ON TOTALLY REAL MINIMAL SUBMANIFOLDS IN $CP^n(c)$

By

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1. Introduction

Recently, in [4] we proved : Let M^n be a minimal $n(\geq 4)$-submanifold in a Euclidean N-sphere S^N of radius 1 (resp. a Euclidean N-space R^N) which has at most two principal curvatures in the direction of any normal which satisfy that if exactly two are distinct, then we assume those multiplicites ≥ 2. Then the second fundamental form is parallel and the length S of the second fundamental form holds $S = 0$ or $n \leq S \leq n^2/4$ (resp. M^n is totally geodesic). And if $n = 4$, then $S = 4$ and M^n is locally isometric to the complex projective 2-space $P^2(4/3)$ with constant holomorphic sectional curvature $4/3$ or the product $S^2(1/\sqrt{2}) \times S^2(1/\sqrt{2})$ of two 2-spheres, where we denote the radius of spheres in the parentheses. Moreover, we obtain that if $S = n^2/4$ and that if $S = n > 4$, then M^n is locally isometric to $S^m(\sqrt{m/n}) \times S^{n-m}(\sqrt{n-m/n})$.

Let $CP^n(c)$ be an n-dimensional complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature $c(> 0)$ and M^n be a totally real n-submanifold isometrically immersed in $CP^n(c)$. Then totally umbilical submanifolds, if there exists, are the simplest submanifolds next to totally geodesic submanifolds in a Riemannian manifold. However, it was proved in [2] that a complex space form of complex dimension ≥ 2 adimits no totally umbilical, totally real submanifolds except the totally geodesic ones. According to [1], a totally real H-umbilical n-submanifold of a Kaehler manifold $\bar{M}_n$ which is introduced as the simplest totally real submanifolds next to the totally geodesic ones in complex space forms is a non-totally geodesic totally real submanifold whose second fundamental form takes the following simple form:

$$h((e_1, e_1) = \lambda Je_1, \quad h(e_2, e_2) = \cdots = h(e_n, e_n) = \mu Je_1,$$

$$h(e_1, e_j) = \mu Je_j, \quad h(e_j, e_k) = 0, \quad j \neq k, j, k = 2, \ldots, n$$

for some suitable functions λ and μ with respect to some suitable orthonormal local frame field, where J is the complex structure of $\bar{M}_n$ and also except some exceptional classes, their totally real H-umbilical n-submanifolds of complex projective spaces or of complex hyperbolic spaces are obtained from Legendre curves via Hopf's fibration in some nature ways. On the othe hand, it is given by O'Neill ([7]) the notion of an isotropic submanifolds of a Riemannian manifold which can be considered as a generalization of the totally geodesic submanifolds. With isotropic totally real n-submanifolds M^n of a complex Kaehler manifold $\bar{M}_n$ Montiel and Urbano ([5]) proved: If $n \geq 3$ and M^n is minimal, then either M^n is totally geodesic or $n = 5, 8, 14$ or 26

The purpose of this paper is to prove the following:

THEOREM. *Let M^n be a totally real, minimal submanifold in $CP^n(c)$ which has at most two principal curvatures in the direction of any normal. Then if M^n is not totally geodesic, then M^n is parallel $(n \geq 4)$, or H-umbilical minimal surface in $CP^2(c)$. In the former case, if n is even (resp. odd), then M^n is isotropic (resp. M^n does not exist). Hence M^n is Einstein and is locally congruent to the following :* $SU(3)$, $n = 8$; $SU(6)/S_p(3)$, $n = 14$ or E_6/F_4, $n = 26$.

2. Preliminaries

Let M be an isometrically immersed in an $(n + p)$-dimensional Riemannian manifold $\bar{M}$. We denote by g the metric of $\bar{M}$ as well as the one induced on M. Let ∇ (resp. $\bar{\nabla}$) denote the covariant differentiation in M (resp. $\bar{M}$), and D the covariant differentiation in the normal bundle. We denote by h and A_ξ the second fundamental form of the immersion and the Weingarten endomorphism associated to a normal vector ξ, respectively. If $\xi_1, \ldots, \xi_p$ are now orthonormal normal vector fields in a neighborhood U of x, then they determine normal connection forms $s_{\alpha\beta}$ in U by

$$D_X \xi_\alpha = \sum_\beta s_{\alpha\beta}(X)\xi_\beta, \quad s_{\alpha\beta} + s_{\beta\alpha} = 0$$

for $X \in T_x M$, where $T_x M$ denotes the tangent space of M at x. Let X and Y be tangent to M and $\xi_1, \ldots, \xi_p$ orthonormal normal vector fields. Then we have the following relationships (in this paper Greek indices run from 1 to p):

$$(1) \qquad (\nabla_X A_\alpha)Y - \sum_\beta s_{\alpha\beta}(X)A_\beta Y = (\nabla_Y A_\alpha)X - \sum_\beta s_{\alpha\beta}(Y)A_\beta X$$

