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ON SECOND SOCLES OF FINITELY COGENERATED INJECTIVE MODULES

By

Kazutoshi KOIKE

In [3, Theorem] Clark and Huynh proved that a right and left perfect right self-injective ring R is QF if and only if the second socle of R_R is finitely generated as a right R-module. In this note, using the technique in the proof of this theorem, we prove that if $E(T)/T$ is finitely cogenerated for every simple right R-module T, then every finitely cogenerated seminoetherian right R-module is of finite length (Theorem 5). Here, seminoetherian modules mean modules whose every nonzero submodule contains a maximal submodule. As a corollary, we obtain the theorem of Clark and Huynh (Corollary 7). Also we point out a condition for certain right perfect rings to have Morita duality (Corollary 10). In the last part of this note, we mention a dual of Theorem 5 (Theorem 13).

Throughout this note, R always denotes a ring with $J = \mathrm{Rad}(R)$. For an R-module X, $\mathrm{Soc}_k(X)$ denotes the kth socle of X for each positive integer k. For notations, definitions and familiar results concerning the ring theory we shall mainly follow [1] and [10].

First we begin with the following lemma.

LEMMA 1. *Let X and Y be right R-modules. Then*

(1) $\mathrm{Soc}_2(X \oplus Y)/\mathrm{Soc}(X \oplus Y)$ *is finitely generated if and only if* $\mathrm{Soc}_2(X)/\mathrm{Soc}(X)$ *and* $\mathrm{Soc}_2(Y)/\mathrm{Soc}(Y)$ *are finitely generated.*

(2) *If $X \leq Y$, then* $\mathrm{Soc}_k(X) = \mathrm{Soc}_k(Y) \cap X$ *for each positive integer k.*

PROOF. (1) This is clear from the fact that

$$\mathrm{Soc}_2(X \oplus Y)/\mathrm{Soc}(X \oplus Y) \cong \mathrm{Soc}_2(X)/\mathrm{Soc}(X) \oplus \mathrm{Soc}_2(Y)/\mathrm{Soc}(Y).$$

(2) This is a special case of [9, Proposition 3.1]. □

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We recall that a right R-module X is said to be *finitely cogenerated* in case for every set $\mathscr{A}$ of submodules of X, $\bigcap \mathscr{A} = 0$ implies $\bigcap \mathscr{F} = 0$ for some finite $\mathscr{F} \subseteq \mathscr{A}$. For finitely cogenerated right R-modules, we note the following.

LEMMA 2 (cf. [1, Proposition 10.7]). *A right R-module X is finitely cogenerated if and only if* $\mathrm{Soc}(X)$ *is finitely generated and is essential in X.*

In order to prove our main result, we need the following two lemmas.

LEMMA 3. *Suppose that* $E(T)/T$ *is finitely cogenerated for every simple right R-module T. If* X_R *is finitely cogenerated, then* $X/\mathrm{Soc}_k(X)$ *is finitely cogenerated for each nonnegative integer k. In this case, each* $\mathrm{Soc}_k(X)$ *is of finite length.*

PROOF. By assumption and Lemmas 1 and 2, for every finitely cogenerated injective module E_R, $E/\mathrm{Soc}(E)$ is finitely cogenerated. Let X_R be finitely cogenerated. We prove that $X/\mathrm{Soc}_k(X)$ is finitely cogenerated by induction on k. If $k = 0$, the statement is trivial. Assume that $X/\mathrm{Soc}_k(X)$ is finitely cogenerated for $k \geq 0$. Let $\overline{X} = X/\mathrm{Soc}_k(X)$. Then $E(\overline{X})$ is finitely cogenerated injective, $\mathrm{Soc}(\overline{X}) = \mathrm{Soc}(E(\overline{X}))$ and $\overline{X}/\mathrm{Soc}(\overline{X}) \leq E(\overline{X})/\mathrm{Soc}(E(\overline{X}))$. As we mentioned above, $E(\overline{X})/\mathrm{Soc}(E(\overline{X}))$ is finitely cogenerated, so $\overline{X}/\mathrm{Soc}(\overline{X})$ is also. Thus $X/\mathrm{Soc}_{k+1}(X) \cong \overline{X}/\mathrm{Soc}(\overline{X})$ is finitely cogenerated. Therefore, by induction, every $X/\mathrm{Soc}_k(X)$ is finitely cogenerated. The last statement of this lemma follows from the fact that $\mathrm{Soc}(X)$, $\mathrm{Soc}_2(X)/\mathrm{Soc}(X), \ldots, \mathrm{Soc}_k(X)/\mathrm{Soc}_{k-1}(X)$ are all finitely generated. □

