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ASYMPTOTIC DISTRIBUTION OF NEGATIVE EIGENVALUES FOR TWO DIMENSIONAL PAULI OPERATORS WITH SPHERICALLY SYMMETRIC MAGNETIC FIELDS

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1. Introduction

We study the asymptotic distribution near the origin of negative eigenvalues for two dimensional Pauli operators with nonconstant magnetic fields. The Pauli operator describes the motion of a particle with spin in a magnetic fields. It acts on the space $L^2(R^3) \otimes C^2$ and is defined as

$$H_P = (-i\nabla_x - A)^2 - \sigma \cdot B$$

under a suitable normalization of units, where $A : R^3 \to R^3$ is a magnetic potential, $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ with components

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

is the vector of 2×2 Pauli matrices and $B = \nabla \times A$ is a magnetic field. If $A(x) = (a_1, a_2, 0)$ with components $a_j = a_j(x_1, x_2)$, $x = (x_1, x_2) \in R^2$, then the magnetic field $B(x) = (0, 0, b(x))$ is directed along the x_3 axis and is identified with the function $b(x) = \partial_1 a_2 - \partial_2 a_1$, $\partial_j = \partial/\partial x_j$. The Pauli operator also takes the simple form

$$H_P = \begin{pmatrix} H_+ - \partial_3^2 & 0 \\ 0 & H_- - \partial_3^2 \end{pmatrix},$$

where

$$H_\pm = \Pi_1^2 + \Pi_2^2 \mp b, \quad \Pi_j = -i\partial_j - a_j. \tag{1.1}$$

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The magnetic field b is represented as the commutator $b = i[\Pi_2, \Pi_1]$ and hence $H_\pm$ can be rewritten as

$$H_\pm = (\Pi_1 \pm i\Pi_2)^*(\Pi_1 \pm i\Pi_2). \tag{1.2}$$

This implies that $H_\pm \geq 0$ is non-negative. If, in particular, $b(x) > c > 0$ is positive, then H_- becomes strictly positive, while it is known ([1]) that H_+ has zero as an eigenvalue with infinite multiplicities. Hence the operator H_+ has the origin as the bottom of its essential spectrum.

We consider the two dimensional Pauli operator

$$H(V) = H_+ - V = \Pi_1^2 + \Pi_2^2 - b - V \tag{1.3}$$

perturbed by an electric potential $V(x)$. If $V(x) > 0$ falls off at infinity, then the operator $H(V)$ has an infinite number of negative discrete eigenvalues accumulating the origin. The aim of the present work is to study the asymptotic distribution near the origin of such negative eigenvalues. This problem has been already discussed by [8, 10] when $b(x) = b$ is a constant magnetic field. We here deal with the case of nonconstant magnetic field $b(x) = b(r)$, $r = |x|$, with spherical symmetry.

We shall formulate the obtained result precisely. Let $\langle x \rangle = (1 + |x|^2)^{1/2}$. We first make several assumptions on the magnetic field $b(x)$ and the electric potential $V(x)$. The magnetic field $b(x) : R^2 \to R$ is assumed to fulfill the following three assumptions.

(b.1) $b(x) > c > 0$ is strictly positive.

(b.2) $b(x) = b(r)$ is spherically symmetric.

(b.3) $b(x)$ is smooth and obeys $|\partial_x^\alpha b(x)| \leq C_\alpha \langle x \rangle^{-|\alpha|}$.

The potential $V(x) : R^2 \to R$ is also assumed to satisfy the following three assumptions. There exists $d > 0$ such that:

(V.1) $V(x) \geq c\langle x \rangle^{-d}$ for some $c > 0$.

(V.2) $V(x)$ is smooth and obeys $|\partial_x^\alpha V(x)| \leq C_\alpha \langle x \rangle^{-d-|\alpha|}$.

(V.3) $-x \cdot \nabla_x V(x) = -r\partial V(x)/\partial r \geq c\langle x \rangle^{-d}$, $c > 0$, for $|x| > R \gg 1$.

Under these assumptions, the operator $H(V)$ formally defined by (1.3) admits a unique self-adjoint realization in the space $L^2 = L^2(R^2)$ with natural domain $\mathscr{D} = \{u \in L^2 : H(V)u \in L^2\}$. We denote by the same notation $H(V)$ this self-adjoint realization. Let $N(H(V) < -\lambda)$, $\lambda > 0$, be the number of negative eigenvalues less than $-\lambda$ of operator $H(V)$. We study the asymptotic behavior as $\lambda \to 0$ of this quantity. The main theorem is formulated as follows.

THEOREM 1.1. *Let the notations be as above. Assume that* (b.1) $\sim$ (b.3) *and* (V.1) $\sim$ (V.3) *are fulfilled. Then*

$$N(H(V) < -\lambda) = (2\pi)^{-1} \int_{V(x)>\lambda} b(r)\, dx(1 + o(1)), \quad \lambda \to 0.$$

The proof is based on the min-max principle and the perturbation theory for singular numbers of compact operators. The idea, in principle, is the same as that in Sobolev [8] where the asymptotic formula above has been obtained in the case of constant magnetic fields as stated above. However the argument there does not apply directly to the case of nonconstant magnetic fields, even if magnetic fields are assumed to be spherically symmetric. We require several technical improvements. As previously stated, the operator H_+ has zero as an eigenvalue with infinite multiplicities. The proof relies on the fact that the spectral function $P(x,y)$ associated with this zero eigenvalue has the rapidly decreasing property

$$P(x,y) = O((|x| + |y|)^{-N}), \quad |x| + |y| \to \infty,$$

for any $N \gg 1$, provided that $x/|x| \neq y/|y|$. This is proved by use of the Poisson summation formula. If magnetic fields are constant, then $P(x,y)$ can be explicitly calculated and the decaying property is easily checked from this representation. This is one of main technical improvements. The theorem above is expected to remain true for a class of magnetic fields without spherical symmetry and it seems to be an interesting open problem. The present method makes an essential use of spherical symmetric property at many stages in the proof and it does not extend to such a general case. Roughly speaking, the difficulty comes from the fact that magnetic potentials which actually appear in Pauli operators undergo a nonlocal change even under a local perturbation of magnetic fields. This makes it difficult to control magnetic fields by an approximate method.

Recently several works have been done on the spectral problems of Pauli operators with nonconstant megnetic fields. For example, the Lieb-Thirring inequality for negative eigenvalues has been discussed in [5, 9] and the asymptotic behavior of ground state densities in the strong field limit has been studied in [4]. The present work is motivated by these works.

2. Perturbation theory for singular numbers

As stated above, we use the perturbation theory for singular numbers of compact operators as a basic tool to prove the main theorem. We here make a brief review on several important properties of singular numbers. We refer to [6] for details.

