In particular, in [5] they studied the Cauchy problem in $\boldsymbol{R}_t \times \boldsymbol{R}_x^n$

$$\begin{gathered} u_{tt} + (-1)^m \sum_{|\alpha|=2m} f_\alpha(\|D_x^{\beta_1} u\|_{L^2}^2, \dots, \|D_x^{\beta_N} u\|_{L^2}^2) D_x^\alpha u = 0 \\ u(0,x) = \varepsilon \cdot u_0(x), \quad u_t(0,x) = \varepsilon \cdot u_1(x) \end{gathered} \tag{1.11}$$

assuming that $u_0(x)$, $u_1(x) \in \mathbf{C}_0^\infty(\boldsymbol{R}_x^n)$; the multi-indices $\beta_i \in \boldsymbol{N}^n$ satisfy the condition

$$|\beta_i| > m - n/2 \quad 1 \leq i \leq N;$$

the $f_\alpha(\lambda_1, \dots, \lambda_N)$ are $\mathbf{C}^2$ real functions on $\boldsymbol{R}^N$ satisfying the *strict hyperbolicity* condition

$$\sum_{|\alpha|=2m} f_\alpha(\lambda_1, \dots, \lambda_N) \xi^\alpha \geq \lambda |\xi|^{2m} \quad (\lambda > 0). \tag{1.12}$$

Under these assumptions, they proved that the initial value problem (1.11) is globally solvable provided the parameter $\varepsilon \in \boldsymbol{R}$ is small enough.

Unfortunately, the technique of [5] does not seem to be directly applicable to the case of the Timoshenko equation (1.1) because of the lower order term.

On the other hand, in [3] the global solvability (for small data) was proved for general $N \times N$ strictly hyperbolic systems with non local nonlinearities. More precisely, the Cauchy problem in $\boldsymbol{R}_t \times \boldsymbol{R}_x^n$

$$U_t = \sum_{i=1}^{n} A_i(s(t))U_{x_i} + s'(t)B(s(t))U, \quad U(0,x) = U_0(x), \tag{1.13}$$

$$s(t) = \int_{R_x^n} \overline{{}^tU(t,x)}SU(t,x)\,dx \quad \text{with } S \text{ self-adjoint}, \tag{1.14}$$

is globally solvable for every small, smooth initial data $U_0(x)$, provided the $N \times N$ matrix $A(s,\xi) = \sum_{i=1}^{n} A_i(s)(\xi_i/|\xi|)$ has N real and distinct eigenvalues $\lambda_1(s,\xi), \dots, \lambda_N(s,\xi)$ for all $\xi \in \boldsymbol{R}_\xi^n \backslash \{0\}$ and $|s| \le \delta$ ($\delta > 0$); $A_i(s)$, $B(s)$ are smooth $N \times N$ matrices.

In this paper, improving the the ideas of [3], we show that the scheme used to prove the global solvability of problem (1.13), (1.14) can be applied to more general situations. For example, by methods of the proof of Theorem 1.1 we can easily consider:

(i) systems of Timoshenko's type equations,

$$\partial_t^2 u_i + a_i(\|D^{\beta_\nu} u_j\|_{L^2}^2)\Delta^2 u_i = m_i(\|D^{\beta_\nu} u_j\|_{L^2}^2)\Delta u_i, \quad 1 \le i \le L \tag{1.15}$$

under conditions similar to (1.2), (1.3);

(ii) higher order hyperbolic equations such as

$$\partial_t^2 u + \sum_{k=1}^{q} (-1)^k a_k(\|D^{\beta_\nu} u\|_{L^2}^2)\Delta^k u = 0 \quad \text{in } \boldsymbol{R}_t \times \boldsymbol{R}_x^n, \tag{1.16}$$

$n \ge 2$, assuming that $a_k(s) \in \mathbf{C}^1$, $a_k(s) \ge \eta > 0$ for $1 \le k \le q$ and

$$a_k(s) = \mu_k a_q(s) \quad \text{for } k > q/2,$$

with $\mu_k \in \boldsymbol{R}\backslash\{0\}$.

2. Reduction to an Equivalent Problem

Let us consider the linear equation

$$u_{tt} + \tilde{a}(t)\Delta^2 u - \tilde{m}(t)\Delta u = 0 \tag{2.1}$$

where $\tilde{a}(t)$, $\tilde{m}(t) \ge \eta > 0$ are bounded $\mathbf{C}^1$ functions. By Fourier transform in the space variables and defining the vector

$$U(t,\xi) \overset{\text{def}}{=} {}^t(|\xi|\hat{u}(t,\xi), \hat{u}_t(t,\xi)),$$

we obtain the ordinary system

$$\frac{\partial U}{\partial t} = i|\xi| \begin{pmatrix} 0 & -i \\ i\tilde{\alpha}(t,|\xi|) & 0 \end{pmatrix} U \overset{\text{def}}{=} i|\xi| A(t,|\xi|) U \tag{2.2}$$

with

$$\tilde{\alpha}(t,|\xi|) \stackrel{\text{def}}{=} |\xi|^2\tilde{a}(t) + \tilde{m}(t). \tag{2.3}$$

Since $\tilde{\alpha}(t,|\xi|) \geq \eta$ the matrix $A(t,|\xi|)$ in (2.2) has real and distinct eigenvalues

$$\Lambda_1 = -\sqrt{\tilde{\alpha}(t,|\xi|)}, \quad \Lambda_2 = \sqrt{\tilde{\alpha}(t,|\xi|)}, \tag{2.4}$$

hence $A(t,|\xi|)$ is *uniformly diagonalizable* for all $\xi \in \boldsymbol{R}^n_\xi$. In fact, setting

$$U(t,\xi) = \mathcal{N}(t,|\xi|)V(t,\xi) \tag{2.5}$$

where

$$\mathcal{N}(t,|\xi|) = \begin{pmatrix} 1 & 1 \\ -i\sqrt{\tilde{\alpha}} & i\sqrt{\tilde{\alpha}} \end{pmatrix} \quad \text{and} \quad \mathcal{N}(t,|\xi|)^{-1} = \frac{1}{2i\sqrt{\tilde{\alpha}}}\begin{pmatrix} i\sqrt{\tilde{\alpha}} & -1 \\ i\sqrt{\tilde{\alpha}} & 1 \end{pmatrix}, \tag{2.6}$$

we easily find the system

$$\frac{\partial V}{\partial t} = i|\xi|DV + \frac{\tilde{\alpha}_t}{4\tilde{\alpha}}BV \tag{2.7}$$

with principal part in the *diagonal* form,

$$D = \begin{pmatrix} -\sqrt{\tilde{\alpha}} & 0 \\ 0 & \sqrt{\tilde{\alpha}} \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \tag{2.8}$$

and where $(\tilde{\alpha}_t/4\tilde{\alpha})B = -\mathcal{N}^{-1}\partial_t\mathcal{N}$.

Let us remark that, having $\tilde{a}(t)$, $\tilde{m}(t) \geq \eta$ in the definition of $\tilde{\alpha}(t,|\xi|)$, the coefficients of system (2.7) are regular functions of $\xi \in \boldsymbol{R}^n_\xi$. In particular, the derivatives of any order of $\sqrt{\tilde{\alpha}}$ with respect to $\rho = |\xi|$ are uniformly bounded on $\boldsymbol{R}^n_\xi$.