—Codazzi equation

Let ∇^* denote the sum of the tangential and the normal connections. ∇^* is the connection in the Whitney sum of the tangent bundle of M and the normal bundle of M induced by ∇ and D. Then we have

$$(2) \qquad \nabla_X^* A_\alpha = \nabla_X A_\alpha - \sum_\beta s_{\alpha\beta}(X) A_\beta,$$

$$(3) \qquad (\nabla_X^* A_\alpha) Y = (\nabla_Y^* A_\alpha) X.$$

We say that M is λ-*isotropic* provided that $|h(v,v)| = \lambda$ for all unit vector v in M. And we call M an *totally real* submanifold of a Kaehler manifold $\bar{M}$ if M admits an isometric immersion into $\bar{M}$ such that for all $x, J(T_xM) \subset T_x^\perp M$, where $T_x^\perp M$ denotes the normal space at x and J the complex structure of $\bar{M}$.

DEFINITION. For $x \in M$, the *first normal space*, $N_1(x)$, is the orthogonal complement in $T_x^\perp M$ of the set

$$N_0(x) = \{\xi \in T_x^\perp M \,|\, A_\xi = 0\}.$$

We define a new inner product, $\langle \,,\, \rangle$ on $N_1(x)$ ([4]) by

$$\langle \xi, \eta \rangle = \text{trace } A_\xi A\eta \text{ for } \xi, \eta \in N_1(x).$$

One easily checks that $\langle \,,\, \rangle$ is a positive definite inner product on $N_1(x)$. The following lemmas can be proved in the case of totally real, minimal submanifolds in $CP^n(c)$ ([4]):

LEMMA 1. *Let M^n be a totally real, minimal submanifolds in $CP^n(c)$. If at each point x of M, the second fundamental form of M in the direction of any normal ξ has two distinct eigenvalues $\lambda(\xi) \neq \mu(\xi)$, then we have the following:*

(i) *The distribution $T_{\lambda(\xi)} = \{X \,|\, A_\xi X = \lambda(\xi)X\}$ is differentiable.*

(ii) *If $X \in T_{\lambda(\xi)}$, then $A_\eta X \in T_{\mu(\xi)}$ for any normal η which is orthogonal to ξ with respect to the inner product $\langle \,,\, \rangle$.*

(iii) *If ξ is a unit normal with respect to the Riemannian metric g of $\bar{M}$ and $\dim T_{\lambda(\xi)} > 1$, then $X \cdot \lambda(\xi) = 0$ for $X \in T_{\lambda(\xi)}$. Thus if the multiplicities of $\lambda(\xi)$ and $\mu(\xi) \geq 2$, then $\lambda(\xi) = const.$ and $\mu(\xi) = const.*

Let k dimension of $N_1(x)$ of a totally real submanifold M in $CP^n(c)$, where k is constant by means of (iii) of Lemma 1. Then:

LEMMA 2. *Let M^n be a totally real, minimal submanifold in $CP^n(c)$ which has at most two principal curvatures (If exactly two are distinct, then we assume*

those multiplicities ≥ 2) *in the direction of any normal. Then* $\nabla^* A_\alpha = 0$ *for* $1 \leq \alpha \leq k$.

The following lemma also holds ([3]):

LEMMA 3. *Let* M *be a totally real, minimal submanifold in* $CP^n(c)$. *If* M *is* λ-*isotropic and* $n \geq 3$, *then* M *is Einstein so that* λ *is constant.*

3. Proof of theorem

Let M be a totally real, minimal submanifold in $CP^n(c)$ which has at most two principal curvatures in the direction of any normal. At first, we suppose that if exactly two are distinct, then those multiplicities ≥ 2. Since M is totally real, it holds the same Codazzi equation of M as one of submanifolds of a Riemannian manifold, and the normal curvature tensor $R^\perp$ satisfies

$$R^\perp(X, Y)\xi = h(X, A_\xi Y) - h(Y, A_\xi X) + \frac{c}{4}\{g(JY, \xi)JX - g(JX, \xi)JY\}.$$

Let k the dimension of $N_1(x)$. From Lemma 2 we, at first, see that the second fundamental form A_α, $\xi_\alpha \in N_1(x)$, is parallel. Assume that $\gamma > k$. Then $A_\gamma = 0$ on M, $\nabla A_\gamma = 0$ and $[A_\gamma, A_\beta] = 0$, $\gamma \neq \beta$, $1 \leq \beta \leq n$. Thus $R^\perp(X, Y)\xi_\gamma = c/4\{g(JY, \xi_\gamma)JX - g(JX, \xi_\gamma)JY\}$. Therefore for any tangent vector X orthogonal to $J\xi_\gamma$ the normal connection form $s_{\gamma\beta}$ satisfies

$$s_{\gamma\beta}(X) = 0, \quad \beta \neq \gamma.$$

By Codazzi equation

$$0 = (\nabla_X A_\gamma)J\xi_\gamma - \sum s_{\gamma\beta}(X)A_\beta J\xi_\gamma$$

$$= (\nabla_{J\xi_\gamma} A_\gamma)X - \sum s_{\gamma\beta}(J\xi_\gamma)A_\beta X.$$