Let $T : X \to X$ be a compact operator (not necessarily self-adjoint) acting on a separable Hilbert space X. We write $|T|$ for $\sqrt{TT^*}$. The singular numbers $s_n(T)$, $n \in N$, of T are defined as the non-increasing sequence of eigenvalues of $|T|$ and they have the following properties: $s_n(T) = s_n(T^*)$ and

$$\begin{aligned} &s_{n+m-1}(T_1 + T_2) \leq s_n(T_1) + s_m(T_2), \\ &s_{n+m-1}(T_1 T_2) \leq s_n(T_1) s_m(T_2) \end{aligned} \tag{2.1}$$

for two compact operators T_1 and T_2. We now define

$$n(\lambda; T) = \#\{n \in N : s_n(T) > \lambda\}, \quad \lambda > 0.$$

The next proposition is obtained as an immedaite consequence of (2.1) and is repeatedly used in proving the main theorem.

PROPOSITION 2.1. (1) *If* $\lambda_1, \lambda > 0$ *with* $\lambda_1 + \lambda_2 = \lambda$, *then*

$$n(\lambda; T_1 + T_2) \leq n(\lambda_1; T_1) + n(\lambda_2; T_2).$$

(2) *If* $\lambda_1, \lambda_2 > 0$ *with* $\lambda_1 \lambda_2 = \lambda$, then

$$n(\lambda; T_1 T_2) \leq n(\lambda_1; T_1) + n(\lambda_2; T_2).$$

(3) *Let* $g(\lambda)$, $\lambda > 0$, *be a function such that* $\lambda^{-\sigma}/c \leq g(\lambda) \leq c\lambda^{-\sigma}$, $c > 1$, *for some* $\sigma > 0$. *If* $\lim_{\lambda \to 0} n(\lambda; T_2)/g(\lambda) = 0$, *then one has*

$$\limsup_{\lambda \to 0} n(\lambda; T_1 + T_2)/g(\lambda) \leq \lim_{\varepsilon \downarrow 0} \limsup_{\lambda \to 0} n((1 - \varepsilon)\lambda; T_1)/g(\lambda),$$

$$\liminf_{\lambda \to 0} n(\lambda; T_1 + T_2)/g(\lambda) \geq \lim_{\varepsilon \downarrow 0} \liminf_{\lambda \to 0} n((1 + \varepsilon)\lambda; T_1)/g(\lambda).$$

We end the section by introducing another new notation. Let $T : X \to X$ be a compact self-adjoint operator. We denote by $N(T > \lambda)$ and $N(T < \lambda)$ the number of eigenvalues greater than λ and less than λ, respectively. By definition, it immediately follows that

$$n(\lambda; T) = N(T > \lambda) + N(T < -\lambda), \quad \lambda > 0.$$

If, in particular, $T \geq 0$ is non-negative, then $n(\lambda; T) = N(T > \lambda)$.

3. Spectral properies of Pauli operators

Let $H_\pm$ be defined by (1.1). In this section, we mention some basic spectral properies of these operators, which are also required to prove the main theorem. In particular, the important property is that the spectral function associated with zero eigenvalue of H_+ decreases rapidly (Proposition 3.1).

We can choose the magnetic potential to be divergenceless, so that it takes the form

$$a_1(x) = -\partial_2\varphi, \quad a_2(x) = \partial_1\varphi, \tag{3.1}$$

where $\varphi(x) = \varphi(r)$ satisfies $\Delta\varphi = b$ and is given as

$$\varphi(r) = \int_0^r r^{-1}a(r)\,dr, \quad a(r) = \int_0^r r\,b(r)\,dr. \tag{3.2}$$

The function φ is smooth and obeys the estimates

$$r^2/c \le \varphi(r) \le cr^2, \quad |\partial_x^\alpha\varphi(x)| \le C_\alpha\langle x\rangle^{2-|\alpha|} \tag{3.3}$$

for some $c > 1$, which follows from assumptions (b.1) $\sim$ (b.3). Throughout the entire discussion, we fix the magnetic potential as in (3.1) and use the notations $\varphi(r)$ and $a(r)$ with the meanings ascribed in (3.2).

We denote by (r, θ) the polar coordinate system and we often identify the unit circle with $[0, 2\pi]$. Let Π_1 and Π_2 be as in (1.1). If the magnetic potential is chosen as above, then it follows that

$$\Pi_1 + i\Pi_2 = -i\exp(-\varphi(r))(\partial_1 + i\partial_2)\exp(\varphi(r))$$

and hence we see from (1.2) that the eigenspace associated with zero eigenvalue of H_+ is spanned by the family of functions

$$u_m(x) = r^m\exp(im\theta)\exp(-\varphi(r)), \quad m \in N_* = N \cup \{0\}. \tag{3.4}$$

This is known as Aharonov-Casher theorem ([1]).

Let $P : L^2 \to L^2$ be the eigenprojection associated with the zero eigenspace of H_+. We write Q for $Id - P$, Id being the identity operator. It is also known (see, for example, [3]) that the non-zero spectra of the operators H_+ and H_- coincide with each other. Since H_- is strictly positive, we have

$$QH_+Q \ge \beta_0 Q \quad \beta_0 = \inf b(r) > 0, \tag{3.5}$$

in the form sense. The family of eigenfunctions $\{u_m\}$ forms an orthogonal system and hence the integral kernel $P(x, y)$ of the eigenprojection P is given by

$$P(x, y) = \sum_{m=0}^{\infty} v_m(x)\bar{v}_m(y), \quad v_m(x) = u_m(x)/\sqrt{e_m}, \tag{3.6}$$

where

$$(3.7)\qquad e_m = \int |u_m(x)|^2\,dx = 2\pi \int_0^\infty r^{2m+1} \exp(-2\varphi(r))\,dr.$$

PROPOSITION 3.1. *Let $x = (r, \theta)$ and $y = (r', \theta')$. If $|\theta - \theta'| > \delta > 0$, then*

$$|\partial_x^\alpha \partial_y^\beta P(x, y)| \le C_{\alpha\beta N}(1 + |x| + |y|)^{-N}$$

for any $N \gg 1$ large enough, where $C_{\alpha\beta N}$ also depends on δ.

The proof is rather long. We will prove this proposition in section 8. The proof uses the Poisson summation formula. If $b(r) = b > 0$ is constant, then $\varphi(r) = br^2/4$ and e_m is calculated as $e_m = (2\pi/b)m!(2/b)^m$, so that $P(x, y)$ has the explicit representation

$$P(x, y) = (b/2\pi) \exp(-(b/4)(|x|^2 + |y|^2 - 2|x|\,|y| \exp(i(\theta - \theta')))).$$

Thus the proposition follows at once in the case of constant magnetic fields. The lemma below is obtained as a simple application of Proposition 3.1.

LEMMA 3.2. *Let $\Gamma_j \subset [0, 2\pi]$, $1 \le j \le 2$, and let $S_j = \{x : x/|x| \in \Gamma_j\}$ be the sector generated by Γ_j. Denote by $\chi_j(x)$ the characteristic function of S_j. If the distance $d(\Gamma_1, \Gamma_2)$ between Γ_1 and Γ_2 is strictly positive, then*

$$n(\lambda; \chi_1 P \chi_2) = O(\lambda^{-\sigma}), \quad \lambda \to 0,$$

for any $\sigma > 0$ small enough.

4. Proof of Theorem 1.1

The theorem below plays a basic role in proving the main theorem. We here complete the proof of Theorem 1.1, accepting this theorem as proved.