3. The Fixed Point Argument

Let us consider now the equation (1.1) with the assumptions (1.2) on $a(s)$, $m(s)$ ($s \in \boldsymbol{R}^N$) and (1.3) on the multi-indices β_ν for $1 \leq \nu \leq N$. Thanks to the results of §2, the nonlinear Cauchy problem

$$\begin{gathered} u_{tt} + a\left(\int_{\boldsymbol{R}^n_x} |D^{\beta_\nu}u|^2\,dx\right)\Delta^2 u + m\left(\int_{\boldsymbol{R}^n_x} |D^{\beta_\nu}u|^2\,dx\right)\Delta u = 0, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x) \end{gathered} \tag{3.1}$$

is equivalent to the following:

$$\frac{\partial V}{\partial t} = i|\xi|\begin{pmatrix} -\sqrt{\alpha} & 0 \\ 0 & \sqrt{\alpha} \end{pmatrix} V + \frac{\alpha_t}{4\alpha} BV$$
$$V(0,\xi) = V_0(\xi) \tag{3.2}$$

with the conditions

$$\alpha(s(t),|\xi|) = |\xi|^2 a(s(t)) + m(s(t)),$$
$$V_0(\xi) = \mathcal{N}(s(0),|\xi|)^{-1} U_0(\xi) \tag{3.3}$$

where

$$s(t) = (s_1(t),\dots,s_N(t)) \quad \text{with } s_\nu(t) = \int_{\boldsymbol{R}_x^n} |D^{\beta_\nu} u(t,x)|^2 \, dx \tag{3.4}$$

for $1 \le \nu \le N$,

$$U_0(\xi) = {}^t(|\xi|\hat{u}_0(\xi), \hat{u}_1(\xi));$$

$\mathcal{N}(s(t),|\xi|)$ is the 2×2 matrix defined in (2.6), with $\tilde{\alpha}(t,|\xi|)$ replaced by $\alpha(s(t),|\xi|)$. Finally, from the properties of the Fourier transform, we have the relations

$$s_\nu(t) = \int_{\boldsymbol{R}_\xi^n} {}^t\overline{V}(t,\xi) S_\nu(t,\xi) V(t,\xi) \, d\xi, \quad 1 \le \nu \le N, \tag{3.5}$$

where $S_\nu(t,\xi)$ are self-adjoint matrices

$$S_\nu(t,\xi) = \frac{1}{(2\pi)^n} \mathcal{N}(s(t),|\xi|)^* \tilde{S}_\nu(\xi) \mathcal{N}(s(t),|\xi|), \tag{3.6}$$

with, recalling (1.3), $\tilde{S}_\nu(\xi)$ given by

$$\tilde{S}_\nu(\xi) = \frac{1}{|\xi|^2} \begin{pmatrix} \xi^{2\beta_{\nu,x}} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{if } \beta_{\nu,t} = 0 \tag{3.7}$$

for and $\xi \in \boldsymbol{R}^n \backslash \{0\}$;

$$\tilde{S}_\nu(\xi) = \begin{pmatrix} 0 & 0 \\ 0 & \xi^{2\beta_{\nu,x}} \end{pmatrix} \quad \text{if } \beta_{\nu,t} = 1. \tag{3.8}$$

Moreover, setting

$$V(t,\xi) = \exp\left(i|\xi| \int_0^t D(s(\tau),|\xi|) \, d\tau \right) W(t,\xi) \tag{3.9}$$

where $D = D(s(t), |\xi|) = diag(-\sqrt{\alpha}, \sqrt{\alpha})$, we can rewrite the system (3.2) and the relations (3.5) in the more convenient form

$$\begin{cases} \dfrac{\partial W}{\partial t} = \dfrac{\alpha_t}{4\alpha} e^{-i|\xi| \int_0^t D\,d\tau} B e^{i|\xi| \int_0^t D\,d\tau} W \\ W(0, \xi) = V_0(\xi) \end{cases} \tag{3.10}$$

with

$$s_\nu(t) = \int_{\boldsymbol{R}^n_\xi} {}^t\overline{W}(t, \xi) e^{-i|\xi| \int_0^t D\,d\tau} S_\nu(t, \xi) e^{i|\xi| \int_0^t D\,d\tau} W(t, \xi)\, d\xi \tag{3.11}$$

for $1 \le \nu \le N$.

Now, we will prove the *global* solvability for small initial data of problem (3.10), (3.11) on $\boldsymbol{R}^+_t \times \boldsymbol{R}^n_\xi$, applying a classical fixed point argument in the function $s(t) = (s_1(t), \ldots, s_N(t))$.

More precisely, for fixed $k > 1$, $\varepsilon > 0$ and any $\mathbf{C}^1$ function $s(t) : \boldsymbol{R}^+_t \to \boldsymbol{R}^N$ such that

$$|s(t)| \le \delta, \quad |s'(t)| \le \frac{\varepsilon}{1 + t^k} \tag{3.12}$$

we set $\alpha = \alpha(s(t), |\xi|)$, $D = D(s(t), |\xi|)$ and we consider the *linear* problem:

$$P_0 \quad \begin{cases} \dfrac{\partial W}{\partial t} = \dfrac{\partial_t \alpha}{4\alpha} e^{-i|\xi| \int_0^t D\,d\tau} B e^{i|\xi| \int_0^t D\,d\tau} W \\ W(0, \xi) = \tilde{W}(\xi) \end{cases} \tag{3.13}$$

where, to simplify the notation, $\tilde{W}(\xi) = V_0(\xi)$. Obviously, the initial value problem P_0 has a unique global solution $W(t, \xi)$ in $\boldsymbol{R}^+_t \times \boldsymbol{R}^n_\xi$. This defines the nonlinear map:

$$s(t) = (s_1(t), \ldots, s_N(t)) \mapsto \Gamma_1(t) = (\Gamma_1(t), \ldots, \Gamma_N(t)) \tag{3.14}$$

by the relations

$$\Gamma_\nu(t) = \int_{\boldsymbol{R}^n_\xi} {}^t\overline{W}(t, \xi) e^{-i|\xi| \int_0^t D\,d\tau} S_\nu(t, \xi) e^{i|\xi| \int_0^t D\,d\tau} W(t, \xi)\, d\xi, \tag{3.15}$$

for $t \in \boldsymbol{R}^+_t$ and $1 \le \nu \le N$.

Clearly, if $s(t) = \Gamma(t)$, i.e. $s(t)$ is a fixed point of the map (3.14), then the function $W(t, \xi)$ is also a solution of the original *nonlinear* problem (3.10), (3.11).

In the following, we will show that the nonlinear map $s(t) \mapsto \Gamma(t)$ defined in (3.14) is a contraction in appropriate spaces. To begin with, we shall prove *a priori* estimates for $|\Gamma(t)|$ and $|\Gamma'(t)|$.

