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# REACHABLE SETS IN LIE GROUPS

By

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**Abstract.** In this paper, we deal with the right invariant control system on Lie group using norm cost, which is an alternative notion of the controllability described in [2], and local reachable sets in Lie groups of this conception was studied in [4].

## 1. Introduction

Let $G$ denote a Lie group with its Lie algebra $L(G)$. We identify $L(G)$ with the set of right invariant vector fields on $G$. We note that $L(G)$ is linearly isomorphic to the tangent space $T_eG$. Since $T_eG$ can be given the structure of a Banach space, $L(G)$ may be given the structure of a Banach space. Let $\Omega$ be a subset of $L(G)$. We consider the right invariant control system on $G$ given by

$$(*) \qquad \dot{x}(t) = U(t)(x(t)), \quad x(0) = g,$$

where $U$ belongs to the class $\mathscr{U}(\Omega)$ of measurable functions from $\boldsymbol{R}^+ = [0, \infty)$ into $\Omega$ which are locally bounded, and we denote the solution $x(\cdot)$ of $(*)$ by $\pi(g, \cdot, U)$, i.e., $\pi(g, 0, U) = g$ and $\pi(g, t, U) = x(t)$ for all $t \geq 0$. If there exists $U \in \mathscr{U}(\Omega)$ such that $h = \pi(g, t, U)$, then we say that $h$ is *attainable from $g$ at time $t$ for the system $\Omega$*. The set of such elements attainable from $g$ at time $t$ is denoted by $A(g, t, \Omega)$. We also employ the notation

$$\mathbf{A}(g, T, \Omega) = \bigcup_{0 \leq t \leq T} A(g, t, \Omega)$$

$$\mathbf{A}(g, \Omega) = \bigcup_{0 \leq t \leq \infty} A(g, t, \Omega).$$

The set $\mathbf{A}(g, \Omega)$ is called the *attainability set from $g$*.

Let $L$ be a Dynkin algebra and let $B$ be an open neighborhood of 0 which is symmetric and star-shaped in $L$ such that for all $x, y \in B$ the Campbell-

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Hausdorff series

$$x * y = x + y + \frac{1}{2}[x, y] + \cdots$$

converges absolutely. For $y \in L$, define a left invariant vector field $X_y$ on $B$ by

$$X_y(x) = g(\operatorname{ad} x)(y)$$

where

$$g(T) = 1 + \frac{T}{2} + \sum_{n=1}^{\infty}(b_{2n}/(2n)!)T^{2n}$$

with the Bernoulli numbers $b_{2n}$. Then $y \mapsto X_y$ is a Lie algebra isomorphism from $L$ to the left invariant vector fields on $B$. Let $\Omega \subseteq L$, let $u$: $[0, T] \to \Omega$ be a control function, and let $U_t = X_{u(t)}$ for $0 \le t \le T$. Then the left invariant control system on $B$ given by

$$x'(t) = U_t(x(t)), \; x(0) = 0.$$

A point $x \in B$ is *reachable at norm cost $\delta$ by means of the control function $U$* if $U$: $[0, T] \to \Omega$ is regulated, and admits a principal solution $X_U(t)$ with $X_U(T) = x$, and $\delta = c(U) = \int_0^T \|U(t)\| \, dt$. For detail discussion of local reachable sets in (local) Lie groups, we refer to [3, 4].

In this article, we define an alternative notion of controllability in the right invariant control system on the Lie group $G$ given by $(*)$ using norm cost as follows: A point $h \in G$ is *reachable from $g$ at norm cost $\delta$ for the system $\Omega$* if there exists $U \in \mathscr{U}(\Omega)$ such that $h = \pi(g, t, U)$ and that

$$\delta = \int_0^t \|U(s)\| \, ds.$$

We denote the set of all elements reachable from $g$ at norm cost $\delta$ for the system $\Omega$ by $R(g, \delta, \Omega)$. For convenience, we also use the following notation

$$\mathbf{R}(g, \delta, \Omega) = \bigcup_{0 \le s \le \delta} R(g, s, \Omega)$$

$$\mathbf{R}(g, \Omega) = \bigcup_{0 \le \delta \le \infty} R(g, \delta, \Omega).$$

We say that the set $\mathbf{R}(g, \Omega)$ is the *reachability set from $g$*. For the case that only piecewise constant controls into $\Omega$, we use the subcript "pc". For instance, the set of all elements reachable from $g$ at norm cost $\delta$ using only piecewise constant controls into $\Omega$ is denoted by $R_{pc}(g, \delta, \Omega)$.