THEOREM 4.1. *Let P again denote the eigenprojection associated with zero eigenspace of H_+. Assume that $W(x)$ fulfills* (V.1) $\sim$ (V.3). *Then*

$$\lim_{\lambda \to 0} N(PWP > \lambda)/Z(\lambda; W) = 1,$$

where

$$Z(\lambda; W) = (2\pi)^{-1} \int_{W(x) > \lambda} b(r)\,dx.$$

PROOF OF THEOREM 1.1. We first note that

$$\lambda^{-2/d}/c \le Z(\lambda; W) \le c\lambda^{-2/d}$$

for some $c > 1$ and that

$$Z((1 \pm \varepsilon)\lambda; W) = Z(\lambda; W)(1 + O(\varepsilon)), \quad \varepsilon \to 0, \tag{4.1}$$

uniformly in $\lambda > 0$ small enough. These properties follow from (V.1) $\sim$ (V.3). Let $Q = Id - P$ and $\beta_0 = \inf b(r)$ again. The operator $H(V)$ under consideration satisfies the form inequalities

$$H(V) \lesseqgtr Q(H_+ - V \pm c)Q - P(V \mp V^2/c)P \tag{4.2}$$

for any $c > 0$. We now choose c as $0 < c < \beta_0$ and define

$$T_\pm = P(V \pm V^2/c)P, \quad S_\pm = Q(H_+ - V \mp c)Q.$$

Then it follows from (4.2) that

$$N(H(V) < -\lambda) \lesseqgtr N(T_\pm > \lambda) + N(S_\pm < -\lambda).$$

The multiplication operator V is relatively compact with respect to H_+. Hence we have by (3.5) that the number $N(S_\pm < 0)$ of negative eigenvalues of $S_\pm$ is finite. We shall show that

$$\limsup_{\lambda \to 0} N(T_+ > \lambda)/Z(\lambda; V) \le 1. \tag{4.3}$$

To prove this, we decompose T_+ into $T_+ = T_1 + T_2$ and use Theorem 4.1 with $W = V$ or $W = V^2$, where $T_1 = PVP$ and $T_2 = PV^2P/c$. Since

$$\lim_{\lambda \to 0} N(T_2 > \lambda)/Z(\lambda; V) = 0$$

by Theorem 4.1, (4.3) is obtained from Proposition 2.1 and (4.1). By Proposition 2.1 again,

$$N(T_1 > (1 + \varepsilon)\lambda) \le N(T_- > \lambda) + N(T_- < -\lambda) + N(T_2 > \varepsilon\lambda)$$

for any $\varepsilon > 0$ small enough. Since $T_- = T_1 - T_2 \ge -T_2$ in the form sense, it follows that $N(T_- < -\lambda) \le N(T_2 > \lambda)$. Hence the lower bound

$$\liminf_{\lambda \to 0} N(T_- > \lambda)/Z(\lambda; V) \ge 1$$

can be also proved in a similar way. Thus the proof is complete. □

The main body of the present work is devoted to proving Theorem 4.1. Let $\{v_m\}$ be as in (3.6) and let $(\ , \ )$ denote the L^2 scalar product. Then the operator PWP in Theorem 4.1 is represented as the infinite matrix with components (Wv_m, v_l). Thus the proof of the main theorem is reduced to the study on the asymptotic distribution of eigenvalues of such an infinite matrix.

5. Spherically symmetric potentials

In this section, we first prove Theorem 4.1 for the special case that $W = W(r)$ is spherically symmetric.

LEMMA 5.1. *Assume that $W = W(r)$ is spherically symmetric and satisfies the same assumptions as in Theorem 4.1. Then*

$$\lim_{\lambda \to 0} N(PWP > \lambda)/Z(\lambda; W) = 1.$$

As an immediate consequence, we can obtain the following lemma, which is used to prove Theorem 4.1 for the general case.

LEMMA 5.2. *Assume that $W(x) \le c\langle x\rangle^{-d}$ for some $c > 0$. Then*

$$\limsup_{\lambda \to 0} \lambda^{2/d} N(PWP > \lambda) \le C$$

for some constant $C > 0$.

PROOF OF LEMMA 5.1. Let e_m be defined by (3.7). If W is spherically symmetric, then PWP is represented as a diagonal matrix and it has $\lambda_m = \alpha_m/e_m$ as eigenvalues, where

$$\alpha_m = (Wu_m, u_m) = 2\pi \int_0^\infty r^{2m+1} W(r) \exp(-2\varphi(r))\, dr.$$

Hence we have

$$N(PWP > \lambda) = \#\{m \in N_* : \alpha_m/e_m > \lambda\}.$$

We study the asymptotic behavior as $m \to \infty$ of e_m and α_m. If we make a change of variable $r \to m^{1/2}t$, then

$$e_m = 2\pi m^{m+1} \int_0^\infty t \exp(-2mg(t; m))\, dt,$$

where

$$g(t;m) = \varphi(m^{1/2}t)/m - \log t. \tag{5.1}$$

Let $a(r)$ be as in (3.2). By definition, $a(r)$ is a monotone increasing function and hence it has the inverse function $a^{-1}(r)$. Since $\varphi'(r) = a(r)/r$, the stationary point τ_m of phase function $g(t;m)$ is defined as a root to the equation $a(m^{1/2}\tau_m) = m$. The root τ_m to this equation uniquely exists and it is represented as

$$\tau_m = m^{-1/2}a^{-1}(m). \tag{5.2}$$

As is easily seen, τ_m satisfies $1/c \le \tau_m \le c$ with some $c > 1$ independent of $m \gg 1$. The function $\varphi(r)$ obeys $\Delta\varphi = b$, so that $\varphi''(r) = b(r) - a(r)/r^2$. Hence the value $g''(t;m)$ at point τ_m is calculated as

$$g''(\tau_m; m) = b(m^{1/2}\tau_m) \ge \beta_0 > 0,$$

so that $g(t;m)$ attains a minimum at $t = \tau_m$. Let $I_m = [\tau_m - \delta, \tau_m + \delta] \subset (0, \infty)$, $0 < \delta \ll 1$, be a small interval around τ_m. The family of phase functions $\{g(t;m)\}$ depends on the parameter m. It follows from (3.3) that this family is bounded in $C^\infty(I_m)$ and $|t - \tau_m|/g'(t;m)$, $t \in I_m$, is also bounded uniformly in $m \gg 1$. This enables us to apply the stationary phase method (Theorem 7.7.5 in [7]) and we obtain that

$$e_m = F_m\tau_m(1 + O(m^{-1})), \quad m \to \infty, \tag{5.3}$$

where

$$F_m = (2\pi)^{3/2}(2b(m^{1/2}\tau_m))^{-1/2}m^{m+1/2}\exp(-2mg(\tau_m; m)). \tag{5.4}$$

