INDUCED MAPPINGS ON HYPERSPACES II

By

Hiroshi Hosokawa

Abstract. A mapping $f: X \to Y$ between continua induces a mapping $C(f): C(X) \to C(Y)$ between hyperspaces. In section 2, we shall give a condition under which $C(f)$ becomes confluent whenever f is confluent. In section 3, we consider the image and the inverse image of an order arc under the mapping $C(f)$ and characterizations of a confluent mapping and a hereditarily confluent mapping. In the last section, we treat about particular subcontinua (which are like fat Whitney levels) and inverse image of them under $C(f)$.

1. Introduction

In this paper, continua are compact connected metric spaces and mappings are continuous functions. The letters X and Y will always denote nondegenerate continua. The *hyperspaces* of X are the metric spaces $2^X = \{K \subset X: K$ is nonempty and compact$\}$ and $C(X) = \{K \in 2^X: K$ is connected$\}$ with the Hausdorff metric H_d (see [10] for the definition of the Hausdorff metric and basic properties of hyperspaces). A mapping $f: X \to Y$ induces a mapping $C(f): C(X) \to C(Y)$, where $C(f)(K) = f(K)$ for each $K \in C(K)$. Clearly, $C(g \circ f) = C(g) \circ C(f)$. In [4], we have proved that if Y is locally connected and f is confluent, then $C(f)$ is confluent. In section 2, we show that the condition of locally connectedness can be weakened (Theorem 2.7).

An *order arc* in $C(X)$ is an arc $\alpha \subset C(X)$ such that for $K, L \in \alpha$, $K \subset L$ or $L \subset K$. An order arc with the end points X and $\{x\}$ for some $x \in X$ is called a *large order arc*. Let $\Gamma(X) = \{\alpha: \alpha$ is an order arc in $C(X)\} \cup F_1(X)$, where

$F_1(X) = \{\{K\}: K \in C(X)\}$ and $L\Gamma(X) = \{\alpha: \alpha \text{ is a large order arc in } C(X)\}$. If K is a subcontinuum of X, then we regard $\Gamma(K)$ as a subspace of $\Gamma(X)$. We show that if f is confluent, then $C(f)(\Gamma(X)) = \Gamma(Y)$ and $C(f)(L\Gamma(X)) = L\Gamma(Y)$. If f is hereditary confluent, then $\cup\{\alpha: \alpha \in \Gamma(X) \text{ and } C(f)(\alpha) = \beta\} = [C(f)]^{-1}(\beta)$ for each $\beta \in \Gamma(Y)$. This is true for large order arcs. Modifying these facts, we obtain characterizations of confluent mappings and hereditarily confluent mappings (Theorems 3.3 and 3.5).

Some results between Whitney levels and order arc spaces are given in [9]. In the last section, we consider for subcontinua which intersect each large order arc. Each neighborhood of such continuum contains a Whitney level. Moreover under some additional condition, such a subcontinuum contains a Whitney level. These continua are preserved by the inverse image of $C(f)$.

2. The mapping induced from a confluent mapping

A mapping $f: X \to Y$ is said to be *confluent* if for each subcontinuum L of Y, each component of $f^{-1}(L)$ is mapped by f onto L. If there is a component of $f^{-1}(L)$ which is mapped onto L by f, then f is said to be *weakly confluent*. For properties of these mappings, see [10].

We have proved in [4] the following three lemmas.

LEMMA 2.1. *If $f: X \to Y$ is a confluent mapping and $\mathscr{L}$ is an arc in $C(Y)$, then each component of $[C(f)]^{-1}(\mathscr{L})$ is mapped by $C(f)$ onto $\mathscr{L}$.*

LEMMA 2.2. *If $f: X \to Y$ is a confluent mapping and $\mathscr{L}$ is an arcwise connected subcontinuum of $C(Y)$, then each component of $[C(f)]^{-1}(\mathscr{L})$ is mapped by $C(f)$ onto $\mathscr{L}$.*

In other words, *if $f: X \to Y$ is confluent and $\mathscr{L}$ is a subcontinuum of $C(Y)$, then each component of $[C(f)]^{-1}(\mathscr{L})$ is mapped by $C(f)$ onto the union of some arc components of $\mathscr{L}$.*

LEMMA 2.3. *If $f: X \to Y$ is a confluent mapping and Y is locally connected, then $C(f): C(X) \to C(Y)$ is confluent.*

There is a confluent mapping $f: X \to Y$ such that $C(f)$ is not weakly confluent (see [4]). The condition of locally connectedness in lemma 2.3 can be weakened. We use the following lemmas which were proved in corollary 7 and corollary 13 of [1] respectively.