First of all, let us observe that for a suitable positive constant M

$$\eta \leq |a(s)|, |m(s)| \leq M, \quad |a'(s)|, |m'(s)| \leq M, \quad \text{for } |s| \leq \delta. \tag{3.16}$$

Thus, since $\|e^{i|\xi|\int_0^t D\,d\tau}\| = 1$ and

$$\begin{aligned} S_\nu(t,\xi) &= \frac{1}{(2\pi)^n}\frac{\xi^{2\beta_{\nu,x}}}{|\xi|^2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{for } \beta_{\nu,t} = 0 \quad (\text{with } |\beta_{\nu,x}| \geq 1) \\ S_\nu(t,\xi) &= \frac{1}{(2\pi)^n}\xi^{2\beta_{\nu,x}}\alpha(s(t),|\xi|)\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad \text{for } \beta_{\nu,t} = 1, \end{aligned} \tag{3.17}$$

from (3.13) we have the inequality

$$\begin{aligned} |\Gamma_\nu(t)| &\leq \int_{R_\xi^n} \|S_\nu(t,\xi)\|\,|W(t,\xi)|^2\,d\xi \\ &\leq C(M)\int_{R_\xi^n} |W(t,\xi)|^2\Big(1 + |\xi|^{2|\beta|}\Big)\,d\xi \\ &\leq C(W)\exp\Big(C(\eta,M)\int_0^t |s'(\tau)|\,d\tau\Big)\int_{R_\xi^n} |\tilde{W}(\xi)|^2\Big(1 + |\xi|^{2|\beta|}\Big)\,d\xi. \end{aligned} \tag{3.18}$$

Differentiating the expression (3.15) of $\Gamma_\nu(t)$ with respect to t, we find:

$$\begin{aligned} \frac{d}{dt}\Gamma_\nu(t) &= 2\mathrm{Re}\int_{R_\xi^n} {}^t\overline{W}(t,\xi)e^{-i|\xi|\int_0^t D\,d\tau}S_\nu(t,\xi)e^{i|\xi|\int_0^t D\,d\tau}\partial_t W(t,\xi)\,d\xi \\ &\quad + \int_{R_\xi^n} {}^t\overline{W}(t,\xi)e^{-i|\xi|\int_0^t D\,d\tau}\partial_t S_\nu(t,\xi)e^{i|\xi|\int_0^t D\,d\tau}W(t,\xi)\,d\xi \\ &\quad + i\int_{R_\xi^n} {}^t\overline{W}(t,\xi)e^{-i|\xi|\int_0^t D\,d\tau}|\xi|(S_\nu D - DS_\nu)e^{i|\xi|\int_0^t D\,d\tau}W(t,\xi)\,d\xi \\ &= I_{\nu,1}(t) + I_{\nu,2}(t) + K_\nu(t) \end{aligned} \tag{3.19}$$

where $I_{\nu,1}(t)$, $I_{\nu,2}$, $K_\nu(t)$ correspond to the three terms above.

Substituting the equations of system (3.13) into the expressions of $I_{\nu,1}(t)$, $I_{\nu,2}(t)$ and using (3.12) we have immediately the estimates

$$\begin{aligned} |I_{\nu,1}(t)|, |I_{\nu,2}(t)| &\leq \frac{C(\eta,M)\varepsilon}{1+t^k}\int_{R_\xi^n} |W(t,\xi)|^2(1 + |\xi|^{2|\beta|})\,d\xi \\ &\leq \frac{C(\eta,M)\varepsilon}{1+t^k}\exp\Big(C(\eta,M)\int_0^t |s'(\tau)|\,d\tau\Big) \\ &\quad \cdot\int_{R_\xi^n} |\tilde{W}(\xi)|^2(1 + |\xi|^{2|\beta|})\,d\xi \end{aligned} \tag{3.20}$$

for all $t \in [0, \infty)$, $1 \le \nu \le N$. Finally, calculating explicitly the commutators $[S_\nu, D]$, we obtain for $K_\nu(t)$ the following expressions:

$$K_\nu(t) = \frac{4}{(2\pi)^n} \operatorname{Im} \int_{\mathbf{R}^n_\xi} \alpha(s(t), |\xi|)^{1/2} w_1 \overline{w_2} e^{-2i|\xi| \int_0^t \sqrt{\alpha}\, d\tau} |\xi|^{-1} \xi^{2\beta_{\nu,x}}\, d\xi \tag{3.21}$$

in the case $\beta_{\nu,t} = 0$;

$$K_\nu(t) = \frac{-4}{(2\pi)^n} \operatorname{Im} \int_{\mathbf{R}^n_\xi} \alpha(s(t), |\xi|)^{3/2} w_1 \overline{w_2} e^{-2i|\xi| \int_0^t \sqrt{\alpha}\, d\tau} |\xi| \xi^{2\beta_{\nu,x}}\, d\xi \tag{3.22}$$

if we have $\beta_{\nu,t} = 1$. Clearly, in (3.21), (3.22) the functions $w_1(t,\xi)$, $w_2(t,\xi)$ are the components of the vector $W(t,\xi)$ solution of problem P_0.

4. Estimate of $K_\nu(t)$ in the case $\beta_{\nu,t} = 0$

We assume that $s(t) : \mathbf{R}^+_t \to \mathbf{R}^N$ satisfies the conditions (3.12), but now we require that

$$k > 2. \tag{4.1}$$

Then, we define

$$S_\infty = \lim_{t \to \infty} s(t), \quad \mu = \frac{m(s_\infty)}{a(s_\infty)}. \tag{4.2}$$

According to the conditions (3.12), (3.16) we have

$$|s_\infty| \le \delta, \quad \frac{\eta}{M} \le \mu \le \frac{M}{\eta} \tag{4.3}$$

and

$$|m(s(t)) - \mu a(s(t))| \le C|s(t) - s_\infty| \le \frac{C\varepsilon}{1 + t^{k-1}}, \tag{4.4}$$

where $C = C(k, \eta, M)$. Thus, with $a = a(s(t))$ and $m = m(s(t))$, the quantity

$$\begin{aligned} \gamma(t, |\xi|) &\stackrel{\text{def}}{=} |\xi| \int_0^t \sqrt{|\xi|^2 a + m}\, d\tau - |\xi| \sqrt{|\xi|^2 + u} \int_0^t \sqrt{a}\, d\tau \\ &= \int_0^t \frac{|\xi|(m - \mu a)}{\sqrt{|\xi|^2 a + m} + \sqrt{a(|\xi|^2 + \mu)}}\, d\tau, \end{aligned} \tag{4.5}$$

is *uniformly* bounded, with bounded derivatives of any order with respect to $\rho = |\xi|$, over $\mathbf{R}^+_t \times \mathbf{R}^n_\xi$.

More precisely, having $k > 2$, from (3.16) and (4.4) we have easily

$$\left|\frac{\partial^l}{\partial \rho^l}\gamma(t,\rho)\right| \leq C_l\ \varepsilon, \quad \forall l \in \boldsymbol{N} \tag{4.6}$$

for suitable constants $C_l = C_l(k, \eta, M)$. Thus, setting

$$\psi_\mu(\rho) \stackrel{\text{def}}{=} \sqrt{\rho^2 + \mu}, \quad g_\mu(t,\rho) \stackrel{\text{def}}{=} \sqrt{\frac{\alpha(s(t),\rho)}{\psi_\mu(\rho)}} \tag{4.7}$$

we can rewrite $K_\nu(t)$, in the case $\beta_{\nu,t} = 0$, in the following form:

$$K_\nu(t) = \frac{4}{(2\pi)^n}\,\mathrm{Im}\bigg(\int_{S^{n-1}}\int_0^\infty g_\mu(t,\rho) w_1 \overline{w_2} e^{-2i\rho\psi_\mu(\rho)\int_0^t \sqrt{a}\,d\tau}$$
$$\cdot\, e^{-2i\gamma(t,\rho)}\rho^{n+2|\beta_{\nu,x}|-2}\psi_\mu(\rho)\omega^{2\beta_{\nu,x}}\,d\rho\,d\omega\bigg). \tag{4.8}$$

REMARK 4.1. Let us observe that the functions $\gamma(t,\rho)$, $g_\mu(t,\rho)$ have uniformly bounded derivatives of any order with respect to $\rho = |\xi|$ and, in particular, the inequality (4.6) does not depend explicitly on $s(t)$. The same holds for $g_\mu(t,\rho)$. In fact, it is easy to see that

$$\left|\frac{\partial^l}{\partial \rho^l} g_\mu(t,\rho)\right| \leq \tilde{C}_l, \quad \forall l \in \boldsymbol{N} \tag{4.9}$$

for suitable positive constants $\tilde{C}_l = \tilde{C}_l(\eta, M)$, provided (3.12), (3.16) hold. □