Similarly we can get

$$\alpha_m = F_m\tau_m W(m^{1/2}\tau_m)(1 + O(m^{-1})).$$

Hence the eigenvalue λ_m behaves like

$$\lambda_m = W(m^{1/2}\tau_m)(1 + O(m^{-1})), \quad m \to \infty.$$

Let $\varepsilon > 0$ be small enough. Then it follows from (5.2) that there exists $m_\varepsilon \gg 1$ such that

$$(1+\varepsilon)^{-1}W(a^{-1}(m)) \le \lambda_m \le (1-\varepsilon)^{-1}W(a^{-1}(m))$$

for $m \ge m_\varepsilon$. By assumption (V.3), W also has the inverse function $W^{-1}(r)$ for $0 < r \ll 1$ small enough and we have $a(W^{-1}(\lambda)) = Z(\lambda; W)$ for $\lambda > 0$ small enough. We now define

$$l_{\pm\varepsilon}(\lambda) = \#\{m \in N_* : m \geq m_\varepsilon,\ m < Z((1 \mp \varepsilon)\lambda; W)\}.$$

Then the quantity $N(PWP > \lambda)$ in question obeys the estimate

$$l_{-\varepsilon}(\lambda) \leq N(PWP > \lambda) \leq l_{+\varepsilon}(\lambda) + m_\varepsilon.$$

This, together with (4.1), proves the theorem. □

For later references, we make further comments on the asymptotic behavior of e_m. We consider the continuous version of e_m, $m \geq 1$. We define $e(\sigma)$ as

$$e(\sigma) = 2\pi\sigma^{\sigma+1} \int_0^\infty t \exp(-2\sigma g(t;\sigma))\, dt, \quad \sigma \geq 1, \tag{5.5}$$

where

$$g(t;\sigma) = \varphi(\sigma^{1/2}t)/\sigma - \log t.$$

As is easily seen, $e(m)$ coincides with e_m for integer $m \geq 1$ and also the stationary point $\tau(\sigma)$ of phase function $g(t;\sigma)$ is given by a unique root to the equation

$$a(\sigma^{1/2}\tau(\sigma)) = \sigma, \quad a(r) = r\varphi'(r). \tag{5.6}$$

The stationary point $\tau(\sigma)$ is smooth as a function of σ and it has the properties $1/c \leq \tau(\sigma) \leq c$, $c > 1$, and

$$(d/d\sigma)^k \tau(\sigma) = O(\sigma^{-k}), \quad \sigma \to \infty.$$

By repeating an argument similar to that used in the proof of Lemma 5.1, we obtain the lemma below, which is used for the proof of Proposition 3.1.

LEMMA 5.3. *Let $e(\sigma)$ be defined above. Then $1/e(\sigma)$ takes the form*

$$1/e(\sigma) = \sigma^{-1/2} G(\sigma) e^{-\sigma \log \sigma} \exp(2\sigma g(\tau(\sigma);\sigma)), \quad \sigma \geq 1,$$

where $G(\sigma)$ is a smooth function and satisfies

$$(d/d\sigma)^k G(\sigma) = O(\sigma^{-k}), \quad \sigma \to \infty.$$

If we further set $E = \sup_{\sigma \geq 1} \exp(2g(\tau(\sigma);\sigma))$, then $1/e(\sigma)$ obeys

$$1/e(\sigma) \leq c\sigma^{-(\sigma+1/2)} E^\sigma$$

for some $c > 1$.

6. Min-max principle

The proof of Theorem 4.1 is done by localizing the potential over small sectors with vertex at the origin. In this section we study a bound as $\lambda \to 0$ of the quantity $n(\lambda; W^{1/2}P)$ for a class of non-negative potentials $W(x)$ with support in such sectors by use of the min-max principle. The aim is to prove the following lemma.

LEMMA 6.1. *Let $\Gamma \subset [0, 2\pi]$ and let $\chi(\theta; \Gamma)$ be the chatacteristic function of Γ. Assume that $W(x) \geq 0$ is non-negative and obeys $W(x) \leq \chi(\theta; \Gamma)\langle x\rangle^{-d}$ for some $d > 0$. Then there exists $C > 0$ indenpendent of Γ such that*

$$\limsup_{\lambda \to 0} \lambda^{4/d} n(\lambda; W^{1/2}P) \leq C|\Gamma|^{\rho}$$

with $\rho = \min(1, 1/2d)$, where $|\Gamma|$ denotes the length of Γ.

The proof relies on the lemma below. We accept this lemma as proved and complete the proof of Lemma 6.1.

LEMMA 6.2. *Let $\chi(\theta) > 0$ be a positive smooth function over $[0, 2\pi]$. Assume that $W(x) > 0$ is also a positive smooth function and takes the form*

$$W(x) = \chi(\theta)\langle x\rangle^{-d}, \quad |x| \gg 1,$$

for $0 < d < 2$. Let $K_0 = H(W) = H_+ - W$. Then

$$\limsup_{\lambda \to 0} \lambda^{2/d} N(K_0 < -\lambda) \leq C \int_0^{2\pi} \chi(\theta)^{2/d}\, d\theta$$

for another $C > 0$ independent of $\chi(\theta)$.

PROOF OF LEMMA 6.1. The proof is divided into three steps.

(1) Let $W(x)$ fulfill the assumption in Lemma 6.2. Then we show that

$$\limsup_{\lambda \to 0} \lambda^{2/d} N(PWP > \lambda) \leq C \int_0^{2\pi} \chi(\theta)^{2/d}\, d\theta. \tag{6.1}$$

This is proved in the same way as in the proof of Theorem 1.1. We define

$$T = P(W - W^2/c)P, \quad T_1 = PWP, \quad T_2 = c^{-1}PW^2P$$

for $0 < c < \beta_0 = \inf b(r)$. Then $T_1 = T + T_2$ and it follows from Proposition 2.1 that

$$N(T_1 > \lambda) = n(\lambda; T_1) \leq n((1-\varepsilon)\lambda; T) + n(\varepsilon\lambda; T_2)$$

for any $\varepsilon > 0$ small enough. Hence we have by Lemma 5.2 that

$$\limsup_{\lambda \to 0} \lambda^{2/d} N(T_1 > \lambda) \leq \lim_{\varepsilon \to 0} \limsup_{\lambda \to 0} \lambda^{2/d} n((1-\varepsilon)\lambda; T).$$

Since $N(T > \lambda) \leq N(K_0 < -\lambda)$ by (4.2) and since $N(T < -\lambda) \leq N(T_2 > \lambda)$, we obtain

$$n(\lambda; T) = N(T > \lambda) + N(T < -\lambda) \leq N(K_0 < -\lambda) + N(T_2 > \lambda).$$

This, together with Lemmas 5.2 and 6.2, implies (6.1).

(2) The second step is to show the lemma for the case $0 < d < 2$. It suffices to prove the lemma for $W(x) = \chi(\theta; \Gamma)\langle x \rangle^{-d}$ with $0 < d < 2$. The proof is done by approximation. We approximate $W(x)$ by a monotone decreasing sequence $\{W_k(x)\}$ of positive smooth functions. The function $W_k(x)$ takes the form $W_k(x) = \chi_k(\theta)\langle x \rangle^{-d}$ for $|x| \geq 1$ and $\chi_k(\theta)$ converges to $\chi(\theta; \Gamma)$ as $k \to \infty$. Then it follows that

$$n(\lambda; W^{1/2}P) = N(PWP > \lambda^2) \leq N(PW_kP > \lambda^2).$$

Hence the lemma follows from (6.1) for the case $0 < d < 2$.