LEMMA 2.4. *Let* $X = \varprojlim\{X_n, \varphi_n\}_{n=1}^{\infty}$ *where* $\{X_n, \varphi_n\}_{n=1}^{\infty}$ *is an inverse sequence with all bonding mappings* $\varphi_n : X_{n+1} \to X_n$ *are confluent. Then all projections* $p_n : X \to X_n$ *are confluent.*

LEMMA 2.5. *Let* $\{X_n, \varphi_n\}_{n=1}^{\infty}$ *and* $\{Y_n, \psi_n\}_{n=1}^{\infty}$ *be inverse sequences and let* $\{\sigma, h_n\}_{n=1}^{\infty}$ *be a mapping between the inverse sequences. If* $h_n: X_{\sigma(n)} \to Y_n$ *is confluent for each* n, *then the limit mapping* $h_\infty : X \to Y$ *is confluent.*

Further we use the following lemma proved by J. Segal in [13] (see also Theorem (1.169) in [12]).

LEMMA 2.6. *Let* $Y = \varprojlim\{Y_n, \psi_n\}_{n=1}^{\infty}$, *where each of the spaces* Y_n *is a continuum. Then the mapping* $h : C_\infty(Y) = \varprojlim\{C(Y_n), C(\psi_n)\}_{n=1}^{\infty} \to C(Y)$ *defined by* $h(B_1, B_2, \ldots, B_n, \ldots) = \varprojlim\{B_n, \psi_n|B_{n+1}\}_{n=1}^{\infty}$ *is a homeomorphism.*

We are now ready to prove the following Theorem.

THEOREM 2.7. *Let* Y *be the inverse limit of an inverse sequence* $\{Y_n, \psi_n\}_{n=1}^{\infty}$ *such that each of the spaces* Y_n *is a locally connected continuum and each of the bonding mappings* $\psi_n: Y_{n+1} \to Y_n$ *is confluent. If* $f: X \to Y$ *is confluent, then* $C(f)$ *is also confluent.*

PROOF. Let $h: C_\infty(Y) \to C(Y)$ be the homeomorphism defined in lemma 2.6 and for each n, $p_n : Y \to Y_n$ be the projection. Since the composition of two confluent mappings is confluent (cf. [10]) and p_n is confluent by lemma 2.4, $p_n \circ f: X \to Y_n$ is confluent. Since Y_n is locally connected, we can apply lemma 2.3 and obtain that the mappings $C(\psi_n)$ and $C(p_n \circ f)$ are confluent for each $n = 1, 2, \ldots$. On the otherhand, since $C(\psi_n) \circ C(p_n \circ f) = C(\psi_n \circ p_n \circ f) = C(p_{n-1} \circ f)$, the sequence of the mappings $\{C(p_n \circ f)\}_{n=1}^{\infty}$ induces a mapping $g: C(x) \to C_\infty(Y)$ (i.e., $g(A) = \{p_n \circ f(A)\}_{n=1}^{\infty} \in C_\infty(Y)$ for each $A \in C(X)$). Then by lemma 2.5, g is confluent. It is easy to see from the definition of g, that $h \circ g = C(f)$. Therefore the induced mapping $C(f): C(X) \to C(Y)$ is confluent.

COROLLARY 2.8. *If* Y *is a solenoid or the buckethandle continuum and* $f: X \to Y$ *is confluent, then* $C(f): C(X) \to C(Y)$ *is also confluent.*

PROOF. This follows from Theorem 2.7 since Y is an inverse limit of circles or arcs with confluent bonding mappings.

3. Order arcs

Recall that an arc α in $C(X)$ is an order arc if $K, L \in \alpha$, then $K \subset L$ or $L \subset K$. Conversely it is known that if $K, L \in C(X)$ and $K \subset L$, then there exists an order arc in $C(X)$ which contains K and L (see [6] or [12]). If α is an order arc, then there is a homeomorphism $\varphi_\alpha : [0,1] \to \alpha$ such that $\varphi_\alpha(0) = \bigcap \alpha$, $\varphi_\alpha(1) = \bigcup \alpha$. For convenient, we shall write $\psi_\alpha(t) = \alpha(t)$ for some fixed homeomorphism $\psi_\alpha : [0,1] \to \alpha$ for each order arc α. We say that $\{\alpha(t)\}_{t \in [0,1]}$ is a parameterization of α. Note that $\Gamma(X) = \{\alpha : \alpha \text{ is an order arc in } C(X)\} \cup F_1(X)$ is compact as the subspace topology of the continuum $C(C(X))$ (see [11]), connected (see [3]), and hence it is a continuum. D. Curtis and M. Lynch determined continua whose order arc space is homeomorphic to the Hilbert cube ([2], Theorem 1.2). Clearly $L\Gamma(X)$ is closed in $\Gamma(X)$.