Taking into account this considerations, we define:

DEFINITION 4.1. *Let $W(t,\xi) = (w_1(t,\xi), w_2(t,\xi))$ be the solution of the linear system P_0 and $k \in \boldsymbol{N}$ such that $0 \leq k \leq n+1$. Then, for $1 \leq i, j \leq 2$ we define the functionals*:

$$J_{ij}^k(t) \stackrel{\text{def}}{=} \sup_{0\leq\tau\leq t}\ \sup_{\zeta\in\boldsymbol{R},\, g\in\mathscr{G}} (1 + |\zeta|^k)\bigg|\int_{S^{n-1}}\int_0^\infty g(\rho) w_i \overline{w_j}(\tau,\omega,\rho)$$
$$\cdot\, e^{i\rho\psi_\mu(\rho)\zeta}\rho^{n+2|\beta_{\nu,x}|-2}\psi_\mu(\rho)\omega^{2\beta_{\nu,x}}\,d\rho\,d\omega\bigg| \tag{4.10}$$

where $\mathscr{G}$ is the set of functions

$$\mathscr{G} \stackrel{\text{def}}{=} \{g(\rho) \in \mathbf{C}^k([0,\infty)) \mid \|g(\rho)\|_{\mathbf{C}^k} \leq 1\}. \tag{4.11}$$

REMARK 4.2. Having $|\beta_{\nu,x}| \geq 1$ (thanks to the condition (1.3)) and taking

$$\tilde{W}(\xi) = \mathcal{N}(s(0), |\xi|)^{-1} U_0(\xi) \tag{4.12}$$

where $U_0(\xi) = {}^t(|\xi|\hat{u}_0(\xi), \hat{u}_1(\xi))$ with $u_0(x)$, $u_1(x) \in \mathbf{C}_0^\infty(\boldsymbol{R}_x^n)$, then it is not difficult to prove that for $0 \leq k \leq n+1$

$$J_{ij}^k(t) : \boldsymbol{R}_t^+ \to \boldsymbol{R}$$

are continuous non decreasing functions. See the Appendix, Lemmas A.1 and A.2.

Next, setting

$$\begin{aligned} \lambda_1(s,\rho) &\overset{\text{def}}{=} -\sqrt{\alpha(s,\rho)}, \quad \lambda_2(s,\rho) \overset{\text{def}}{=} \sqrt{\alpha(s,\rho)} \\ \tilde{\lambda}_1(s) &\overset{\text{def}}{=} -\sqrt{a(s)}, \quad \tilde{\lambda}_2(s) \overset{\text{def}}{=} \sqrt{a(s)} \end{aligned} \tag{4.13}$$

we can rewrite the system (3.13) in the form (for $i = 1, 2$)

$$\frac{\partial}{\partial t} w_i(t,\xi) = s'(t) \cdot \sum_{h=1}^{2} q_{ih}(s(t), |\xi|) w_h(t,\xi) \exp\left(-i|\xi| \int_0^t (\lambda_i - \lambda_h)(s(\tau))\, d\tau\right) \tag{4.14}$$

where $s'(t) = (s_1'(t), \ldots, s_N'(t))$ and (recalling the definition (2.8) of B) the $q_{ij}(s,\rho) \in \boldsymbol{R}^N$ are the elements of the 2×2 matrix with N dimensional entries

$$Q(s,\rho) = \frac{1}{4\alpha} \begin{pmatrix} -\partial_s \alpha & \partial_s \alpha \\ \partial_s \alpha & -\partial_s \alpha \end{pmatrix}. \tag{4.15}$$

Then, for $1 \leq i$, $j \leq 2$ we have, with $\tilde{W}(\xi) = (\tilde{w}_1(\xi), \tilde{w}_2(\xi))$,

$$\begin{aligned} w_i(t,\xi)\overline{w}_j(t,\xi) &= \tilde{w}_i(\xi)\overline{\tilde{w}_j}(\xi) \\ &+ \sum_{h=1}^{2} \int_0^t s'(\tau) q_{ih}(s(\tau), |\xi|) w_h \overline{w_j}(\tau,\xi) \exp\left(-i|\xi| \int_0^\tau (\lambda_i - \lambda_h)\, dy\right) d\tau \\ &+ \sum_{h=1}^{2} \int_0^t s'(\tau) q_{jh}(s(\tau), |\xi|) w_i \overline{w}_h(\tau,\xi) \exp\left(i|\xi| \int_0^\tau (\lambda_j - \lambda_h)\, dy\right) d\tau. \end{aligned} \tag{4.16}$$

Now, let us introduce the simplified notations

$$\sup_{\{*\}} \overset{\text{def}}{=} \sup_{0 \leq \tau \leq t}\ \sup_{\zeta \in \boldsymbol{R}, g \in \mathcal{G}}$$

and

$$\phi_{ij}(t,\rho) \overset{\text{def}}{=} \int_0^t (\lambda_i - \lambda_j)(s(y),\rho)\,dy, \quad \tilde{\phi}_{ij}(t) \overset{\text{def}}{=} \int_0^t (\tilde{\lambda}_i - \tilde{\lambda}_j)(s(y))\,dy, \tag{4.17}$$

for $1 \le i,\ j \le 2$. Then, we have

$$\rho\phi_{ij}(t,\rho) = \rho\psi_\mu(\rho)\tilde{\phi}_{ij}(t) + 2\chi_{i,j}\gamma(t,\rho) \tag{4.18}$$

where $\chi_{i,j} = -1$ if $i < j$; $\chi_{i,j} = 0$ if $i = j$ and $\chi_{i,j} = 1$ if $i > j$.

Then, using (4.16) from the definition of $J_{ij}^k(t)$ we obtain

$$\begin{aligned} J_{ij}^k(t) \le{} & \sup_{\{*\}}(1+|\zeta|^k)\left|\int_{S^{n-1}}\int_0^\infty g(\rho)\tilde{w}_i(\xi)\overline{\tilde{w}_j(\xi)}e^{i\rho\psi(\rho)\zeta}\rho^{n+2|\beta_{\nu,x}|-2}\psi_\nu(\rho)\omega^{2\beta_{\nu,x}}\,d\rho\,d\omega\right| \\ & + \sup_{\{*\}}(1+|\zeta|^k)\left|\int_{S^{n-1}}\int_0^\infty\sum_{h=1}^2\int_0^\tau s'(y)q_{ih}g(\rho)w_h\overline{w}_je^{i\rho\psi_\mu(\rho)(\zeta-\tilde{\phi}_{ih}(y))}e^{-2i\chi_{i,h}\gamma(y,\rho)}\right. \\ & \left.\cdot\,\rho^{n+2|\beta_{\nu,x}|-2}\psi_\mu(\rho)\omega^{2\beta_{\nu,x}}\,dy\,d\rho\,d\omega\right| \\ & + \sup_{\{*\}}(1+|\zeta|^k)\left|\int_{S^{n-1}}\int_0^\infty\sum_{h=1}^2\int_0^\tau s'(y)q_{jh}g(\rho)w_i\overline{w}_he^{i\rho\psi_\mu(\rho)(\zeta+\tilde{\phi}_{jh}(y))}e^{2i\chi_{j,h}\gamma(y,\rho)}\right. \\ & \left.\cdot\,\rho^{n+2|\beta_{\nu,x}|-2}\psi_\mu(\rho)\omega^{2\beta_{\nu,x}}\,dy\,d\rho\,d\omega\right| \end{aligned} \tag{4.19}$$

where $q_{ij} = q_{ij}(s(y),\rho)$.