(3) The final step is to prove the lemma for the case $d \geq 2$. We again assume $W(x)$ to take the form as in step (2) and decompose W as $W = W_1^{1/2} W_2^{1/2}$, where

$$W_1(x) = \chi(\theta; \Gamma)\langle x \rangle^{-1}, \quad W_2(x) = \chi(\theta; \Gamma)\langle x \rangle^{-2d+1}.$$

If we define $T_1 = PW_1^{1/2}$ and $T_2 = W_2^{1/2}P$, then

$$n(\lambda; W^{1/2}P) = N(PWP > \lambda^2) = n(\lambda^2; T_1T_2)$$

and hence it follows from Proposition 2.1 that

$$n(\lambda; W^{1/2}P) \leq n(\mu; T_1) + n(\nu; T_2),$$

where $\mu = \lambda^{1/d}/L$ and $\nu = L\lambda^{(2d-1)/d}$ for $L > 0$. By Lemma 5.2, we have

$$\limsup_{\lambda \to 0} \lambda^{4/d} n(\nu; T_2) = \limsup_{\lambda \to 0} \lambda^{4/d} N(PW_2P > \nu^2) \leq CL^{-4/(2d-1)}$$

with some $C > 0$ independent of L. On the other hand, we have already shown in step (2) that

$$\limsup_{\lambda\to 0} \lambda^{4/d} n(\mu; T_1) \le CL^4|\Gamma|$$

with another $C > 0$. If we take $L = |\Gamma|^{-(2d-1)/8d}$, then

$$L^4|\Gamma| = L^{-4/(2d-1)} = |\Gamma|^{1/2d}$$

and hence the lemma is also proved for the case $d \ge 2$. □

We shall prove Lemma 6.2. The proof is based on the min-max principle and uses the following lemma due to Colin de Verdière [2].

LEMMA 6.3. *Let $Q(R)$ be a cube with side R. Let H_γ be the Schrödinger operator with constant magnetic field $\gamma > 0$. We consider the operator H_γ over the cube $Q(R)$ under zero Dirichlet boundary conditions and denote by $N_D(H_\gamma < \mu; Q(R))$, $\mu > 0$, the number of eigenvalues less that μ. Then*

$$N_D(H_\gamma < \mu; Q(R)) \le (2\pi)^{-1}\gamma R^2 \nu(\mu/\gamma),$$

where

$$\nu(\mu) = \#\{n \in N_* : 2n + 1 \le \mu\}.$$

PROOF OF LEMMA 6.2. The proof is divided into three steps. Throughout the proof, $\lambda > 0$ is assumed to be small enough and we use the notation $|G|$ to denote the measure of $G \subset R^2$.

(1) Let $W(x)$ be as in the lemma. We define

$$G_{j\lambda} = \{x \in R^2 : W(x) > \lambda/(j+1)\}, \quad 1 \le j \le 3.$$

Then $G_{1\lambda} \subset G_{2\lambda} \subset G_{3\lambda}$ in the strict sense. We introduce a smooth non-negative partition $\{\psi_0, \psi_1\}$ with the following properties: (1) $\psi_0(x;\lambda)^2 + \psi_1(x;\lambda)^2 = 1$ on R^2. (2) ψ_0 is suported in $G_{2\lambda}$ and $\psi_0 = 1$ on $G_{1\lambda}$. (3) ψ_0 and ψ_1 obey the estimate

$$|\partial_x^\alpha \psi_0(x;\lambda)| + |\partial_x^\alpha \psi_1(x;\lambda)| \le C_\alpha \lambda^{|\alpha|/d}$$

for C_α independent of λ. A simple calculation shows that

$$K_0 = H(W) = H_+ - W = \sum_{j=0,1} \psi_j(K_0 - Y_1)\psi_j \tag{6.2}$$

in the form sense, where

$$Y_1 = Y_1(x;\lambda) = \sum_{j=0,1} |\nabla_x \psi_j(x;\lambda)|^2 = O(\lambda^{2/d}), \quad \lambda \to 0.$$

We define

$$K_1 = H(W_1) = H_+ - W_1, \quad W_1(x;\lambda) = W(x) + Y_1(x;\lambda)$$

and use the notation $N_D(K_1 < -\lambda; G)$ with the same meaning as in Lemma 6.3 for domain $G \subset R^2$. By relation (6.2), the min-max principle implies that

$$N(K_0 < -\lambda) \leq N_D(K_1 < -\lambda; G_{2\lambda}) + N_D(K_1 < -\lambda; \Omega_\lambda),$$

where $\Omega_\lambda = R^2 \backslash \bar{G}_{1\lambda}$. Since $0 < d < 2$ by assumption, it follows that that $W_1(x;\lambda) < \lambda$ for $x \in \Omega_\lambda$ and $Y_1(x;\lambda) < \lambda/2$ for $x \in G_{2\lambda}$. This yields that $N_D(K_1 < -\lambda; \Omega_\lambda) = 0$ and

$$\text{(6.3)} \qquad N(K_0 < -\lambda) \leq N_D(K_0 < -\lambda/2; G_{2\lambda}).$$

(2) We take $M \gg 1$ large enough and denote by Q_k the cube with center at z_k and side M. We cover $G_{2\lambda}$ with a family of such cubes

$$G_{2\lambda} \subset \bigcup_{1 \leq k \leq l} Q_k, \quad l = l_\lambda.$$

This can be done in such a way that $\bigcup_{1 \leq k \leq l} Q_k \subset G_{3\lambda}$ and

$$\text{(6.4)} \qquad \sum_{k=1}^{l} |Q_k| \leq 2|G_{3\lambda}|.$$

We further introduce a non-negative smooth partition $\{\varphi_k\}_{k=1}^{l}$ subject to the covering above. The partition has the following properties: (1) $\sum_{k=1}^{l} \varphi_k(x)^2 = 1$ on $G_{2\lambda}$. (2) φ_k is supported in Q_k and obeys $|\partial_x^\alpha \varphi_k(x)| \leq C_\alpha M^{-|\alpha|}$ for C_α independent of M. Then we again obtain the form equality

$$K_0 = \sum_{k=1}^{l} \varphi_k (K_0 - Y_2) \varphi_k$$

in $C_0^\infty(G_{2\lambda})$, where

$$Y_2(x) = \sum_{k=1}^{l} |\nabla_x \varphi_k(x)|^2 \leq CM^{-2}.$$

Hence we have

$$\text{(6.5)} \qquad N_D(K_0 < -\lambda/2; G_{2\lambda}) \leq \sum_{k=1}^{l} N_D(K_2 < -\lambda/2; Q_k),$$

where

$$K_2 = H(W_2) = H_+ - W_2, \quad W_2(x) = W(x) + Y_2(x).$$

(3) We now expand the magnetic potential $a_j(x)$ as

$$a_j(x) = a_j(z_k) + \nabla_x a_j(z_k) \cdot (x - z_k) + r_{jk}(x), \quad 1 \le j \le 2,$$

in Q_k, where the remainder term r_{jk} satisfies

$$|r_{jk}(x)| \le C\langle z_k \rangle^{-1} M^2, \quad x \in Q_k,$$

for C independent of M and k. We set

$$\Lambda_{jk} = -i\partial_j - (a_j - r_{jk}) = \Pi_j + r_{jk}, \quad 1 \le j \le 2,$$