LEMMA 3.1. *The space $L\Gamma(X)$ is connected. Hence it is a continuum.*

PROOF. We first show that the set $L\Gamma_x(X) = \{\alpha \in L\Gamma(X) : \alpha(0) = \{x\}\}$ is arcwise connected for each $x \in X$. Let $\alpha, \beta \in L\Gamma_x(X)$. Define $\gamma_t \in L\Gamma_x(X)$ for each $t \in [0,1]$ by

$$\gamma_t = \{\beta(s) : 0 \le s \le t\} \cup \{\beta(t) \cup \alpha(s) : 0 \le s \le 1\}.$$

Note that $\gamma_0 = \alpha$ and $\gamma_1 = \beta$ because $\{\beta(1) \cup \alpha(s) : 0 \le s \le 1\} = \{\beta(1)\}$. It is easy to see that $\{\gamma_t\}_{t \in [0,1]}$ is an arc from α to β in $L\Gamma_x(X)$.

Now suppose $L\Gamma(X) = \mathscr{A}_1 \cup \mathscr{A}_2$, where $\mathscr{A}_1, \mathscr{A}_2$ are closed and disjoint. Let $X_i = \{x \in X : L\Gamma_x(X) \cap \mathscr{A}_i \ne \phi\}$ for $i = 1, 2$. Since $L\Gamma_x(X)$ is connected, X_1, X_2 are disjoint. The mapping $h : L\Gamma(X) \to X$ defined by $h(\alpha) = x$ if $\alpha \in L\Gamma_x(X)$ is continuous (cf. Theorem 2.2 in [3]) and $h(\mathscr{A}_i) = X_i$ for $i = 1, 2$. Therefore X_1, X_2 are closed. Clearly $X = X_1 \cup X_2$ and hence $X_1 = \phi$ or $X_2 = \phi$. This implies that $\mathscr{A}_1 = \phi$ or $\mathscr{A}_2 = \phi$. Hence $L\Gamma(X)$ is connected.

We now return to the induced mapping. The following proposition is easy to prove.

PROPOSITION 3.2. *Let $f : X \to Y$ be a mapping and $\alpha \in \Gamma(X)$. Then $C(f)$ is monotone on α and $C(f)(\alpha) \in \Gamma(Y)$. If f is onto and $\alpha \in L\Gamma(X)$, then $C(f)(\alpha) \in L\Gamma(Y)$.*

The following Theorem is a characterization of confluent mappings.

THEOREM 3.3. *Let $f : X \to Y$ be an onto mapping. Then the following assertions are equivalent:*

(a) f is confluent.

(b) For each $\beta \in \Gamma(Y)$, each $L \in \beta$ and each component K of $f^{-1}(L)$, there exists $\alpha \in \Gamma(X)$ such that $K \in \alpha$ and $C(f)(\alpha) = \beta$.

(c) For each $\beta \in L\Gamma(Y)$, each $L \in \beta$ and each component K of $f^{-1}(L)$, there exists $\alpha \in L\Gamma(X)$ such that $K \in \alpha$ and $C(f)(\alpha) = \beta$.

PROOF. (a) implies (b). Suppose f is confluent, $L \in \beta \in \Gamma(Y)$ and K is a component of $f^{-1}(L)$. Note that $\beta(0) \subset L = f(K)$. Choose a component K_0 of $f^{-1}(\beta(0))$ contained in K. For each $t \in [0, 1]$, let K_t be the component of $f^{-1}(\beta(t))$ such that $K_0 \subset K_t$ and let $\mathscr{F}$ be the closure of the set $\{K_t : t \in [0, 1]\}$ in $C(X)$. Then $\mathscr{F}$ is a linearly ordered set by inclusion and $K \in \mathscr{F}$. By the Maximal Principle, there is a maximal linearly orderd set $\alpha \subset C(X)$ such that $\mathscr{K} \subset \alpha$ and $K_0 \subset A \subset K_1$ for each $A \in \alpha$. Then as in the proof of Theorem 1.8 of [12], α is an order arc. It is easy to see that α satisfies the required conditions.

(b) implies (c). Let $L \in \beta \in L\Gamma(Y)$ and let K be a component of $f^{-1}(L)$. There is $\gamma \in \Gamma(X)$ such that $K \in \gamma$ and $C(f)(\gamma) = \beta$. Choose $\gamma_1, \gamma_2 \in \Gamma(X)$ as follows. If $\gamma(1) = X$, then $\gamma_1 = \{X\}$. If $\gamma(1) \neq X$, then let γ_1 be any order arc from $\gamma(1)$ to X. Similarly choose $\gamma_2 \in \Gamma(X)$ so that $\gamma_2(0) \in F_1(X)$ and $\gamma_2(1) = \gamma(0)$. Then $\alpha = \gamma_2 \cup \gamma \cup \gamma_1$ is a required large order arc.