To begin with, in view of Lemma A.2 of the Appendix, we have

$$\begin{aligned} \sup_{\zeta\in\boldsymbol{R},\,g\in\mathscr{G}}(1+|\zeta|^k)&\left|\int_{S^{n-1}}\int_0^\infty g(\rho)\tilde{w}_i\overline{\tilde{w}_j}e^{i\rho\psi_\mu(\rho)\zeta}\right. \\ &\left.\cdot\,\rho^{n+2|\beta_{\nu,x}|-2}\psi_\mu(\rho)\omega^{2\beta_{\nu,x}}\,d\rho\,d\omega\right| \overset{\text{def}}{=} R_\nu^k(i,j) < \infty, \end{aligned} \tag{4.20}$$

by the assumptions on the initial data $\tilde{W}(\xi)$ and the condition $k \le n+1$.

Besides, for $h = 1, 2$ from the definition of the $J_{ij}^k(t)$, changing the order of integration, one has

$$\begin{aligned} \sup_{\{*\}}(1+|\zeta|^k)&\left|\int_{S^{n-1}}\int_0^\infty\int_0^\tau s'(y)q_{ih}g(\rho)w_h\overline{w}_j(y,\rho,\omega)e^{i\rho\psi_\mu(\rho)(\zeta-\tilde{\phi}_{ih}(y))}e^{-2i\chi_{i,h}\gamma(y,\rho)}\right. \\ &\left.\cdot\,\rho^{n+2|\beta_{\nu,x}|-2}\psi_\mu(\rho)\omega^{2\beta_{\nu,x}}\,dy\,d\rho\,d\omega\right| \\ \le{} & \mathscr{C}_1\sup_{\{*\}}(1+|\zeta|^k)\int_0^\tau|s'(y)|\frac{J_{hj}^k(y)}{1+|\zeta-\tilde{\phi}_{ih}(y)|^k}\,dy, \end{aligned} \tag{4.21a}$$

$$\sup_{\{*\}}(1+|\zeta|^k)\left|\int_{S^{n-1}}\int_0^\infty\int_0^\tau s'(y)q_{jh}g(\rho)w_i\bar{w}_h(y,\rho,\omega)e^{i\rho\psi_\mu(\rho)(\zeta+\tilde{\phi}_{jh}(y))}e^{2i\chi_{i,h}\gamma(y,\rho)}\right.$$

$$\left.\cdot\,\rho^{n+2|\beta_{v,x}|-2}\psi_\mu(\rho)\omega^{2\beta_{v,x}}\,dy\,d\rho\,d\omega\right|$$

$$\leq \mathscr{C}_1\sup_{\{*\}}(1+|\zeta|^k)\int_0^\tau |s'(y)|\frac{J_{ih}^k(y)}{1+|\zeta+\tilde{\phi}_{jh}(y)|^k}dy \tag{4.21b}$$

where, recalling the Definition 4.1,

$$\mathscr{C}_1=\sup_{|s|\leq\delta}\|q_{ij}(s,\cdot)\|_{\mathbf{C}^k}\cdot\sup_{t\in\mathbf{R}_t^+}\|e^{2i\gamma(t,\cdot)}\|_{\mathbf{C}^k}. \tag{4.22}$$

Thanks to the estimates (4.6), (4.9) and formula (4.15) it is clear that

$$\mathscr{C}_1=\mathscr{C}_1(k,\eta,M)<\infty.$$

Now, to estimate the right hand side of (4.21a), (4.21b) we need the following:

LEMMA 4.1. *Let $\varphi(\tau):\mathbf{R}\to\mathbf{R}$ be a $\mathbf{C}^1$ function such that $\varphi(0)=0$ and*

$$\lambda\leq|\varphi'(\tau)|\leq\Lambda$$

for suitable real constants $\Lambda\geq\lambda>0$. Then, for any real number $k>1$, there exists a constant $\mathscr{C}_2=\mathscr{C}_2(k,\lambda,\Lambda)$, $1\leq\mathscr{C}_2<\infty$, such that

$$\sup_{\zeta\in\mathbf{R}}\int_{-\infty}^{\infty}\frac{1+|\zeta|^k}{(1+|\tau|^k)(1+|\zeta-\varphi(\tau)|^k)}d\tau\leq\mathscr{C}_2. \tag{4.23}$$

PROOF. Inequality (4.23) follows by elementary computations. See also [3], Lemma 5.3. □

To apply Lemma 4.1, we define:

$$\lambda\overset{\text{def}}{=}\inf_{|s|\leq\delta}|\tilde{\lambda}_i(s)-\tilde{\lambda}_j(s)|\quad\text{for } i\neq j, \tag{4.24}$$

$$\Lambda\overset{\text{def}}{=}\sup_{|s|\leq\delta}|\tilde{\lambda}_i(s)-\tilde{\lambda}_j(s)|\quad\text{for } i\neq j. \tag{4.25}$$

Hence, from the definitions of $\tilde{\lambda}_i(s)$ and $\tilde{\phi}_{ij}(t)$, it is clear that $\tilde{\phi}_{ii}(t)\equiv 0$ and

$$0<\lambda\leq|\tilde{\phi}'_{ij}(t)|\leq\Lambda\quad\text{for } i\neq j, \tag{4.26}$$

with $2\sqrt{\eta}\leq\lambda\leq\Lambda\leq 2\sqrt{M}$, thanks to (3.16). Thus, we can state the following:

LEMMA 4.2. *Let $n \geq 2$ and let k be a fixed integer, $2 < k \leq n+1$. Let $s(t) : \boldsymbol{R}_t^+ \to \boldsymbol{R}^N$ be a $\mathbf{C}^1$ function satisfying the conditions* (3.12). *Moreover, assume that the initial data $\tilde{W}(\xi)$ of problem P_0 satisfies the conditions* (4.20), *i.e.* $R_\nu^k(i,j) < \infty$. *Finally, let us define*

$$R_\nu^k \stackrel{\text{def}}{=} \max_{1 \leq i,j \leq 2}(R_\nu^k(i,j)) \quad \text{and} \quad J^k(t) \stackrel{\text{def}}{=} \max_{1 \leq i,j \leq 2}(J_{ij}^k(t)). \tag{4.27}$$

Then, for any $t \geq 0$, we have

$$J^k(t) \leq R_\nu^k + 4\varepsilon\mathscr{C}_1\mathscr{C}_2 J^k(t) \tag{4.28}$$

where $\mathscr{C}_1$, $\mathscr{C}_2$ are the constants appearing respectively in (4.22) *and in the statement of Lemma* 4.1.

PROOF. According to (4.19), (4.20), (4.21a), (4.21b) we have the inequalities

$$J_{ij}^k(t) \leq R_\nu^k(i,j) + \mathscr{C}_1 \sum_{h=1}^{2} \sup_{0 \leq \tau \leq t} \sup_{\zeta \in \boldsymbol{R}} (1 + |\zeta|^k) \int_0^\tau |s'(y)| \frac{J_{hj}^k(y)}{1 + |\zeta - \tilde{\phi}_{ih}(y)|^k} dy$$

$$+ \mathscr{C}_1 \sum_{h=1}^{2} \sup_{0 \leq \tau \leq t} \sup_{\zeta \in \boldsymbol{R}} (1 + |\zeta|^k) \int_0^\tau |s'(y)| \frac{J_{ih}^k(y)}{1 + |\zeta + \tilde{\phi}_{jh}(y)|^k} dy. \tag{4.29}$$