and define $H_k = \Lambda_{1k}^2 + \Lambda_{2k}^2$ as an operator acting on $L^2(Q_k)$. This operator has $b_k = b(z_k) > 0$ as a constant magnetic field and satisfies the form inequality

$$H_k/2 - q_k \le \Pi_1^2 + \Pi_2^2 \tag{6.6}$$

in $C_0^\infty(Q_k)$, where $q_k(x) = r_{1k}(x)^2 + r_{2k}(x)^2$. We may assume that $|z_k| \to \infty$ as $k \to \infty$. Thus, if we choose M large enough, then there exists $k_M \gg 1$ independent of λ such that

$$2(b(x) + W_2(x) + q_k(x) - \lambda/2) < 5b_k/2, \quad x \in Q_k,$$

for $k > k_M$. Hence it follows from (6.6) that

$$N_D(K_2 < -\lambda/2; Q_k) \le N_D(H_k < 5b_k/2; Q_k)$$

for k as above. We now use Lemma 6.3 to obtain that

$$N_D(K_2 < -\lambda/2; Q_k) \le (2\pi)^{-1} b_k |Q_k|.$$

This, together with (6.3) $\sim$ (6.5), yields that

$$N(K_0 < -\lambda) \le 2(2\pi)^{-1}\beta|G_{3\lambda}| + C$$

for C independent of λ, where $\beta = \sup b(r)$. Since

$$\limsup_{\lambda \to 0} \lambda^{2/d} |G_{3\lambda}| \le C \int_0^{2\pi} \chi(\theta)^{2/d}\, d\theta$$

for some $C > 0$, the proof is now complete. □

7. Proof of Theorem 4.1

In this section we prove Theorem 4.1. The proof further uses the following two lemmas, which are proved after completing the proof of Theorem 4.1.

LEMMA 7.1. *Assume that* $W(x)$ *satisfies* (V.1) $\sim$ (V.3). *Let* $W_j(x)$, $1 \le j \le 2$, *be defined by* $W_j = \chi(\theta; \Gamma_j) W(x)$ *for open intervals* $\Gamma_j \subset [0, 2\pi]$. *If* $\Gamma_1 \cap \Gamma_2 = \emptyset$, *then*

$$\lim_{\lambda \to 0} \lambda^{2/d} n(\lambda; W_1^{1/2} P W_2^{1/2}) = 0.$$

LEMMA 7.2. *Assume that* $W(r)$ *is spherically symmetric and satisfies* (V.1) $\sim$ (V.3). *Let* $\Gamma_L = (0, 2\pi/L)$ *for interger* L *and let* $W_L(x)$ *be defined by* $W_L = \chi(\theta; \Gamma_L) W(r)$. *Then*

$$n(\lambda; W_L^{1/2} P W_L^{1/2}) = Z(\lambda; W_L) + o(\lambda^{-2/d}), \quad \lambda \to 0,$$

where

$$Z(\lambda; W_L) = (2\pi)^{-1} \int_{W_L(x) > \lambda} b(r)\, dx = Z(\lambda; W)/L.$$

PROOF OF THEOREM 4.1. We prove only the upper bound

$$\limsup_{\lambda \to 0} N(PWP > \lambda)/Z(\lambda : W) \le 1. \tag{7.1}$$

A similar argument shows the lower bound

$$\liminf_{\lambda \to 0} N(PWP > \lambda)/Z(\lambda : W) \ge 1.$$

Let $\Gamma_{jL} = (2(j-1)\pi/L, 2j\pi/L)$, $1 \le j \le L$, for integer $L \gg 1$ and let $W_{jL}(x) = \chi(\theta; \Gamma_{jL}) W(x)$. We further define

$$T_{1L} = \sum_{j=1}^{L} W_{jL}^{1/2} P W_{jL}^{1/2}, \quad T_{2L} = \sum_{1 \le j,k \le L, j \ne k} W_{jL}^{1/2} P W_{kL}^{1/2}.$$

Then we have

$$N(PWP > \lambda) = n(\lambda; W^{1/2} P W^{1/2}) = n(\lambda; T_{1L} + T_{2L}).$$

By Proposition 2.1, it follows from Lemma 7.1 that

$$\lim_{\lambda \to 0} \lambda^{2/d} n(\lambda; T_{2L}) = 0.$$

Let $F_{jL} = W_{jL}^{1/2} P W_{jL}^{1/2}$ and let $S_{jL} = (0, \infty) \times \Gamma_{jL}$ be the sector generated by Γ_{jL}. Then F_{jL} can be regarded as an operator from $L^2(S_{jL})$ into itself and hence

$$n(\lambda; T_{1L}) = \sum_{j=1}^{L} n(\lambda; F_{jL}).$$

Thus we obtain again by Proposition 2.1 that

$$\limsup_{\lambda\to 0} N(PWP > \lambda)/Z(\lambda; W) \le \lim_{L\to\infty} \limsup_{\lambda\to 0} \sum_{j=1}^{L} n((1-1/L)\lambda; F_{jL})/Z(\lambda; W).$$

We now write $W(r,\theta)$ for $W(x)$ and denote by $\theta_{jL} = (2j-1)\pi/L$ the midpoint of interval Γ_{jL}. We set $\tilde{W}_{jL}(r) = W(r,\theta_{jL})$. We further define the operator E_{jL} as $E_{jL} = U_{jL}^{1/2} P U_{jL}^{1/2}$ with $U_{jL}(x) = \chi(\theta; \Gamma_{jL})\tilde{W}_{jL}(r)$. Then it follows from (V.1) and (V.2) that

$$W_{jL}(x) \le (1 + cL^{-1})U_{jL}(x), \quad x \in S_{jL},$$

for some $c > 0$ independent of L and hence

$$n((1-1/L)\lambda; F_{jL}) \le n((1-\varepsilon_L)\lambda; E_{jL})$$

for some $\varepsilon_L > 0$, where ε_L satisfies that $\varepsilon_L \to 0$ as $L \to \infty$. By Lemma 7.2, we have

$$n((1-\varepsilon_L)\lambda; E_{jL}) = Z((1-\varepsilon^L)\lambda; U_{jL}) + o(\lambda^{-2/d})$$

and also it follows from (V.2) and (V.3) that

$$\sum_{j=1}^{L} Z((1-\varepsilon_L)\lambda; U_{jL}) = Z(\lambda; W)(1+o(1)), \quad L \to \infty,$$

uniformly in λ small enough. Thus (7.1) is obtained and the proof of the theorem is complete. □

Proof of Lemma 7.1. If the distance $d(\Gamma_1, \Gamma_2) > 0$ is strictly positive, then it follows from Lemma 3.2 that

$$\lim_{\lambda\to 0} \lambda^{\sigma} n(\lambda; W_1^{1/2} P W_2^{1/2}) = 0$$

for any $\sigma > 0$ small enough. If $d(\Gamma_1, \Gamma_2) = 0$, then the lemma is proved by approximation. Let Γ_ε be an interval such that $\Gamma_\varepsilon \subset \Gamma_1$ with $d(\Gamma_\varepsilon, \Gamma_2) > 0$ and

$$|\Sigma_\varepsilon| = |\Gamma_1 \backslash \Gamma_\varepsilon| \le \varepsilon$$

for any $\varepsilon > 0$ small enough. We decompopse W_1 into

$$W_1(x) = \chi(\theta; \Sigma_\varepsilon)W_1(x) + \chi(\theta; \Gamma_\varepsilon)W_1(x) = U_{1\varepsilon}(x) + U_{2\varepsilon}(x).$$