(c) implies (a). Let L be a subcontinuum of Y and let K be a component of $f^{-1}(L)$. There is $\beta \in L\Gamma(Y)$ such that $L \in \beta$. By the assumption, there is $\alpha \in L\Gamma(X)$ such that $K \in \alpha$ and $C(f)(\alpha) = \beta$. Let $M \in \alpha$ be such that $C(f)(M) = L$. Then $M \cap K \neq \phi$ and hence $M \subset K$. Thus $L = C(f)(M) = f(M) \subset f(K) \subset L$. This implies that f is confluent.

One can easily fined a confluent mapping $f: X \to Y$ and some $\beta \in \Gamma(Y)$ such that for some $K \in [C(f)]^{-1}(\beta)$, there is no order arc α which contains K and mapped onto β by $C(f)$.

An onto mapping $f: X \to Y$ is said to be *hereditarily confluent* if for each $K \in C(X)$, the restriction $f|K: K \to f(K)$ is confluent.

PROPOSITION 3.4. If $f: X \to Y$ is hereditarily confluent, then for each $\beta \in \Gamma(Y)$,

$$\cup \{\alpha: \alpha \in \Gamma(X) \quad and \quad C(f)(\alpha) = \beta\} = [C(f)]^{-1}(\beta).$$

If $\beta \in L\Gamma(Y)$, then

$$\cup \{\alpha : \alpha \in L\Gamma(X) \quad and \quad C(f)(\alpha) = \beta\} = [C(f)]^{-1}(\beta).$$

PROOF. We only prove that if $\beta \in \Gamma(Y)$ and $K \in [C(f)]^{-1}(\beta)$, then there is $\alpha \in \Gamma(X)$ such that $K \in \alpha$ and $C(f)(\alpha) = \beta$. Since $f|K: K \to f(K)$ is confluent and $\beta(0) \subset f(K) \in \beta$, we can apply Theorem 3.3 for a subarc of β. Thus there is $\gamma_1 \in \Gamma(K)$ such that $K \in \gamma_1$ and $C(f|K)(\gamma_1) = \{L : L \in \beta \text{ and } L \subset f(K)\}$. Let M be the component of $f^{-1}(f(K))$ containing K. Again applying Theorem 3.3, there is an order arc γ_2 such that $M \in \gamma_2$ and $C(f)(\gamma_2) = \{L: L \in \beta$ and $f(M) \subset L\}$. We may assume that $M = \gamma_2(0)$. If $K = M$, then we put $\alpha = \gamma_1 \cup \gamma_2$. If $K \neq M$, then choose an order arc γ_0 from K to M and we put $\alpha = \gamma_1 \cup \gamma_0 \cup \gamma_2$. Then either case, we have $K \in \alpha \in \Gamma(X)$ and $C(f)(\alpha) = \gamma$.

The following Theorem is a characterization of hereditarily confluent mappings.

THEOREM 3.5. *Let $f: X \to Y$ be an onto mapping. Then the following assertions are equivalent*:

(a) *f is hereditarily confluent.*

(b) *For each $\beta \in \Gamma(Y)$, each pair $L_1, L_2 \in \beta$ such that $L_1 \subset L_2$ and each $K_i \in [C(f)]^{-1}(L_i)$ for $i = 1, 2$, such that $K_1 \subset K_2$, there exists $\alpha \in \Gamma(X)$ such that $K_1, K_2 \in \alpha$ and $C(f)(\alpha) = \beta$.*

(c) *For each $\beta \in L\Gamma(Y)$, each pair $L_1, L_2 \in \beta$ such that $L_1 \subset L_2$ and each $K_i \in [C(f)]^{-1}(L_i)$ for $i = 1, 2$, such that $K_1 \subset K_2$, there exists $\alpha \in L\Gamma(X)$ such that $K_1, K_2 \in \alpha$ and $C(f)(\alpha) = \beta$.*

PROOF. (a) implies (b). Let f be hereditarily confluent, $\beta \in \Gamma(Y)$, L_1, $L_2 \in \beta$ and let $K_1 \in [C(f)]^{-1}(L_1)$, $K_2 \in [C(f)]^{-1}(L_2)$ be such that $K_1 \subset K_2$. Since $f|K_2$ is hereditarily confluent and $\beta_1 = \{B \in \beta: B \subset L_2\} \in \Gamma(L_2) \subset \Gamma(Y)$, $\beta_2 = \{B \in \beta: L_2 \subset B\} \in \Gamma(Y)$, we can apply proposition 3.4 so that there are $\alpha_1 \in \Gamma(K_2) \subset \Gamma(X)$, $\alpha_2 \in \Gamma(X)$ such that $K_i \in \alpha_i$ for each $i = 1, 2$ and $C(f|K_2)(\alpha_1) = \beta_1$, $C(f)(\alpha_2) = \beta_2$. We may assume that $\alpha_2(0) = K_2$. If $\alpha_1(1) = \alpha_2(0)$, then define $\alpha = \alpha_1 \cup \alpha_2$. If $\alpha_1(1) \neq \alpha_2(0)$, then $\alpha_1(1) \subset K_2 = \alpha_2(0)$. Hence we can choose an order arc α_0 from $\alpha_1(1)$ to $\alpha_2(0)$ and define $\alpha = \alpha_1 \cup \alpha_0 \cup \alpha_2$. It is easy to see that $K_1, K_2 \in \alpha \in \Gamma(X)$ and $C(f)(\alpha) = \beta$.