Therefore, from the assumptions on $s(t)$ and using Lemma 4.1, in the case $i \neq j$, we find

$$J_{ij}^k(t) \leq R_\nu^k(i,j) + \varepsilon\mathscr{C}_1\mathscr{C}_2 \sum_{h=1}^{2} (J_{hj}^k(t) + J_{ih}^k(t)). \tag{4.30}$$

with $\mathscr{C}_2 = \mathscr{C}_2(k, \eta, M)$. Hence, taking the maximum for $1 \leq i$, $j \leq 2$, estimate (4.30) immediately gives (4.28). □

Summarizing the above results, we have:

PROPOSITION 4.3. *Let $n \geq 2$ and let k be a fixed integer, $2 < k \leq n+1$. Assume that the function $s(t)$ satisfies the conditions* (3.12), *that the initial data $\tilde{W}(\xi)$ of the Cauchy problem* (3.9) *belongs to $\mathbf{C}^k$ and*

$$\int_{S^{n-1}} \int_0^\infty |\partial_\rho^h \tilde{W}(\rho,\omega)|^2 (1 + \rho^{n+2|\beta|-1}) \, d\rho \, d\omega < \infty, \tag{4.31}$$

for $0 \le h \le k$ *(see the Appendix). Finally, assume that*

$$\varepsilon \le \frac{1}{8\mathscr{C}_1\mathscr{C}_2}. \tag{4.32}$$

Then, the function $K_\nu(t)$ *defined in* (3.21) *(in the case* $\beta_{\nu,t} = 0$*) satisfies for* $t \in [0, \infty)$ *the estimate*

$$|K_\nu(t)| \le \frac{C}{1+t^k} R_\nu^k, \tag{4.33}$$

for a suitable constant $C = C(k, \eta, M)$*. The term* R_ν^k*, defined in* (4.20), (4.27), *depends only on the initial data* $\tilde{W}(\xi)$.

Proof. Thanks to (4.32) we have

$$J^k(t) \le \frac{R_\nu^k}{1 - 4\mathscr{C}_1\mathscr{C}_2\varepsilon} \le 2R_\nu^k \quad \forall t \ge 0. \tag{4.34}$$

Thus, going back to the expression (4.8) of $K_\nu(t)$ (in the case $\beta_{\nu,t} = 0$) and recalling the Definition 4.1 of $J_{ij}^k(t)$, it follows that

$$\begin{aligned} |K_\nu(t)| \le \frac{4}{(2\pi)^n} \bigg| \int_{S^{n-1}} \int_0^\infty & g_\mu(t,\rho) w_1 \overline{w_2}(t,\rho,\omega) e^{-2i\rho\psi_\mu(\rho)\int_0^t \sqrt{a}\,d\tau} \\ & \cdot e^{-2i\gamma(t,\rho)} \psi_\mu(\rho) \rho^{n+2|\beta_{\nu,x}|-2} \omega^{2\beta_{\nu,x}}\, d\rho\, d\omega \bigg| \\ \le \mathscr{C}_3 & \frac{J^k(t)}{1 + |2\int_0^t \sqrt{a(s(\tau))}\,d\tau|^k}, \end{aligned} \tag{4.35}$$

where

$$\mathscr{C}_3 = \frac{4}{(2\pi)^n} \sup_{t \in \boldsymbol{R}_t^+} \|g_\mu(t,\cdot) e^{-2i\gamma(t,\cdot)}\|_{\mathbf{C}^k} < \infty. \tag{4.36}$$

Thus, taking into account that $a(s) \ge \eta$ and using the estimate (4.34) of $J^k(t)$, we get the inequality

$$|K_\nu(t)| \le \mathscr{C}_3 \frac{J^k(t)}{1 + (2\sqrt{\eta}t)^k} \le 2\mathscr{C}_3 C(\eta) \frac{R_\nu^k}{1+t^k}. \tag{4.37}$$

This ends the proof of the inequality (4.33) for $|K_\nu(t)|$. □

5. Estimate of $K_\nu(t)$ in the case $\beta_{\nu,t} = 1$

We can estimate the terms $K_\nu(t)$ for $\beta_{\nu,t} = 1$ following the same lines of the estimates in the previous section. More precisely, we define the functionals:

$$Y_{ij}^k(t) \overset{\text{def}}{=} \sup_{0\le\tau\le t} \sup_{\zeta\in\boldsymbol{R},g\in\mathscr{G}} (1+|\zeta|^k)\left|\int_{S^{n-1}}\int_0^\infty g(\rho)w_i\bar{w}_j(\tau,\omega,\rho)\right.$$
$$\left.\cdot\, e^{i\rho\psi_\mu(\rho)\zeta}\rho^{n+2|\beta_{\nu,x}|}\psi_\mu(\rho)^{3/2}\omega^{2\beta_{\nu,x}}\,d\rho\,d\omega\right| \tag{5.1}$$

where again $\mathscr{G} = \{g(\rho)\in \mathbf{C}^k([0,\infty))|\,\|g\|_{\mathbf{C}^k}\le 1\}$.

For $0\le k\le n+1$ the functionals $Y_{ij}^k(t)$ are continuous non decreasing functions, if we assume that the initial data satisfies a condition similar to (4.31), namely

$$\int_{S^{n-1}}\int_0^\infty |\partial_\rho^h \tilde{W}(\rho,\omega)|^2(1+\rho^{n+2|\beta|+1})\,d\rho\,d\omega < \infty, \tag{5.2}$$

for $0\le h\le k$. Setting by definition

$$\tilde{R}_\nu^k = \max_{1\le i,j\le 2} \sup_{\zeta\in\boldsymbol{R},g\in\mathscr{G}} (1+|\zeta|^k)\left|\int_{S^{n-1}}\int_0^\infty g(\rho)\tilde{w}_i\overline{\tilde{w}_j}e^{i\rho\psi_\mu(\rho)\zeta}\right.$$
$$\left.\cdot\,\rho^{n+2|\beta_{\nu,x}|}\psi_\mu(\rho)^{3/2}\omega^{2\beta_{\nu,x}}\,d\rho\,d\omega\right|, \tag{5.3}$$

we can prove that there exists $\mathscr{C}_3 = \mathscr{C}_3(k,\eta,M) > 0$ such that

$$\varepsilon \le \frac{1}{\mathscr{C}_3} \Rightarrow |K_\nu(t)| \le \frac{C}{1+t^k}\tilde{R}_\nu^k \quad \forall t\ge 0 \tag{5.4}$$

for a suitable $C = C(k,\eta,M)$, provided $s(t)$ satisfies (3.12) with $k>2$.

6. Conclusion of the Proof of Theorem 1.1

Now, summarising up the estimates of the previous section, it is not difficult to show that the map $s(t)\mapsto\Gamma(t)$ is a contraction in a appropriate spaces.

