REMARKS ON UNICOHERENCE AT SUBCONTINUA

By

Janusz J. Charatonik, Włodzimierz J. Charatonik,
and Alejandro Illanes

Abstract. Examples are presented showing that some statements (and the main results) of the paper [12] "On unicoherence at subcontinua" by Zhou Youcheng are not true. For some other assertions from that paper, which are correctly stated but whose proofs contain essential gaps, the proper arguments are supplied.

All *spaces* in this paper are assumed to be metric, and *mappings* are continuous. A *continuum* means a compact, connected space. A continuum X is said to be:

    – *unicoherent* if the intersection of every two of its subcontinua whose union is X is connected;

    – *unicoherent at a subcontinuum* $Y \subset X$ if for each pair of proper subcontinua A and B of X such that $A \cup B = X$ the intersection $A \cap B \cap Y$ is connected.

Denote by C the complex plane, by R the real line, and by $S = \{z \in C : |z| = 1\}$ the unit circle. Let X be any space. A mapping $f : X \to S$ is said to be

    – *inessential* $(f \sim 1)$ if there exists a mapping $\varphi : X \to R$ such that $f(x) = e^{i\varphi(x)}$ for each $x \in X$;

    – *inessential on a subspace* Y of X $(f \sim 1$ *on* $Y)$ if there exists a mapping $\varphi : Y \to R$ such that $f(x) = e^{i\varphi(x)}$ for each $x \in Y$.

A continuum X is said to have

    – *property* (b) if the condition $f \sim 1$ holds for each mapping $f : X \to S$ (it is known that for locally connected continua X having property (b) is equivalent to unicoherence of X);

– *property* (b) *on a subcontinuum* Y of X if the condition $f \sim 1$ *on* Y holds for each mapping $f : X \to S$.

The concept of the unicoherence of a continuum at a subcontinuum is due to Owens [9], and it is related to Bennett's strong unicoherence [1, 2], and Maćkowiak's weak hereditary unicoherence [6] (see also [7] and Section 2 of [9]). Some mapping properties of this concept are investigated in [3]. Finally, in [12], a further study of this concept is provided. Unfortunately, some assertions formulated (and proved) in that paper are false, and some steps in proofs of other results are stated without any sufficient argument. In the present paper we construct suitable examples illustrating falsehood of certain implications from [12], and we repair proofs of some results either by filling up the gaps, or by giving another argument. We start with a sequence of remarks concerning results presented in [12].

REMARK 1. In the first section of [12], in Lemma 1 on p. 258 it is stated that if a continuum X has property (b) on a subcontinuum Y of X, and Z is a subcontinuum of Y, then Y has property (b) on Z . The following example shows that this implication is not true even for locally connected continua. Let $X = \{z \in C : |z| \leq 1\}$ and let $Y = Z = S$ be its boundary in the plane C. Then X has property (b) on Y, while Y does not have property (b) (on $Z = Y$) as the identity mapping shows (or because Y is locally connected and not unicoherent).

REMARKS 2. (A) In the second section of [12], Theorem 8 on p. 259 says that if Y is a subcontinuum of a continuum X and if for each pair of proper subcontinua A and B of X the continuum X has property (b) on $A \cap B \cap Y$, then X is unicoherent at Y. The formulation of this theorem is confusing, because property (b) on a subset Y of the continuum X is defined only in the case when Y is connected, i.e., when Y is a subcontinuum of X. Thus it is (tacitly) assumed in Theorem 8 that the intersection $A \cap B \cap Y$ is connected, but then it is nothing to prove, because this is just the definition of unicoherence of X at Y. Trying to repair this inaccuracy, one can extend the definition of unicoherence of X at Y to any (closed) subset Y of X claiming that the same condition is satisfied, i.e., that $f \sim 1$ *on* Y holds for each mapping $f : X \to S$. Precisely in this sense the author uses the concept in his proof of the theorem. But then the function $f : X \to S$ considered in the proof is not well-defined (contrary to the author's statement) as it can be seen by the following example. Let $X = S$ be supplied with the metric d between two points defined as the length of the shortest arc contained in S and joining the points. Further, let $Y = \{z \in S : \arg z \in [0, \pi]\}$, $A = \{z \in S : \arg z \in$

$[-\pi, 0]\}$, and $B = \{z \in S : \arg z \in [-\pi/3, 4\pi/3]\}$. Then $C = \{1\}$ and $D = \{-1\}$. For the point $z = e^{-i\pi/3} \in A \cap B$ we have $d(x, C) = \pi/3$, $d(x, D) = 2\pi/3$, so $\varphi(x) = \pi/3$, and thereby $f(x)$ takes two different (conjugate) values according to each of the two parts of its definition.