We may easily show that $R(g,\delta,\Omega) = R(e,\delta,\Omega)g$, $\mathbf{R}(g,\delta,\Omega) = \mathbf{R}(e,\delta,\Omega)g$ and $\mathbf{R}(g,\Omega) = \mathbf{R}(e,\Omega)g$ from the right invariance of the control system.

The exponential map from $L(G)$ to $G$ is denoted by exp. For $X \in L(G)$, the integral curve $x(\cdot)$ for the constant function $X$ with initial value $x(t_0) = e \in G$ is given by $x(t) = \exp((t - t_0)X)$ for all $t \in \boldsymbol{R}$. By right invariance of $X$, the solution with initial value $x(t_0) = g$ is given by $x(t) = \exp((t - t_0)X)g$.

## 2. Reachable sets in Lie groups

We begin with the following proposition which is a variant of Proposition 1.3 of [2].

PROPOSITION 1. *Let $\Omega \subseteq L(G)$. Then the set $R_{pc}(e,\delta,\Omega)$ consists of all $g \in G$ of the form*

$$g = \exp(\tau_1 X_1)\exp(\tau_2 X_2)\cdots\exp(\tau_n X_n),$$

*where*

$$\sum_{i=1}^{n} \tau_i \|X_i\| = \delta \quad and \quad X_1, \ldots, X_n \in \Omega.$$

*The set $\mathbf{R}_{pc}(e,\Omega)$ is equal to the semigroup generated by the set* $\exp(\boldsymbol{R}^+\Omega)$.

PROOF. Let $h$ be reachable at norm cost $\delta$ for the system $\Omega$. Then there exists $U \in \mathcal{U}_{pc}(\Omega)$ such that $h = \pi(e,T,U)$ and that $\delta = \int_0^T \|U(t)\|\,dt$ for some $T > 0$. Since $h = \pi(e,T,U)$, there exists $s_1, \ldots, s_n \in \mathbf{R}^+$ and $Y_1, \ldots, Y_n \in \Omega$ such that

$$h = \exp(s_1 Y_1)\exp(s_2 Y_2)\cdots\exp(s_n Y_n),$$

where $T = \sum_{i=1}^n s_i$. Let $t_i = \sum_{k=1}^i s_k$, $\tau_i = s_{n+1-i}$ and let $X_i = Y_{n+1-i}$ for $i = 1, 2, \ldots, n$. Then $0 = t_0 < t_1 < \cdots < t_n = T$ and

$$\delta = \int_0^T \|U(t)\|\,dt = \sum_{i=1}^n (t_i - t_{i-1})\|Y_i\| = \sum_{i=1}^n \tau_i \|X_i\|.$$

Moreover, $h = \exp(\tau_1 X_1)\exp(\tau_2 X_2)\cdots\exp(\tau_n X_n)$. By the proceeding calculation, $\mathbf{R}_{pc}(e,\Omega)$ is a subset of the semigroup $\mathbf{A}_{pc}(e,\Omega)$ which is the semigroup generated by the set $\exp(\boldsymbol{R}^+\Omega)$. Conversely, if $g \in \mathbf{A}_{pc}(e,\Omega)$, then $g$ is of the form

$$\exp(s_1 X_1)\exp(s_2 X_2)\cdots\exp(s_n X_n), X_1, X_2, \ldots X_n \in \Omega.$$

Thus $g \in R_{pc}(e,\delta,\Omega) \subseteq \mathbf{R}_{pc}(e,\Omega)$, where $\delta = \sum_{i=1}^n s_i \|X_i\|$. □