It follows again from Lemma 3.2 that

$$\lim_{\lambda\to 0} \lambda^{2/d} n(\lambda; U_{2\varepsilon}^{1/2} P W_2^{1/2}) = 0$$

and also we have by Lemma 6.1 that

$$\limsup_{\lambda\to 0} \lambda^{2/d} n(\lambda; U_{1\varepsilon}^{1/2} P W_2^{1/2}) = o(1), \quad \varepsilon \to 0.$$

This can be shown by repeating the same argument as used in step (3) of the proof of Lemma 6.1. Thus the proof of the lemma is complete. □

PROOF OF LEMMA 7.2. The proof is done in almost the same way as in the proof of Theorem 4.1, so we give only a sketch for a proof. We use the notations S_{jL}, F_{jL} and T_{1L} with the meanings ascribed in the proof of Theorem 4.1. Since both $b(r)$ and $W(r)$ are spherically symmetric, $F_{jL} : L^2(S_{jL}) \to L^2(S_{jL})$ are all unitarily equivalent to the operator $F_L = W_L^{1/2} P W_L^{1/2}$ and hence

$$n(\lambda; T_{1L}) = \sum_{j=1}^{L} n(\lambda; F_{jL}) = Ln(\lambda; F_L).$$

We repeat the same argument as used in the proof of Theorem 4.1. Then we obtain by Lemmas 5.1 and 7.1 that

$$\limsup_{\lambda\to 0} n(\lambda; T_{1L})/Z(\lambda; W) \le \lim_{\varepsilon\downarrow 0} \limsup_{\lambda\to 0} N(PWP > (1-\varepsilon)\lambda)/Z(\lambda; W) = 1.$$

This implies that

$$\limsup_{\lambda\to 0} n(\lambda; F_L)/Z(\lambda; W) \le 1/L.$$

Similarly we can show that

$$\liminf_{\lambda\to 0} n(\lambda; F_L)/Z(\lambda; W) \ge 1/L.$$

Thus the proof is complete. □

8. Proof of Proposition 3.1

In this section we prove Proposition 3.1, which has played a basic role in the proof of Theorem 4.1.

PROOF OF PROPOSITION 3.1. The proof is divided into several steps. We begin by recalling the definition (3.6)

$$P(x, y) = \exp(-\eta(r, r')) \sum_{m=0}^{\infty} \rho^m \exp(im(\theta - \theta'))/e_m,$$

where $x = (r, \theta)$, $y = (r', \theta')$, $\rho = rr'$ and $\eta(r, r') = \varphi(r) + \varphi(r')$. The function $\varphi(r)$ has the properties in (3.3) and hence $\eta(r, r')$ satisfies

$$\eta(r, r') \geq c(r + r')^2$$

for some $c > 0$. Throughout the proof, we assume that $r + r' > 1$.

(1) Assume that $\rho \leq K$ for $K \gg 1$ fixed. Then we have

$$|P(x, y)| \leq \exp(-\eta(r, r')) \sum_{m=0}^{\infty} K^m / e_m.$$

It follows from Lemma 5.3 that $\sum K^m / e_m < \infty$. Thus $P(x, y)$ is shown to be rapidly deceasing

$$P(x, y) = O((r + r')^{-N}), \quad N \gg 1,$$

provided that $\rho \leq K$.

(2) Next we assume that $\rho > K \gg 1$. We introduce a smooth nonnegative cut-off function $\psi \in C_0^\infty([0, \infty))$ such that $\psi(\sigma) = 1$ for $0 \leq \sigma \leq 1$ and $\psi(\sigma) = 0$ for $\sigma \geq 2$. We fix $0 < \delta \ll 1$ small enough and define

$$\psi_1(\sigma; \rho) = \psi(\sigma/\delta\rho), \quad \psi_2(\sigma; \rho) = \psi(\delta\sigma/\rho) - \psi(\sigma/\delta\rho), \quad \psi_3(\sigma; \rho) = 1 - \psi(\delta\sigma/\rho).$$

Then $P(x, y)$ is decomposed into the sum $P(x, y) = \sum_{j=1}^{3} P_j(x, y)$, where

$$P_j(x, y) = \exp(-\eta(r, r')) \sum_{m=0}^{\infty} \psi_j(m; \rho)\rho^m \exp(im(\theta - \theta'))/e_m.$$

We shall show that each function $P_j(x, y)$ has rapidly decreasing property.

(3) We first consider $P_1(x, y)$ and $P_3(x, y)$. By definition, $P_1(x, y)$ obeys the estimate

$$|P_1(x, y)| \leq \exp(-\eta(r, r')) \sum_{m=0}^{[2\delta\rho]} \rho^m / e_m,$$

where [ ] denotes the Gauss notation. By Lemma 5.3, we have

$$\rho^m / e_m \leq cm^{-1/2} e^{m(\log E\rho - \log m)}, \quad m \geq 1.$$

If we take δ so small that $[2\delta\rho] < E\rho/e$, then $m(\log E\rho - \log m)$ is monotone increasing in m, $1 \leq m \leq [2\delta\rho]$. Since

$$[2\delta\rho](\log E\rho - \log[2\delta\rho]) = [2\delta\rho] \log E\rho/[2\delta\rho] = o(1)\rho, \quad \delta \to 0,$$

uniformly in $\rho > K$, this yields that $P_1(x,y)$ is rapidly deceasing. It is also easy to prove that $P_3(x,y)$ is rapidly deceasing. We may assume that $[\rho/\delta] > 2E\rho$. Then it follows again from Lemma 5.3 that

$$\rho^m/e_m \le cm^{-1/2}(E\rho/m)^m \le 2^{-m}, \quad m \ge [\rho/\delta].$$

This shows that $P_3(x,y)$ is also rapidly deceasing.