(B) Furthermore, as an argument in the proof of the above discussed Theorem 8, Proposition 5 (of [12, p. 258]) is used, in which the conclusion of the theorem (that is, unicoherence of X at Y) is assumed.

REMARK 3. Still in the same second section of [12], Theorem 9 is formulated saying that if a continuum X is locally connected and Y is a subcontinuum of X, then

(1) X is unicoherent at Y

if and only if

(2) X has property (b) on Y.

Our next example shows that the equivalence does not hold. In fact, let (as previously) $X = S$ and Y be the upper semi-circle. Then X has property (b) on Y (since Y is an arc), while it is not unicoherent at Y, because taking $A = Y$ and B as the lower semicircle, we get $A \cap B \cap Y = \{1, -1\}$. Thus (2) does not imply (1).

REMARK 4. In the proof of the opposite implication of Theorem 9 of [12] the author factorizes the mapping $f : X \to S$ as $f = f_2 \circ f_1$, where $f_1 : X \to X'$ is monotone and $f_2 : X' \to S$ is light (by Whyburn's factorization theorem [11, (2.3), p. 297]). Applying Hurewicz's theorem (see e.g. [4, §45, VI, Theorem 1, p. 114]) it follows that X' is one-dimensional. Then, for an arbitrary subcontinuum Y of X at which X is unicoherent, it is stated without any argument that its image $Y' = f_1(Y) \subset X'$ is locally connected, and so it is a dendrite. (Recall that a *dendrite* is defined as a locally connected continuum that contains no simple closed curve.) Since this statement is not obvious, we conclude that this is a gap in the proof of the implication. The gap is fulfilled below, and, as the reader will see, it needs some work to complete proper arguments.

We start with three auxiliary assertions. The first two of them are easy to prove. The second one is formulated as Exercise 5.28 in [8, p. 85].

ASSERTION 1. *If Z is a locally connected space and C is a component of a subset A of Z, then $\operatorname{bd} C \subset \operatorname{bd} A$.*

ASSERTION 2. *If Z is a connected space, U is a connected subset of Z and C is a component of $Z \setminus U$, then $Z \setminus C$ is connected.*

Assertion 3. *If Z is a continuum and U and V are two nonempty open subsets of Z such that* $(\mathrm{cl}\, U) \cap (\mathrm{cl}\, V) = \emptyset$, *then there exists a component C of* $Z \setminus (U \cup V)$ *such that* $C \cap \mathrm{cl}\, U \neq \emptyset \neq C \cap \mathrm{cl}\, V$.

Proof. Fix a point $p \in U$. Let D be a component of $Z \setminus V$ which contains p. By the boundary bumping theorem (see e.g. [8, Theorem 5.7, p. 75]) there exists a point $q \in D \cap \mathrm{cl}\, V \neq \emptyset$. Let E be the component of $D \setminus (D \cap U)$ which contains q. Applying the boundary bumping theorem to the continuum D we infer that there exists a point $r \in E \cap \mathrm{cl}_D(D \cap U)$. Then $E \subset Z \setminus (U \cup V)$ and $E \cap \mathrm{cl}\, U \neq \emptyset \neq E \cap \mathrm{cl}\, V$. Then the component C of $Z \setminus (U \cup V)$ which contains r (and thus E) satisfies the required conditions. $\square$

In the proof of the implication from (1) to (2) (see Remark 3) of Theorem 9 of [12, pp. 259 and 260]) the following theorem is used (compare Remark 4).