EXAMPLE. Let $G = Aff(\boldsymbol{R})$ and let its Lie algebra $L(G) = aff(\boldsymbol{R})$. Then we may identify $G$ and $L(G)$ with the sets

$$\left\{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a, b \in \boldsymbol{R}, a > 0\right\}, \quad \left\{\begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} : c, d \in \boldsymbol{R}\right\}$$

respectively. For $\Omega = \{X, Y\}$, where

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

we have that the set of all elements reachable from $e$ at norm cost $\delta$ using only piecewise constant controls into $\Omega$

$$\begin{aligned} R_{pc}(e, \delta, \Omega) &= \left\{\begin{pmatrix} e^{\alpha} & \beta e^{\alpha} \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} e^{\alpha} & \beta \\ 0 & 1 \end{pmatrix} : \alpha, \beta \geq 0, \alpha + \beta = \delta\right\} \\ &= A_{pc}(e, T, \Omega) \end{aligned}$$

where $T = \delta$. For $\Omega = \{X, Y\}$, where

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},$$

we also have that the set $R_{pc}(e, \delta, \Omega)$ of all elements reachable from $e$ at norm cost $\delta$ using only piecewise constant controls into $\Omega$ is the set

$$\left\{\begin{pmatrix} e^{\alpha+\beta} & e^{\alpha}(e^{\beta} - 1) \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} e^{\alpha+\beta} & e^{\beta} - 1 \\ 0 & 1 \end{pmatrix} : \alpha, \beta \geq 0, \alpha + \sqrt{2}\beta = \delta\right\}.$$

The following proposition is the relationship between attainable sets and reachable sets.

PROPOSITION 2. *Let $G$ be a Lie group with its Lie algebra $L(G)$ and let $\Omega \subseteq L(G)$.*

(1) $\mathbf{A}(e, T, \Omega) \subseteq \mathbf{R}(e, NT, \Omega)$ *if* $\|X\| \leq N$ *for all* $X \in \Omega$. *In particular, if* $\Omega$ *is a compact subset of* $L(G)$, *then* $\mathbf{A}(e, T, \Omega) \subseteq \mathbf{R}(e, \delta, \Omega)$ *for some* $\delta \geq 0$.

(2) $\mathbf{A}(e, T, \Omega) = \mathbf{R}(e, T, \Omega)$ *if* $\Omega = \{X \in L(G) \colon \|X\| = 1\}$.

(3) *If there exists* $\varepsilon > 0$ *such that* $X \in \Omega$ *implies* $X = rY$ *for some* $r > 0$ *and some* $Y \in \Omega$ *with* $\varepsilon \geq \|Y\|$, *then* $\mathbf{R}_{pc}(e, \delta, \Omega) \subseteq \mathbf{A}_{pc}(e, \varepsilon^{-1}\delta, \Omega)$.

PROOF. The proof of (1) and (2) is immediate from the definition.

(3) This is straightforward.
(4) See the proof of Proposition 1 in order to prove this statement. □

REMARK. Let $G$ be a matrix Lie group with matrix Lie algebra $L(G)$ and exponential mapping the usual matrix exponential. If $\Omega$ is a subset of $L(G)$ consisting of matrices which all have the same non-zero trace $b$ and all have the same norm $N$, then an reachable matrix $M$ is reachable only at the norm cost $(1/b)\log(\det(M))N$. For detail, see also Corollary 2.12 in [2].

The class of *regulated functions* which may be characterized either as being uniform limits of step functions if their domain of definition is compact interval or else as having limits from the right and left in all points of their domain of definition (whenever such limits make sense). This class includes the piecewise continuous functions. A regulated function exhibits many properties akin to continuity; for instance, it is bounded and is, in fact, continuous on the complement of a countable subset of its domain.

Suppose that the dimension of $L(G)$ is $n$. We choose some basis $X_1, \dots, X_n$ of $L(G)$. Each $U\colon [0,T] \to \Omega$ has a unique representation in the form $U = (u_1, \dots, u_n)$, where $U(t) = \sum_{i=1}^{n} u_i(t)X_i \in \Omega$ for each $t \in [0,T]$. Note in this case that $U$ is regulated if and only if each of its coordinate functions $u_i(\cdot)$ with respect to $X_i$ is regulated. We consider the Hilbert space of regulated functions from $[0,T]$ into $L(G)$. Then

$$\int_0^T \|U(t)\|\,dt = \int_0^T \sqrt{u_1(t)^2 + \cdots + u_n(t)^2}\,dt.$$

Let $\mathscr{U}_r(\Omega)$ denote the set of all regulated functions from $\boldsymbol{R}^+$ into $\Omega$ and let $\mathscr{U}_r(T)$ denote the class of regulated functions from $[0,T]$ to $\Omega$. For convenience, we use the following notations; for given $\delta > 0$,