The proof of (b) implies (c) is same as that of (b) implies (c) in Theorem 3.3.

(c) implies (a). Let K be a subcontinuum of X, L a subcontinuum of $f(K)$ and C a component of $(f|K)^{-1}(L)$. We must show that $f(C) = L$. Let β be a large order arc in $C(Y)$ such that $\{f(C), L, f(K)\} \subset \beta$ (note that such a β exists since $f(C) \subset L \subset f(K)$). It follows from the hypothesis and $C, K \in [C(f)]^{-1}(\beta)$, $C \subset K$, there exists $\alpha \in L\Gamma(X)$ such that $C, K \in \alpha$ and $C(f)(\alpha) = \beta$. Let $M \in \alpha$

be the maximum element in $\alpha \cap [C(f)]^{-1}(L)$. Then $f(C) \subset L \subset f(K)$ and $C, M, K \in \alpha$ implies $C \subset M \subset K$. But since C is a component of $f^{-1}(L) \cap K$, we have $C = M$. Therefore $f(C) = L$.

For an onto mapping $f: X \to Y$, we can regard $C(f)$ as a mapping from $L\Gamma(X)$ to $L\Gamma(Y)$. It is interest to know a condition when $C(f): L\Gamma(X) \to L\Gamma(Y)$ is an open mapping (cf. [4], Theorem 4.3).

4. Whitney levels

A *Whitney map* for $C(X)$ is a mapping $\omega: C(x) \to [0,1]$ such that $\omega(\{x\}) = 0$ for each $x \in X$, $\omega(X) = 1$ and if $K, L \in C(X)$ and $K \subset L \neq K$, then $\omega(K) < \omega(L)$. Such a mapping always exists (see [15], [16] or [12]). A *Whitney level* is a set of the form $\omega^{-1}(t)$, $t \in [0, 1)$, for any Whitney map ω. It is known that every Whitney level is a subcontinuum of $C(X)$.

A subset $\mathscr{A} \subset C(X)$ is said to be an *anti-chain* if $A, B \in \mathscr{A}$ and $A \subset B$, then $A = B$. A. Illanes has given a characterization of Whitney levels as follows (see [5], Theorem 1.2).

PROPOSITION 4.1. *Let* $\mathscr{A} \subset C(X) - (\{X\} \cup F_1(X))$. *Then the following assertions are equivalent*:

(a) $\mathscr{A}$ *is a Whitney level.*

(b) $\mathscr{A}$ *is a compact anti-chain which intersects every large order arc in* $C(X)$.

(c) $\mathscr{A}$ *is an anti-chain and separates* $C(X)$.

For any continuum X, consider the following conditions for a closed subset $\mathscr{K} \subset C(X)$.

(∗) $\mathscr{K} \cap (\{X\} \cup F_1(X)) = \phi$.

(∗∗) $\alpha \cap \mathscr{K}$ is nonempty and connected for every $\alpha \in L\Gamma(X)$.

(∗∗∗) $\alpha \cap \mathscr{K}$ is nonempty for every $\alpha \in L\Gamma(X)$ and the mapping $\Phi_{\mathscr{K}}: L\Gamma(X) \to 2^{C(X)}$ difined by $\Phi_{\mathscr{K}}(\alpha) = \alpha \cap \mathscr{K}$ is continuous.

THEOREM 4.2. *Let* $\mathscr{K}$ *be a closed subset of* $C(X)$. *If* $\mathscr{K}$ *satisfies* (∗∗), *then* $\mathscr{K}$ *is connected and each neighborhood of* $\mathscr{K}$ *contains a Whitney level. If in addition,* $\mathscr{K}$ *satisfies* (∗), *then* $C(X) - \mathscr{K}$ *has just two components.*

PROOF. Suppose $\mathscr{K}$ satisfies (∗∗). If $X \in \mathscr{K}$, then for each $K \in \mathscr{K} - \{X\}$, there exists a large order arc $\alpha \in L\Gamma(X)$ such that $K \in \alpha$. By the assumption,