Theorem 1. *Let a continuum X be locally connected and one-dimensional. If Y is a subcontinuum of X such that X is unicoherent at Y, then Y is a dendrite.*

Proof. First we show that Y is locally connected. So, suppose on the contrary that there exists a point $y \in Y$ at which y is not connected im kleinen. Then there is an open subset W of Y such that $y \in W$ and if D is a component of W which contains y, then $y \notin \mathrm{int}_Y D$. Since X is one-dimensional, there exists an open subset R of X such that $y \in R$, $\mathrm{cl}_Y(R \cap Y) \subset W$ and $\mathrm{bd}\, R$ is zero-dimensional. Let S be a component of R which contains y. By Assertion 1 $\mathrm{bd}\, S$ is zero-dimensional. Since X is locally connected, S is open in X. By the choice of W, $\mathrm{cl}_Y(S \cap Y)$ is not connected. Then there exist two closed nonempty disjoint subsets H and K of Y such that $\mathrm{cl}_Y(S \cap Y) = H \cup K$. Let $U = (S \cap Y) \setminus K$ and $V = (S \cap Y) \setminus H$. Thus U and V are nonempty open subsets of Y such that $U \subset H$, $V \subset K$ and $S \cap Y = U \cup V$. Therefore $\mathrm{cl}_Y U \cap \mathrm{cl}_Y V = \emptyset$. By Assertion 3 there exists a component C of $Y \setminus (U \cup V)$ such that $C \cap \mathrm{cl}_Y U \neq \emptyset \neq C \cap \mathrm{cl}_Y V$. Notice that $C \cap S = \emptyset$.

Let D be the component of $X \setminus S$ with $C \subset D$. By Assertion 2 the set $X \setminus D$ is connected. Put $A = D$ and $B = \mathrm{cl}(X \setminus D)$. Then A and B are closed connnected subsets of X such that $X = A \cup B$ and $A \cap B = \mathrm{bd}\, D \subset \mathrm{bd}\, S$. Thus $A \cap B$ is zero-dimensional. Therefore $A \cap B \cap Y$ is totally disconnected. On the other hand, since X is unicoherent at Y, the intersection $A \cap B \cap Y$ is connected, whence it follows that it a one-point set. This is impossible because $A \cap B \cap Y$ contains the disjoint nonempty sets $C \cap \mathrm{cl}\, U$ and $C \cap \mathrm{cl}\, V$. This contradiction proves that Y is locally connected.

Now we prove that Y contains no simple closed curve. Suppose on the contrary that there is a simple closed curve T contained in Y. Fix two points p and q in T. Let R be a connected open subset of X such that $p \in R$, $q \notin \mathrm{cl}\, R$ and $\mathrm{bd}\, R$ is zero-dimensional. Let E be the component of $X \backslash R$ which contains q. Then E and $\mathrm{cl}(X \backslash E)$ are connected closed subsets of X. By the unicoherence of X at Y the intersection $E \cap \mathrm{cl}(X \backslash E) \cap Y = (\mathrm{bd}\, E) \cap Y$ is a connected subset of Y. On the other hand, it follows from Assertion 1 that $\mathrm{bd}\, E \subset \mathrm{bd}(X \backslash R) = \mathrm{bd}\, R$, and thus $\mathrm{bd}\, E$ is zero-dimensional. Therefore $(\mathrm{bd}\, E) \cap Y$ is a one-point set. But each of the two arcs in T joining p and q contains points in $(\mathrm{bd}\, E) \cap Y \backslash \{p, q\}$. This contradiction shows that Y contains no simple closed curve. Therefore Y is a dendrite, as needed. $\square$

Since the rest part of the proof of the implication from (1) to (2) in Theorem 9 in [12] is correct, we infer from Remark 3 and Theorem 1 that the mentioned Theorem 9 of [12] has to be reduced to one implication only. This is stated below.

COROLLARY 1 (Zhou Youcheng). *If a continuum X is locally connected, then the unicoherence of X at a subcontinuum Y of X implies that X has property* (b) *on Y.*

A mapping $f : X \to Y$ between spaces X and Y is called a *local homeomorphism* provided that for each point x of X there exists an open neighborhood U of x such that $f(U)$ is an open subset of Y and that the restriction $f|U : U \to f(U)$ is a homeomorphism. It is known that a mapping between compact spaces is a local homeomorphism if and only if it is open and k-to-one for some positive integer k (see [11, (6.2), p. 200] and [5, Theorem 4, p. 856]). Thus the class of local homeomorphisms of continua is situated between homeomorphisms and finite-to-one mappings. Since homeomorphisms obviously preserve unicoherence at subcontinua, while finite-to-one mappings do not (even if the domain space X is a linear graph, see [3, Statement 14, p. 213]), it is natural to ask whether or not unicoherence at subcontinua is preserved under local homeomorphisms [3, Question 15, p. 213].