(4) We shall prove that $P_2(x,y)$ is rapidly deceasing. The proof uses the Poisson summation formula. Recall the definition (5.5). The function $e(\sigma)$ is defined by

$$e(\sigma) = 2\pi\sigma^{\sigma+1}\int_0^\infty t\exp(-2\sigma g(t;\sigma))\,dt, \quad \sigma \ge 1,$$

where

$$g(t;\sigma) = \varphi(\sigma^{1/2}t)/\sigma - \log t.$$

By definition, $\psi_2(\sigma;\rho)$ has support in $(\delta\rho, 2\rho/\delta)$. If we take $K > 1/\delta$ large enough, then $\psi_2(\sigma;\rho)$ vanishes over the interval $(-\infty, 1)$ for $\rho > K$. Thus

$$q(\sigma;\rho) = \psi_2(\sigma;\rho)\rho^\sigma e^{i\sigma(\theta-\theta')}/e(\sigma)$$

can be defined as a function of the Schwartz class over $(-\infty,\infty)$ and $P_2(x,y)$ is represented as

$$P_2(x,y) = \exp(-\eta(r,r'))\sum_{m=-\infty}^{\infty} q(m;\rho).$$

Hence the Poisson summation formula yields

$$P_2(x,y) = (2\pi)^{1/2}\exp(-\eta(r,r'))\sum_{m=-\infty}^{\infty}\hat{q}(2m\pi;\rho),$$

where

$$\hat{q}(2m\pi;\rho) = \int e^{-i2m\pi\sigma}q(\sigma;\rho)\,d\sigma.$$

According to Lemma 5.3, $1/e(\sigma)$ takes the form

$$1/e(\sigma) = G(\sigma)\sigma^{-1/2}e^{-\sigma\log\sigma}\exp(2\sigma g(\tau(\sigma);\sigma)),$$

where $\tau(\sigma)$ is the unique root to equation (5.6). We now rewrite $\hat{q}(2m\pi;\rho)$ as

$$\hat{q}(2m\pi;\rho) = \int e^{-\sigma u(\sigma;\rho)}\exp(i\sigma(\theta-\theta'-2m\pi))v(\sigma;\rho)\,d\sigma,$$

where

$$u(\sigma;\rho) = \log \sigma/\rho - 2\varphi(\sigma^{1/2}\tau(\sigma))/\sigma + 2\log\tau(\sigma),$$

$$v(\sigma;\rho) = \sigma^{-1/2}\psi_2(\sigma;\rho)G(\sigma).$$

We further make a change of variable $\sigma \to \rho s$ to obtain that

$$\hat{q}(2m\pi;\rho) = \int e^{-\rho f(s;\rho)} \exp(i\rho(\theta - \theta' - 2m\pi)s)w(s;\rho)\,ds,$$

where

$$f(s;\rho) = su(\rho s;\rho) = s\log s - 2\varphi((\rho s)^{1/2}\tau(\rho s))/\rho + 2s\log\tau(\rho s),$$

$$w(s;\rho) = \rho v(\rho s;\rho) = \rho^{1/2}s^{-1/2}\psi_2(\rho s;\rho)G(\rho s).$$

The function $w(s;\rho)$ has support in the interval $(\delta, 2/\delta)$ and satisfies

$$|(d/ds)^k w(s;\rho)| \leq C_k\rho^{1/2}$$

for C_k independent of ρ. We look at the stationary point of $f(s;\rho)$. Since

$$\varphi'((\rho s)^{1/2}\tau(\rho s)) = (\rho s)^{1/2}/\tau(\rho s)$$

by (5.6), $f'(s;\rho)$ is calculated as

$$f'(s;\rho) = \log s + 2\log\tau(\rho s) = \log s\tau(\rho s)^2. \tag{8.1}$$

Hence the stationary point s_ρ is given as a solution to equation

$$s_\rho\tau(\rho s_\rho)^2 = 1. \tag{8.2}$$

The equation above has a unique solution. In fact, we differentiate the both sides of (5.6)

$$a(\rho^{1/2}s^{1/2}\tau(\rho s)) = \rho s$$

with respect to s. Since $a'(r) = rb(r)$, we obtain

$$b(\rho^{1/2}s^{1/2}\tau(\rho s))(s\tau(\rho s)^2)' = 2. \tag{8.3}$$

This implies that $s\tau(\rho s)^2$ is a monotone increasing function. Thus (8.2) has a unique solution. We calculate the values of $f(s;\rho)$ and $f''(s;\rho)$ at stationary point s_ρ. It follows from relations (8.1) $\sim$ (8.3) that

$$f''(s_\rho;\rho) = 2/b(\rho^{1/2}) > 0$$

and also we have $f(s_\rho;\rho) = -2\varphi(\rho^{1/2})/\rho$ by a simple calculation. Hence $f(s;\rho)$ attains the minimum $-2\varphi(\rho^{1/2})/\rho$ at $s = s_\rho$. We assert that

$$\exp(-\eta(r,r') - \rho f(s_\rho;\rho)) = \exp(-\varphi(r) - \varphi(r') + 2\varphi((rr')^{1/2})) \le 1.$$

To see this, we set $F(t) = \varphi(e^t)$. Then we have

$$\varphi(r) + \varphi(r') - 2\varphi((rr')^{1/2}) = F(t) + F(t') - 2F((t+t')/2)$$

with $t = \log r$ and $t' = \log r'$. Since $F''(t) = e^{2t}b(e^t) > 0$, $F(t)$ is a convex function and hence the above assertion follows at once. By assumption, $\theta \ne \theta'$, so that $\theta - \theta' - 2m\pi \ne 0$ for any integer $m \in Z$. Thus we obtain by repeated use of partial integration that

$$\hat{q}(2m\pi;\rho) = (1 + |m|)^{-N} O(\rho^{-N}), \quad \rho \to \infty,$$

for any $N \gg 1$. This proves that $P_2(x,y)$ is rapidly decreasing when $1/c \le r'/r \le c$, $c > 1$. If $r \gg r'$ or $r' \gg r$, then

$$\exp(-\varphi(r) - \varphi(r') + 2\varphi((rr')^{1/2})) = O((r + r')^{-N}).$$

Hence $P_2(x,y)$ has also rapidly decreasing property in such a case.

We can show by use of the same argument as above that $\partial_x^\alpha \partial_y^\beta P(x,y)$ is also rapidly decreasing. Thus the proof of the proposition is now complete. □

References

[1] Y. Aharonov and A. Casher, Ground state of a spin-1/2 charged particle in a two-dimensional magnetic field, Phys. Rev. A, **19**, 2461–2462 (1979).

[2] Y. Colin de Verdière, L'asymptotique de Weyl pour les bouteilles magnetiques, Commun. Math. Phys., **105**, 327–335 (1986).

[3] H. Cycon, L. R. Froese, W. G. Kirsch and B. Simon, Schrödinger Operators with Application to Quantum Mechanics and Global Geometry, Springer Verlag, 1987.

[4] L. Erdös, Ground-state density of the Pauli operator in the large field limit, Lett. Math. Phys., **29**, 219–240 (1993).

[5] L. Erdös, Magnetic Lieb-Thirring inequalities and stochastic oscillatory integrals, Operator Theory, Advances and Applications, **78**, 127–134 (1994).

[6] I. C. Gohberg and M. G. Krein, Introduction to the Theory of Linear Non-selfadjoint Operators, Translations of Mathematical Monographs, Vol. 18, A. M. S., 1969.

[7] L. Hörmander, The Analysis of Linear Partial Differential Operators I, Springer Verlag, 1983.

[8] A. V. Sobolev, Asymptotic behavior of the energy levels of a quantum particle in a homogeneous magnetic field, perturbed by a decreasing electric field I, J. Soviet Math., **35**, 2201–2211 (1986).

[9] A. V. Sobolev, On the Lieb-Thirring estimates for the Pauli operator, Duke Math. J., **82**, 607–637 (1996).

[10] H. Tamura, Asymptotic distribution of eigenvalues for Schrödinger operators with homogeneous magnetic fields, Osaka J. Math., **25**, 633–647 (1988).

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