LEMMA 4. *Suppose that* $E(T)/T$ *is finitely cogenerated for every simple right R-module T. If* X_R *is finitely cogenerated and* $Y_R \leq X_R$ *such that* Y_R *is of finite length, then* X/Y *is finitely cogenerated.*

PROOF. Since Y is of finite length, by Lemma 1 there exists $k \geq 0$ such that $Y \leq \mathrm{Soc}_k(X)$. Now we have an exact sequence

$$0 \to \mathrm{Soc}_k(X)/Y \to X/Y \to X/\mathrm{Soc}_k(X) \to 0.$$

By Lemma 3, $\mathrm{Soc}_k(X)$ is of finite length; so $\mathrm{Soc}_k(X)/Y$ is finitely cogenerated. On the other hand, $X/\mathrm{Soc}_k(X)$ is finitely cogenerated by Lemma 3 again. Therefore X/Y is finitely cogenerated by [11, 21.4(2)]. □

Recall that a module X is *semiartinian* if and only if every proper factor module of X has a simple submodule (see [10, p. 182]). Dualizing this, we say that a module is *seminoetherian* in case every nonzero submodule has a maximal submodule (see [4]).

THEOREM 5. *Suppose that $E(T)/T$ is finitely cogenerated for every simple right R-module T. Then every finitely cogenerated seminoetherian right R-module is of finite length.*

PROOF. Let X_R be a finitely cogenerated seminoetherian module. First we define a descending chain (X_α) of submodules of X by transfinite induction, where α are ordinals. When $\alpha = 1$, we define X_α as a maximal submodule of X. Assume that we have defined submodules X_β for all $\beta < \alpha$. When α is a limit ordinal, we define $X_\alpha = \bigcap_{\beta<\alpha} X_\beta$. When α is not a limit ordinal with $\alpha = \beta + 1$ and $X_\beta \neq 0$, we define X_α as a maximal submodule of X_β. By transfinite induction, (X_α) is well-defined.

Since X is a set, there exists a minimal ordinal β such that $X_\beta = X_\gamma$ for all $\gamma \geq \beta$. By the definition of $(X_\alpha), X_\beta = 0$. Then, since X is finitely cogenerated, β is not a limit ordinal.

To see that β is finite, we assume that β is infinite. Then, since β is not a limit ordinal and is infinite, it follows that β can be written as $\gamma + n$, where γ is a limit ordinal and n is a positive integer. Now for the descending chain

$$X_\gamma > X_{\gamma+1} > \cdots > X_{\gamma+n} = X_\beta = 0,$$

each composition factor $X_{\gamma+i}/X_{\gamma+i+1}$ is simple by the definition, and so X_γ is of finite length. Thus, by Lemma 4, X/X_γ is finitely cogenerated. On the other hand, since γ is a limit ordinal, $X_\gamma = \bigcap_{\delta<\gamma} X_\delta$. Hence there exists an ordinal $\delta < \gamma$ such that $X_\delta = X_\gamma$. However, this is a contradiction. Therefore β is finite and X is of finite length.

REMARK 6. (1) In [4, Theorem 5] Clark and Smith proved that if

(∗) $\mathrm{Soc}_2(E(T))$ is finitely generated for every simple right R-module T,

then every semiartinian and seminoetherian right R-module with finitely generated socle is of finite length. The assumption (∗) of this result is weaker than that of Theorem 5. However, in Theorem 5 we do not assume that the module is semiartinian (see Lemma 2).

(2) If R is right perfect, $\mathrm{Rad}(X) = XJ$ and $\mathrm{Rad}(X)$ is small in X for each X_R. Thus, every nonzero right R-module has a maximal submodule and is seminoetherian.

(3) If R is left perfect, $\mathrm{Soc}(X) = l_X(J)$ and $\mathrm{Soc}(X)$ is essential in X for each X_R. Thus, every nonzero right R-module has a simple submodule and is semiartinian.

(4) If R has only finite many isomorphism classes of simple right R-modules, then $E(T)/T$ is finitely cogenerated for every simple right R-module if and only if $U/\mathrm{Soc}(U)$ is finitely cogenerated for every finitely cogenerated injective cogenerator U_R if and only if $U/\mathrm{Soc}(U)$ is finitely cogenerated for some finitely cogenerated injective cogenerator U_R.

We recall that a ring R is *right PF* if R_R is an injective cogenerator (see [5, p. 213]). As is well-known (e.g., see [2, Proposition 2.1 and Lemma 2.4]), if R is right *PF*, then $\mathrm{Soc}_k(R_R) = \mathrm{Soc}_k({}_RR)$ for each positive integer k. In this case, we simply write $\mathrm{Soc}_k(R)$ for $\mathrm{Soc}_k({}_RR)$.