REMARK 5. The third section of [12] contains a theorem (viz. Theorem 10, p. 260) which states that unicoherence of a locally connected continuum at its subcontinuum is an invariant property under local homeomorphisms. However, Lemma 1 of [12] (which is known to be false according to Remark 1) is used as an essential argument in the proof of this theorem, and thus the proof is invalid. In this situation Question 15 of [3 p. 213] remains still open.

A unicoherent continuum X is said to be *strongly unicoherent* provided that for every two proper subcontinua A and B of X such that $X = A \cup B$ both A and B are unicoherent (see [2]; compare Section 2 of [9]). A continuum X is said to be *hereditarily unicoherent* provided that every of its subcontinua is unicoherent. Thus strong unicoherence is a property that is situated between unicoherence and hereditary unicoherence. A hereditarily unicoherent and arcwise connected continuum is called *a dendroid*.

REMARK 6. It is stated in Corollary 11 of [12], p. 261, that strong unicoherence of locally connected continua is an invariant property under local homeomorphisms. However, the proof of this corollary is based on the above mentioned Theorem 10 of [12] whose proof is incorrect (see Remark 5). We give here an appropriate argument, even for a much stronger result, where local connectedness of the considered continuum is replaced by a weaker condition of its arcwise connectedness. But it has to be stressed that even under these weakened assumptions the result is trivial, because in the considered circumstances all local homeomorphisms are homeomorphisms.

PROPOSITION 1. *Let a continuum X be arcwise connected and strongly unicoherent, and let a surjective mapping $f : X \to Y$ be a local homeomorphism. Then f is a homeomorphism, and consequently Y is strongly unicoherent.*

PROOF. Since for arcwise connected continua strong unicoherence is equivalent to the property of being a dendroid (see [7, Corollary 2, p. 407]), the domain X of f is a dendroid. Since every local homeomorphism defined on a dendroid is a homeomorphism (see [5, Corollary 10, p. 858]), the conclusion follows.   □

COROLLARY 2 (Zhou Youcheng). *Let a continuum X be locally connected and strongly unicoherent, and let a surjective mapping $f : X \to Y$ be a local homeomorphism. Then (f is a homeomorphism, and consequently) Y is strongly unicoherent.*

REMARK 7. It follows from Proposition 1 that the question on invariance of strong unicoherence of continua under local homeomorphisms is interesting only for such continua that are not arcwise connected.

REMARK 8. Very recently Isabel Puga has shown that if a locally connected continuum X is unicoherent at a subcontinuum Y, then Y is locally connected

and unicoherent. Since each locally connected unicoherent one-dimensional continuum is a dendrite (see e.g. [4, §57, III, Corollary 8, p. 442]), Puga's results are more general than our Theorem 1.

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(J. J. Charatonik and W. J. Charatonik)
Mathematical Institute, University of Wrocław,
pl. Grunwaldzki 2/4, 50-384 Wrocław,
Poland

Current address, J. J. Charatonik:
Instituto de Matemáticas, UNAM,
Circuito Exterior, Ciudad Universitaria, 04510 México, D. F.,
México

Current address, W. J. Charatonik:
Departamento de Matemáticas,
Facultad de Ciencias, UNAM, Circuito Exterior,
Ciudad Universitaria, 04510 México, D. F.,
México

(A. Illanes) Instituto de Matemáticas, UNAM,
Circuito Exterior, Ciudad Universitaria, 04510 México, D. F.,
México

E-mail address, J. J. Charatonik: jjc@hera.math.uni.wroc.pl,
and jjc@gauss.matem.unam.mx

E-mail address, W. J. Charatonik: wjcharat@hera.math.uni.wroc.pl,
and wjcharat@lya.fciencias.unam.mx

E-mail address, A. Illanes: illanes@gauss.matem.unam.mx