$\alpha \cap \mathcal{K}$ is connected and containes K, X. Thus K is in the same component of X in $\mathcal{K}$. Therefore $\mathcal{K}$ is connected. If $X \notin \mathcal{K}$, then define two subsets $\mathcal{A}_1, \mathcal{A}_2$ of $C(X) - \mathcal{K}$ by

$$\mathcal{A}_1 = \{A \in C(X) - \mathcal{K} : \text{there exists } K \in \mathcal{K} \text{ such that } A \subset K\}$$

and

$$\mathcal{A}_2 = \{A \in C(X) - \mathcal{K} : \text{there exists } K \in \mathcal{K} \text{ such that } K \subset A\}.$$

One can easily verify that $F_1(X) - \mathcal{K} \subset \mathcal{A}_1$, $X \in \mathcal{A}_2$. Let $B \in \mathcal{A}_1 \cap \mathcal{A}_2$. Then there are $K_1, K_2 \in \mathcal{K}$ such that $K_1 \subset B \subset K_2$. Let $\alpha \in L\Gamma(X)$ be such that $\{K_1, B, K_2\} \subset \alpha$. Since $B \notin \mathcal{K}$, $\mathcal{K} \cap \alpha = \{K : K \in \mathcal{K} \cap \alpha \text{ and } K \subset B\} \cup \{K : K \in \mathcal{K} \cap \alpha \text{ and } B \subset K\}$ is a separation of $\mathcal{K} \cap \alpha$. This contradicts to (**) and hence $\mathcal{A}_1 \cap \mathcal{A}_2 = \phi$. Simillary we can show that each component of $\mathcal{A}_1$ intersects $F_1(X)$, $\mathcal{A}_2$ is connected and $\mathcal{A}_1 \cup \mathcal{A}_2 = C(X) - \mathcal{K}$.

In order to prove that $\overline{\mathcal{A}}_1 \cap \mathcal{A}_2 = \phi = \mathcal{A}_1 \cap \overline{\mathcal{A}}_2$, let $\{A_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{A}_1$ and $A = \lim_{n \to \infty} A_n$. Choose $K_n \in \mathcal{K}$ such that $A_n \subset K_n$ for $n = 1, 2, \ldots$. Since $\mathcal{K}$ is compact, we may assume that $\lim_{n \to \infty} K_n = K \in \mathcal{K}$. Clearly $A \subset K$ and hence $A \in \mathcal{A}_1$ or $A \in \mathcal{K}$. Thus we have that $\overline{\mathcal{A}}_1 \cap \mathcal{A}_2 = \phi$. Similarly we can show that $\mathcal{A}_1 \cap \overline{\mathcal{A}}_2 = \phi$.

To prove that $\mathcal{K}$ is connected, suppose $\mathcal{K} = \mathcal{K}_1 \cup \mathcal{K}_2$, where $\mathcal{K}_1, \mathcal{K}_2$ are closed and disjoint. By (**), each $\alpha \in L\Gamma(X)$ must intersect only one of the sets $\mathcal{K}_1, \mathcal{K}_2$. Let $L_i = \{\alpha \in L\Gamma(X) : \alpha \cap \mathcal{K}_i \neq \phi\}$ for $i = 1, 2, \ldots$. Then L_1, L_2 are closed disjoint and $L_1 \cup L_2 = L\Gamma(X)$. Since $L\Gamma(X)$ is connected, $L_1 = \phi$ or $L_2 = \phi$. This implies that $\mathcal{K}_1 = \phi$ or $\mathcal{K}_2 = \phi$. Therefore $\mathcal{K}$ is connected.

Let $\mathcal{U}$ be any neighborhood of $\mathcal{K}$ in $C(X)$. If $X \in \mathcal{K}$, then by the continuity of Whitney maps, $\mathcal{U}$ contains a Whitney level near X. If $F_1(X) \subset \mathcal{K}$, then $\mathcal{K}$ itself contains a Whitney level. Therefore suppose $F_1(X) - \mathcal{K} \neq \phi$ and $X \notin \mathcal{K}$. In this case, $\mathcal{A}_1$ and $\mathcal{A}_2$ are nonempty. Let $\mu : C(X) \to [0, 1]$ be a Whitney map. There are $t_1 > 0$ and $t_2 < 1$ such that $\mu(\overline{\mathcal{A}}_1) = [0, t_1]$ and $\mu(\overline{\mathcal{A}}_2) = [t_2, 1]$. Let $\sigma_1 : [0, t_1] \to [0, 1/3], \sigma_2 : [t_2, 1] \to [2/3, 1]$ be homeomorphisms such that $\sigma_1(0) = 0, \sigma_2(1) = 1$. Note that for any pair $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$, $A_2 \subset A_1$ does not hold. Define a mapping $\mu_1 : (\mathcal{A}_1 \cup \mathcal{A}_2) - \mathcal{U} \to [0, 1]$ by

$$\mu_1(A) = \begin{cases} \sigma_1 \circ \mu(A) & \text{if } A \in \mathcal{A}_1 - \mathcal{U} \\ \sigma_2 \circ \mu(A) & \text{if } A \in \mathcal{A}_2 - \mathcal{U}. \end{cases}$$