The following corollary is a generalization of [3, Theorem], since every left perfect right self-injective ring is right *PF* ([5, Definition and Proposition 24.32]).

Corollary 7. *Let R be a right PF ring such that ${}_RR$ is semiartinian. Then the following statements are equivalent:*

(1) *R is QF.*

(2) *$R/\mathrm{Soc}(R)$ is finitely cogenerated as a right R-module.*

(3) *$\mathrm{Soc}_2(R)$ is finitely generated as a right R-module and $\mathrm{Soc}_2(R)/\mathrm{Soc}(R)$ is an essential right R-submodule of $R/\mathrm{Soc}(R)$.*

In particular, if R_R is also semiartinian, R is QF if and only if $\mathrm{Soc}_2(R)$ is finitely generated as a right R-module.

Proof. It is shown in [4, Proposition 2] that a one-sided self-injective ring is right perfect if ${}_RR$ is semiartinian. Thus this corollary follows from Remark 6, Lemma 2 and the fact that $\mathrm{Soc}(R)$ is finitely generated on both sides ([2, Lemma 2.4]). □

Remark 8. It is an open problem whether a one-sided perfect right self-injective ring is *QF*. As we mentioned in the introduction, in [3, Theorem] Clark and Huynh prove that a two-sided perfect right self-injective ring is *QF* if $\mathrm{Soc}_2(R)$ is finitely generated as a right R-module. Concerning this, several results are shown recently. In [4, Corollary 4] the authors point out that the perfect condition can be weakened to semiartinian. Other authors approach to the problem above by investigating the condition that the *left* R-module $\mathrm{Soc}_2(R)$ (or J/J^2) is finitely (countably) generated. These results can be found in [6], [7] and [12].

Applying Theorem 5 to right perfect rings, we have

COROLLARY 9. *Let R be a right perfect ring and let U_R be a finitely cogenerated injective cogenerator. Then the following statements are equivalent*:

(1) *$U/\mathrm{Soc}(U)$ is finitely cogenerated.*

(2) *U is of finite length.*

(3) *Every finitely cogenerated right R-module is of finite length.*

In this case, R is semiprimary.

PROOF. The equivalences of (1), (2) and (3) follow from Theorem 5 and Remark 6. If these equivalent conditions hold, then $UJ^n = 0$ for some positive integer n. Thus, since U_R cogenerates R_R, $J^n = 0$ and R is semiprimary. □

The following corollary gives a condition for certain right perfect rings to have Morita duality.

COROLLARY 10. *Let R be a right perfect ring such that R_R is finitely cogenerated and let U_R be a finitely cogenerated injective cogenerator. Then the following statements are equivalent*:

(1) *$U/\mathrm{Soc}(U)$ is finitely cogenerated.*

(2) *U is finitely generated and R is right artinian.*

(3) *${}_SU_R$ defines a Morita duality, where $S = \mathrm{End}_R(U)$.*

PROOF. (1) $\Rightarrow$ (2) By Corollary 9, U_R is of finite length and R is right artinian.

(2) $\Rightarrow$ (1) Trivial.

(2) $\Leftrightarrow$ (3) This follows from [1, Theorem 30.4, Corollary 30.5 and Exercise 28.8]. □

Finally we note dual results for preceding ones. Their proofs are almost dual and will be omitted. In general, for a right R-module X, $XJ \neq \mathrm{Rad}(X)$ and X/XJ is not semisimple. So we need to suppose that R is semilocal for the results below.

LEMMA 11 (cf. Lemma 3). *Suppose that R is semilocal and J_R is finitely generated. If X_R is finitely generated, then XJ^k is finitely generated for each non-negative integer k. In this case, each X/XJ^k is of finite length.*

Lemma 12 (cf. Lemma 4). *Suppose that R is semilocal and J_R is finitely generated. If X_R is finitely generated and $Y_R \leq X_R$ such that X/Y is of finite length, then Y is finitely generated.*

Theorem 13 (cf. Theorem 5). *Suppose that R is semilocal and J_R is finitely generated. Then every finitely generated semiartinian right R-module is of finite length.*

As a dual of Corollary 9, we can obtain a result for left perfect rings. However, this result is a part of [8, Lemma 11].

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Oshima National College of Maritime Technology
Komatsu, Oshima-cho, Oshima-gun, Yamaguchi 742-2193,
Japan
E-mail: koike@c.oshima-k.ac.jp