From the results of §4, §5 we can easily show that fixed the integer k, $2<k\le n+1$ and taking

$$\varepsilon \le \min\left(1,\frac{1}{8\mathscr{C}_1\mathscr{C}_2},\frac{1}{\mathscr{C}_3}\right)$$

in (3.12), for a sufficiently small initial data $\tilde{W}(\xi)$ we have

$$|s(t)| \leq \delta, \quad |s'(t)| \leq \frac{\varepsilon}{1+t^k} \Rightarrow |\Gamma(t)| \leq \delta, \quad |\Gamma'(t)| \leq \frac{\varepsilon}{1+t^k}. \tag{6.1}$$

In fact, from (3.18) it follows that

$$|\Gamma(t)| \leq C(k,\eta,M) \int_{\boldsymbol{R}_x^n} |\tilde{W}(\xi)|^2 (1+|\xi|^{2|\beta|})\, d\xi \leq \delta \tag{6.2}$$

provided $\tilde{W}(\xi)$ is small enough. Moreover, recalling (3.19), (3.20), Proposition 4.3 and the estimate (5.4) we find the inequality

$$|\Gamma'(t)| \leq \frac{C(\eta,M)\varepsilon}{1+t^k} \int_{\boldsymbol{R}_\xi^n} |\tilde{W}(\xi)|^2 (1+|\xi|^{2|\beta|})\, d\xi$$

$$+ \frac{C}{1+t^k} \sum_{\nu=1}^{N} (R_\nu^k + \tilde{R}_\nu^k). \tag{6.3}$$

Thus, $|\Gamma'(t)| \leq \varepsilon(1+t)^{-k}$ on $\boldsymbol{R}_t^+$ provided the quantities

$$\int_{\boldsymbol{R}_\xi^n} |\tilde{W}(\xi)|^2 (1+|\xi|^{2|\beta|})\, d\xi \quad \text{and} \quad R_\nu^k,\ \tilde{R}_\nu^k \quad (1 \leq \nu \leq N) \tag{6.4}$$

are sufficiently small. Now, following similar arguments of Lemma 3.4 of [3], we define the subset $S_{\delta,\varepsilon}^k$ of $\mathbf{C}^1([0,\infty))^N$ as

$$S_{\delta,\varepsilon}^k([0,\infty)) = \{s(t) \in \mathbf{C}^1([0,\infty))^N \mid |s(t)| \leq \delta, \quad (1+t^k)|s'(t)| \leq \varepsilon\}. \tag{6.5}$$

Using the estimate (6.1) we have

$$\Gamma(S_{\delta,\varepsilon}^k) \subseteq S_{\delta,\varepsilon}^k,$$

if the initial data $\tilde{W}(\xi)$ and its derivatives $\partial_\rho^h \tilde{W}(\rho)$, up to the order k, are small and decay sufficiently fast as $|\xi| \to \infty$.

To proceed, let $\phi(t) : [0,\infty) \to (0,\infty)$ be a continuous weight function such that

$$\phi(t) \geq 1 + \int_0^t \phi(\tau)\, d\tau \quad \forall t \in [0,\infty). \tag{6.6}$$

For example we may choose $\phi(t) = e^t$. Then we define:

Definition 6.1. *For $l \in \mathcal{N}$, let $C_\phi^l([0,\infty))^N$ be the Banach space*

$$C_\phi^1([0,\infty))^N \stackrel{\text{def}}{=} \left\{ s(t) \in \mathbf{C}^l([0,\infty))^N \,\middle|\, \sup_{t\geq 0} \phi(t)^{-1} \left| \frac{d^i}{dt^i} s(t) \right| < \infty,\ \text{for } 0 \leq i \leq l \right\}, \tag{6.7}$$

with the norm $\|\cdot\|_{l,\phi}$ *defined by*

$$\|s(t)\|_{l,\phi} = \sum_{i=0}^{l} \sup_{t\geq 0} \phi(t)^{-1}\left|\frac{d^i}{dt^i}s(t)\right|. \tag{6.8}$$

REMARK 6.1. It is not difficult to prove that $S^k_{\delta,\varepsilon}([0,\infty))$ is a closed subset of $C^1_\phi([0,\infty))^N$. In fact, if

$$s_n(t) \in S^k_{\delta,\varepsilon}([0,\infty)) \quad \text{and} \quad s_n(t) \to s_\infty(t) \quad \text{in } C^1_\phi([0,\infty))^N$$

as $n \to \infty$, then $s_\infty(t) \in C^1([0,\infty))^N$ and

$$s_n(t) \to s_\infty(t), \quad s'_n(t) \to s'_\infty(t) \tag{6.9}$$

uniformly on every bounded interval $[0,T]$ with $T > 0$. This implies that

$$|s_\infty(t)| \leq \delta, \quad (1+t^k)|s'_\infty(t)| \leq \varepsilon \tag{6.10}$$

on $[0,T]$ for and $T > 0$. Hence, $s_\infty(t) \in S^k_{\delta,\varepsilon}([0,\infty))$. On the other hand, the norm of $C^1_\phi([0,\infty))^N$ introduced in (6.8) is rather *weak*, thus it is easier to prove that the map

$$s(t) \mapsto \Gamma(t)$$

is contraction on $S^k_{\delta,\varepsilon}$ with respect to the weighted norm $\|\cdot\|_{1,\phi}$. □

More precisely, given $s_a(t)$, $s_b(t) \in S^k_{\delta,\varepsilon}$ we can solve the corresponding initial value problems

$$\begin{aligned} &P_a \quad \begin{cases} \dfrac{\partial W_a}{\partial t} = \dfrac{\partial_t\alpha_a}{4\alpha_a} e^{-i|\xi|\int_0^t D_a\,d\tau} B\, e^{i|\xi|\int_0^t D_a\,d\tau} W \overset{def}{=} \dfrac{\partial_t\alpha_a}{4\alpha_a} Q_a W_a \\ W_a(0,\xi) = \tilde{W}(\xi), \end{cases} \\ &P_b \quad \begin{cases} \dfrac{\partial W_b}{\partial t} = \dfrac{\partial_t\alpha_b}{4\alpha_b} e^{-i|\xi|\int_0^t D_b\,d\tau} B\, e^{i|\xi|\int_0^t D_b\,d\tau} W \overset{def}{=} \dfrac{\partial_t\alpha_b}{4\alpha_b} Q_b W_b \\ W_b(0,\xi) = \tilde{W}(\xi), \end{cases} \end{aligned} \tag{6.11}$$

in the interval $[0,\infty)$. Here, obviously, $D_\gamma = D(s_\gamma(t),|\xi|)$, $\alpha_\gamma = \alpha(s_\gamma(t),|\xi|)$ for $\gamma = a,b$. Then, from the equality

$$\frac{d}{dt}(W_a - W_b) = \left(\frac{\partial_t\alpha_a}{4\alpha_a} - \frac{\partial_t\alpha_b}{4\alpha_b}\right) Q_a W_a + \frac{\partial_t\alpha_b}{4\alpha_b}(Q_a - Q_b)W_a + \frac{\partial_t\alpha_b}{4\alpha_b} Q_b(W_a - W_b) \tag{6.12}$$

applying Gronwall's Lemma and using the fact that

$$\begin{aligned}\int_0^t |s_a - s_b|\,d\tau &\le \|s_a - s_b\|_{0,\phi}\int_0^t \phi(\tau)\,d\tau \le \phi(t)\|s_a - s_b\|_{0,\phi},\\ \int_0^t |s_a' - s_b'|\,d\tau &\le \|s_a - s_b\|_{1,\phi}\int_0^t \phi(\tau)\,d\tau \le \phi(t)\|s_a - s_b\|_{1,\phi}\end{aligned} \tag{6.13}$$

we can prove that, for any $\xi \in \boldsymbol{R}^n$,

$$\|W_a(t,\xi) - W_b(t,\xi)\|_{1,\phi} \le C\left(\|s_a - s_b\|_{1,\phi} + \frac{\varepsilon(1+|\xi|^2)\|s_a - s_b\|_{0,\phi}}{1+(1+|\xi|^2)\|s_a - s_b\|_{0,\phi}}\right)|\tilde{W}(\xi)| \tag{6.14}$$

where $C = C(k,\varepsilon,\eta,M)$.

Denoting with Γ_ν^a, Γ_ν^b the corresponding value of Γ_ν, for $1 \le \nu \le N$, and proceeding in this way (see also [3], section §6), it is now possible to estimate $\|\Gamma_\nu^a - \Gamma_\nu^b\|_{1,\phi}$ in terms of $\|s_a - s_b\|_{1,\phi}$. We will finally obtain that

$$\|\Gamma_\nu^a - \Gamma_\nu^b\|_{1,\phi} \le C\|s_a - s_b\|_{1,\phi}\int_{\boldsymbol{R}_\xi^n} |\tilde{W}(\xi)|^2(1+|\xi|^{2|\beta_\nu|+4})\,d\xi. \tag{6.15}$$

Hence, the map $s(t) \mapsto \Gamma(t)$ is a contraction in $S_{\delta,\varepsilon}^k$ provided the last integral is sufficiently small.