Then by Theorem 3.1 in [14], μ_1 can be extended to a Whitney map $\tilde{\mu} : C(X) \to [0, 1]$. The set $\tilde{\mu}^{-1}(1/2)$ is a Whitney level contained in $\mathcal{U}$. Fur-

thermore if $\mathcal{K}$ satisfies (*), then $F_1(X) \subset \mathcal{A}_1$ and each component of $\mathcal{A}_1$ intersects $F_1(X)$. Therefore $\mathcal{A}_1$ is connected.

REMARK. If $\mathcal{K} \subset C(X)$ satisfies (*), (**) and $\alpha \cap \mathcal{K}$ is degenerate for each $\alpha \in L\Gamma(X)$, then $\mathcal{K}$ is an anti-chain and separates $C(X)$. Hence it is a Whitney level.

THEOREM 4.3. *Let* $f: X \to Y$ *be an onto mapping,* $\mathcal{L} \subset C(Y)$ *a closed subset and* $\mathcal{K} = [C(f)]^{-1}(\mathcal{L})$.
(a) *If* $\mathcal{L}$ *satisfies* (*), *then so does* $\mathcal{K}$.
(b) *If* $\mathcal{L}$ *satisfies* (**), *then so does* $\mathcal{K}$.

PROOF. It is clear that if $\mathcal{L}$ satisfies (*), then so does $\mathcal{K}$. Suppose $\mathcal{L}$ satisfies (**). Let $\alpha \in L\Gamma(X)$ and $\beta = C(f)(\alpha)$. Then by proposition 3.2, $\beta \in L\Gamma(Y)$ and hence $\beta \cap \mathcal{L}$ is nonempty and connected. Since $C(f)|\alpha: \alpha \to \beta$ is monotone, $\alpha \cap \mathcal{K} = [C(f)|\alpha]^{-1}(\beta) \cap [C(f)]^{-1}(\mathcal{L}) = [C(f)|\alpha]^{-1}(\beta \cap \mathcal{L})$ is nonempty and connected.

For a closed subset $\mathcal{K}$ of $C(X)$, define $\mathcal{K}_+$ by

$$\mathcal{K}_+ = \{K \in \mathcal{K}: \text{if } K' \in \mathcal{K} \text{ and } K \subset K', \text{then } K = K'\}.$$

LEMMA 4.4. *If a closed subset* $\mathcal{K} \subset C(X)$ *satisfies* (*), (**) *and* (***), *then* $\mathcal{K}_+$ *is a Whitney level.*

PROOF. Obviously the set $\alpha \cap \mathcal{K}_+$ is degenerate for each large order arc α if it is nonempty, it is sufficient to prove that $\mathcal{K}_+$ is closed and intersects each large order arc.

ASSERTION 1. If $\alpha \in L\Gamma(X)$, then the maximum element of $\alpha \cap \mathcal{K}$ is an element of $\mathcal{K}_+$.

Note that since $\mathcal{K}$ is compact, $\alpha \cap \mathcal{K}$ is compact. Also $\mathcal{K}$ satisfies (**), $\alpha \cap \mathcal{K}$ is an arc or a degenerate set. Therefore there exists the maximum element in $\alpha \cap \mathcal{K}$ for each $\alpha \in L\Gamma(X)$. Let K be the maximum element of $\alpha \cap \mathcal{K}$, where $\alpha \in L\Gamma(X)$ and suppose $K \subset K'$ for some $K' \in \mathcal{K}$. We shall show that $K = K'$. There is $\beta \in L\Gamma(X)$ such that $K, K' \in \beta$ and $\{B \in \beta: B \subset K\} = \{B \in \alpha: B \subset K\}$. Let $\alpha = \{\alpha(s)\}_{s \in [0,1]}$, $\beta = \{\beta(s)\}_{s \in [0,1]}$ be parametrizations of α, β respectively such that $\alpha(t_0) = \beta(t_0) = K$. Since $K \neq X$, $t_0 < 1$. Choose a sequence $\{t_n\}_{n=1}^{\infty}$ of real numbers such that $1 > t_1 > t_2 \cdots$ and $\lim_{n \to \infty} t_n = t_0$. For each $n = 1, 2, \ldots$,

define $\gamma_n \in L\Gamma(X)$ by

$$\gamma_n = \{\alpha(s): 0 \le s \le t_n\} \cup \{\alpha(t_n) \cup \beta(s): t_0 \le s \le 1\}.$$