Hence,

$$\sum_{\beta=1}^{k} s_{\gamma\beta}(J\xi_\gamma)A_\beta X = 0.$$

From (ii) of Lemma 1 and the assumption of the multiplicity we obtain

$$s_{\gamma\beta}(J\xi_\gamma) = 0, \quad 1 \leq \beta \leq k.$$

Thus A_γ is parallel. Therefore we see that the second fundamental form of M is parallel. From (ii) of Lemma 1 we can choose a normal basis $\xi_1, \ldots, \xi_k$ for

$N_1(x)$ which are orthogonal with respect to $\langle\ ,\ \rangle$ and unit with respect to the Riemannian metric g of $\overline{M}$ so that

$$A_\alpha = \begin{pmatrix} \lambda_\alpha I_p & 0 \\ 0 & \mu_\alpha I_{n-p} \end{pmatrix}, \quad A_\beta = \begin{pmatrix} 0 & B \\ {}^t B & 0 \end{pmatrix}, \quad \beta \neq \alpha,$$

where I_p (resp. I_{n-p}) is a $p \times p$ (resp. an $(n-p) \times (n-p)$) identity matrix.

Now, we define f by $f(v) = |h(v,v)|^2$. Let v be any unit vector in $T_x M$ which satisfies $f(v) \neq 0$. We choose a normal basis $\{\xi_1 = (h(v,v)/|h(v,v)|), \xi_2, \ldots, \xi_n\}$ at x such that $\xi_1, \ldots, \xi_{n-1}$ and ξ_n are unit with respect to the Riemannian metric of $\overline{M}$ and mutually orthogonal with respect to $\langle\ ,\ \rangle$.

Then if n is even, then

$$A_1 = \begin{pmatrix} |h(v,v)|I_{n/2} & 0 \\ 0 & -|h(v,v)|I_{n/2} \end{pmatrix}, \quad A_\beta = \begin{pmatrix} 0 & C \\ {}^t C & 0 \end{pmatrix}, \quad \beta \neq 1$$

Thus we have

$$f(e_1) = \cdots = f(e_n)$$

for an orthonormal basis $\{v = e_1, \ldots, e_n\}$ of $T_x M$ which diagonalizes the matrix A_1. Then we see that for any orthonormal vectors u, v

$$g(h(v,v), h(v,u)) = 0.$$

Hence M is isotropic (for example, see [3]). From Lemma 3 we see that M is Einstein. By [6] we obtain the conclusions.

On the other hand, if n is odd, then the equation

$$|\lambda I_n - A_\beta| = 0, \beta \neq 1$$

has the solution 0. Thus $A_\beta = 0$.

Then,

$$A_1 = \begin{pmatrix} |h(v,v)|I_p & 0 \\ 0 & -\dfrac{p}{n-p}|h(v,v)|I_{n-p} \end{pmatrix}, \quad 2 \leq p < n-p.$$

Note that $k = 1 = $ constant. Then we can choose an orthonormal basis $\{e_1, \ldots, e_n\}$ such that

$$A_{Je_1} = \begin{pmatrix} \lambda I_p & 0 \\ 0 & -\dfrac{p}{n-p}\lambda I_{n-p} \end{pmatrix}, \quad 2 \leq p < n-p, \quad A_{Je_\beta} = 0, \quad \beta \neq 1.$$

Since M^n then is totally real, we see that $\lambda = g(A_{Je_1}e_2, e_2) = g(h(e_2,e_2), Je_1) = g(h(e_1,e_2), Je_2) = 0$.

Next, if at least one, say, A_α of A_β, $1 \le \beta \le n$ has a eigenvalue λ with the multiplicity 1, then

$$A_\alpha = \begin{pmatrix} \lambda & 0 \\ 0 & \mu I_{n-1} \end{pmatrix}, \quad A_\beta = \begin{pmatrix} 0 & D \\ {}^tD & 0 \end{pmatrix}, \quad \beta \ne \alpha,$$

where $D = (a_2, \ldots, a_n)$. If $n \ge 3$, then

$$|\lambda I_n - A_\beta| = 0$$

has 0-solution. Hence $A_\beta = 0$. Then by the similar way with the above we see that M is totally geodesic. It remains the case of $n = 2$. Then from Lemma 1 we see that M is H-umbilical minimal surface. This proves Theorem.

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