This ends the Proof of Theorem 1.1.

Appendix: Decay Estimates for Oscillating Integrals

LEMMA A.1. *Let k, p be fixed integers such that $0 \le k \le p+1$. Let $g(\xi) \equiv g(\rho,\omega)$ be a regular function defined on $\boldsymbol{R}_\xi^n \backslash \{0\}$, satisfying*

$$\int_{S^{n-1}}\int_0^\infty |\partial_\rho^m g(\rho,\omega)|(1+\rho^p)\,d\rho\,d\omega < \infty \tag{A.1}$$

for $0 \le m \le k$. Then, the following estimates hold, for any $\zeta \in \boldsymbol{R}$,

(i) *if $0 \le k \le p$,*

$$|\zeta|^k \int_{S^{n-1}}\left|\int_0^\infty g(\rho,\omega)e^{i\rho\zeta}\rho^p\,d\rho\right|d\omega \le \int_{S^{n-1}}\int_0^\infty |\partial_\rho^k(g(\rho,\omega)\rho^p)|\,d\rho\,d\omega;$$

(ii) *if $k = p+1$,*

$$|\zeta|^{p+1}\int_{S^{n-1}}\left|\int_0^\infty g(\rho,\omega)e^{i\rho\zeta}\rho^p\,d\rho\right|d\omega \le 2\int_{S^{n-1}}\int_0^\infty |\partial_\rho^{p+1}(g(\rho,\omega)\rho^p)|\,d\rho\,d\omega.$$

PROOF. Without loss of generality, in the following we may assume $\zeta \neq 0$ and $k \geq 1$. Let us consider first the case $k \leq p$. Integrating by parts, since

$$\partial_\rho^m(g(\rho,\omega)\rho^p) \to 0,$$

for $0 \leq m \leq k-1$ as $\rho \to 0$, we find

$$\int_0^\infty g(\rho,\omega)e^{i\rho\zeta}\rho^p\,d\rho = \left(\frac{i}{\zeta}\right)^k \int_0^\infty \partial_\rho^k(g(\rho,\omega)\rho^p)e^{i\rho\zeta}\,d\rho. \tag{A.2}$$

Hence, we have the estimates (i) for $0 \leq k \leq p$.

To obtain the estimate (ii), we observe that, given $f(\rho,\omega) : (0,\infty) \times \boldsymbol{S}^{n-1} \to \boldsymbol{R}$ satisfying

$$\int_{\boldsymbol{S}^{n-1}} \int_0^\infty (|f(\rho,\omega)| + |\partial_\rho f(\rho,\omega)|)\,d\rho\,d\omega < \infty \tag{A.3}$$

we have, a.e. for $\omega \in \boldsymbol{S}^{n-1}$,

$$\begin{aligned}\left|\int_0^\infty f(\rho,\omega)e^{i\rho\zeta}\,d\rho\right| &= \left|\frac{1}{i\zeta}\lim_{\rho\to 0} f(\rho,\omega) + \frac{1}{i\zeta}\int_0^\infty \partial_\rho f(\rho,\omega)e^{i\rho\zeta}\,d\rho\right| \\ &\leq \frac{2}{|\zeta|}\int_0^\infty |\partial_\rho f(\rho,\omega)|\,d\rho. \end{aligned}\tag{A.4}$$

Now, applying the estimate (A.4) to the function $\partial_\rho^p(g(\rho,\omega)\rho^p)$, we find

$$\left|\int_0^\infty \partial_\rho^p(g(\rho,\omega)\rho^p)e^{i\rho\zeta}\,d\rho\right| \leq \frac{2}{|\xi|}\int_0^\infty |\partial_\rho(\partial_\rho^p(g(\rho,\omega)\rho^p))|\,d\rho \tag{A.5}$$

and thanks to the equality (A.2), with $k = p$, we immediately have (ii).

Thus we have proved Lemma A.1.

Next, introducing the function

$$\psi_\mu(\rho) = \sqrt{\rho^2 + \mu} \quad \text{with } \mu > 0, \tag{A.6}$$

we may prove similar decay estimates for the oscillating integral

$$\int_0^\infty g(\rho,\omega)e^{i\rho\psi_\mu(\rho)\zeta}\rho^p\,d\rho. \tag{A.7}$$

In fact, with the same hypotheses on $g(\rho,\omega)$, we have:

LEMMA A.2. *Let us assume for $g(\rho,\omega)$ the same hypotheses of Lemma* A.1 *and that $\mu > 0$. Then, the following estimates hold, for any $\zeta \in \boldsymbol{R}$,*

$$|\zeta|^k \int_{S^{n-1}} \left| \int_0^\infty g(\rho,\omega) e^{i\rho\psi_\mu(\rho)\zeta} \rho^p \, d\rho \right| d\omega \le C_k(\mu) \int_{S^{n-1}} \int_0^\infty |\partial_\rho^k (g(\rho,\omega)\rho^p)| \, d\rho \, d\omega, \tag{A.8}$$

for $0 \le k \le p+1$, *where* $C_k(\mu) \ge 0$ *is a suitable constant.*

Proof. We can follow the same ideas of the preceding proof. Observing that

$$\frac{d}{d\rho}(\rho\psi_\mu(\rho)) = \frac{2\rho^2+\mu}{\sqrt{\rho^2+\mu}} \ge C(\mu) > 0 \tag{A.9}$$

we may integrate by parts. For example, in the case $p \ge 1$, since $g(\rho,\omega)\rho^p \to 0$ as $\rho \to 0$, we find

$$\int_0^\infty g(\rho,\omega) e^{i\rho\psi_\mu(\rho)\zeta} \rho^p \, d\rho = \frac{i}{\zeta} \int_0^\infty \partial_\rho \left(\frac{g(\rho,\omega)\rho^p}{(\rho\psi_\mu(\rho))'} \right) e^{i\rho\psi_\mu(\rho)\zeta} \, d\rho. \tag{A.10}$$

Thus, using (A.9) and integrating by parts, we have easily the estimates (A.8) for $0 \le k \le p$.

Finally, to prove the estimate in the case $k = p+1$, let us observe that given $f(\rho,\omega) : (0,\infty) \times \boldsymbol{S}^{n-1} \to \boldsymbol{R}$ satisfying (A.3) we have, a.e. for $\omega \in \boldsymbol{S}^{n-1}$,

$$\left| \int_0^\infty f(\rho,\omega) e^{i\rho\psi_\mu(\rho)\zeta} \, d\rho \right| = \left| \frac{1}{i\zeta} \lim_{\rho\to 0} \frac{f(\rho,\omega)}{(\rho\psi_\mu(\rho))'} + \frac{1}{i\zeta} \int_0^\infty \frac{\partial}{\partial_\rho} \left(\frac{f(\rho,\omega)}{(\rho\psi_\mu(\rho))'} \right) e^{i\rho\psi_\mu(\rho)\zeta}, d\rho \right|$$

$$\le \frac{C_\mu}{|\zeta|} \int_0^\infty (|f(\rho,\omega)| + |\partial_\rho f(\rho,\omega)|) \, d\rho, \tag{A.11}$$

with $C_\mu \ge 0$ a fixed constant. Hence, applying the estimate (A.11) we have, as in the proof of the previous lemma, (A.8) for $k = p+1$. □

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