It is easy to see that $\lim_{n \to \infty} \gamma_n = \beta$ and hence by continuity of $\Phi_{\mathscr{K}}$, $\lim_{n \to \infty} \gamma_n \cap \mathscr{K} = \beta \cap \mathscr{K}$. If $t_0 < s < t_n$, then since $\alpha(t_0)$ is the maximum element of $\alpha \cap \mathscr{K}$ and $\alpha(t_0) \ne \alpha(s)$, $\alpha(s) \notin \mathscr{K}$. Hence by $(**)$, K is the maximum element of $\gamma_n \cap \mathscr{K}$ for each n. Therefore K is the maximum element of $\beta \cap \mathscr{K}$. This implies that $K' = K$ and hence we have $K \in \mathscr{K}_+$.

ASSERTION 2. $\mathscr{K}_+$ is closed in $C(X)$.

Let $\{K_n\}_{n=1}^{\infty}$ be a sequence in $\mathscr{K}_+$ such that $\lim_{n \to \infty} K_n = K$. Since $\mathscr{K}_+ \subset \mathscr{K}$ and $\mathscr{K}$ is closed, $K \in \mathscr{K}$. Let α_n be a large order arc such that $K_n \in \alpha_n$ for $n = 1, 2, \ldots$. Since $L\Gamma(X)$ is compact, we can assume that $\lim_{n \to \infty}(\alpha_n) = \alpha$ for some $\alpha \in L\Gamma(X)$. Since $\Phi_{\mathscr{K}}$ is continuous and K_n is the maximum element of $\alpha_n \cap \mathscr{K}$, K is the maximum element of $\alpha \cap \mathscr{K}$. Therefore by assertion 1, $K \in \mathscr{K}_+$ so that $\mathscr{K}_+$ is closed.

For a closed set $\mathscr{K} \subset C(X)$, define a set $S(\mathscr{K})$ by

$$S(\mathscr{K}) = \{A \in C(X): \text{there exists } K \in \mathscr{K} \text{ such that } A \subset K\}.$$

It is easy to see that $S(\mathscr{K})$ is closed and $S(\mathscr{K})_+ = \mathscr{K}_+$. Moreover if $\mathscr{K}$ satisfies $(***)$, then $S(\mathscr{K})$ satisfies $(**)$ and $(***)$. Therefore we have:

COROLLARY 4.5.   *If $\mathscr{K}$ satisfies $(*)$ and $(***)$, then $\mathscr{K}_+$ is a Whitney level.*

References

[1]   J. J. Charatonik and W. J. Charatonik, On projections and limit mappings of inverse systems of compact spaces, Topology and its Applications, **16** (1983), pp. 1–9.

[2]   D. Curtis and M. Lynch, Spaces of order arcs in hyperspaces of Peano continua, Houston J. Math., **15** (1989), pp. 517–525.

[3]   C. Eberhart, S. B. Nadler, Jr. and W. O. Nowell, Jr., Spaces of order arcs in hyperspaces, Fund. Math., **112** (1981), pp. 111–120.

[4]   H. Hosokawa, Induced mappings on hyperspaces, to appear.

[5]   A. Illanes, The space of Whitney levels, Topology and its Applications, **40** (1991), pp. 157–169.

[6]   J. L. Kelley, Hyperspaces of a continuum, Trans. Amer. Math. Soc., **52** (1942), pp. 22–36.

[7]   K. Kuratowski, Topology, Vol. I, New York-London-Warszawa 1996.

[8]   K. Kuratowski, Topology, Vol. II, New York-London-Warszawa 1998.

[9]   M. Lynch, Whitney levels and certain order arc spaces, Topology and its Applications, **38** (1991), pp. 189–200.

[10]   T. Mackowiak, Continuous mappings on continua, Dissert. Math. (Rozprawy Matem.), **158**, Warszawa (1979).

[11]   S. Mazurkiewicz, Sul l'hyperspace d'un continua, Fund. Math., **18** (1932), pp. 171–177.

[12] S. B. Nadler, Jr., Hyperspaces of sets, Marcel Dekker, Inc., New York, (1978).

[13] J. Segal, Hyperspaces of the inverse limit space, Proc. Amer. Math. Soc., **10** (1959), pp. 706–709.

[14] L. E. Ward Jr., Extending Whitney maps, Pacific J. Math., **93** (1981), pp. 465–469.

[15] H. Whitney, Regular families of curves I, Proc. Nat. Acad. Sci., **18** (1932), pp. 275–278.

[16] H. Whitney, Regular families of curves, Annals of Math., **34** (1933), pp. 244–270.

Department of Mathematics
Tokyo Gakugei Univ.
Nukuikita-machi, koganei-shi
Tokyo, 184, Japan