$$\mathscr{U}_r(\delta) = \{U \in \mathscr{U}_r(\Omega)\colon \int_0^{t_u} \|U(w)\|\,dw = \delta \text{ for some } t_u > 0\}$$

$$T_\delta^u = \inf\{t_u\colon \int_0^{t_u} \|U(w)\|\,dw = \delta\}$$

$$T_\delta = \sup\{T_\delta^u\colon U \in \mathscr{U}_r(\delta)\}$$

Now, for given $(U,s) \in \mathscr{U}_r(\delta) \times [0,\delta]$, we define a mapping $F : [0, T_\delta^u] \to \boldsymbol{R}$ by

$$F(t) = \int_0^t \|U(w)\| \, dw.$$

Then $F$ is continuous, $F(0) = 0$ and $F(T_\delta^u) = \delta$. Thus for given $s(0 < s < \delta)$, by Intermediate Value Theorem, there exists $T_s^u(0 < T_s^u < T_\delta^u)$ such that

$$F(T_s^u) = \int_0^{T_s^u} \|U(w)\| \, dw = s.$$

Throughout, we denote the set of all elements reachable from $g$ at norm cost $\delta$ for the system $\Omega$ by means of regulated functions by $R_r(g,\delta,\Omega)$.

LEMMA 3. Let $\Omega$ be a subset of $L(G)$ such that for all $X \in \Omega$, $\|X\| \geq M$ for some $M > 0$. *The mapping* $\Phi\colon \mathscr{U}_r(\delta) \times [0,\delta] \to G$ *defined by* $\Phi((U,s)) = \pi(g, T_s^u, U)$ *is continuous for each* $g \in G$ *and each* $\delta > 0$, *where* $\mathscr{U}_r(\delta)$ *is given the norm topology.*

PROOF. First, we note that for given $0 < \delta < \infty$ and $U \in \mathscr{U}_r(\delta)$,

$$T_\delta M \leq \int_0^{T_\delta} \|U(w)\| \, dw = \delta < \infty.$$

It follows that $T_\delta < \infty$. Now we define a mapping

$$(**) \qquad \Psi\colon \mathscr{U}_r(\delta) \times [0,\delta] \to \mathscr{U}_r(T_\delta) \times [0, T_\delta]$$

by $\Psi((U,s)) = (V, T_s^u)$, where $V$ is the restriction map of $U$ on $[0, T_\delta]$ and $\mathscr{U}_r(T_\delta)$ is given the topology of weak convergence. Then the map $\Psi$ is well-defined by the proceeding arguments and definitions. In order to prove that $\Psi$ is continuous, let $\{(U_n, s_n)\}_{n=1}^{\infty}$ be any sequence in $\mathscr{U}_r(\delta) \times [0,\delta]$ which converges to $(U,s)$. Then $U_n$ converges to $U$ and also $s_n$ converges to $s$. Since $U_n$ converges to $U$ in the norm topology, $V_n$ also converges to $V$ in the norm topology and hence $V_n$ converges to $V$ in the topology of weak convergence. To complete the proof of continuity of the map $\Psi$, it is sufficent to show $\{T_{s_n}^{u_n}\}_{n=1}^{\infty}$ converges to $T_s^u$, where

$$s_n = \int_0^{T_{s_n}^{u_n}} \|U_n(w)\| \, dw$$

and

$$s = \int_0^{T_s^u} \|U(w)\| \, dw.$$

Now we note that

$$\begin{aligned}
|s_n - s| &= \left| \int_0^{T_{s_n}^{u_n}} \|U_n(w)\|\, dw - \int_0^{T_s^u} \|U(w)\|\, dw \right| \\
&= \left| \int_0^{T_{s_n}^{u_n}} \|U_n(w)\|\, dw - \int_0^{T_s^u} \|U_n(w)\|\, dw + \int_0^{T_s^u} \|U_n(w)\|\, dw - \int_0^{T_s^u} \|U(w)\|\, dw \right| \\
&\geq \left| \int_0^{T_{s_n}^{u_n}} \|U_n(w)\|\, dw - \int_0^{T_s^u} \|U_n(w)\|\, dw \right| - \left| \int_0^{T_s^u} \|U(w)\|\, dw - \int_0^{T_s^u} \|U_n(w)\|\, dw \right| \\
&= \left| \int_{T_{s_n}^{u_n}}^{T_s^u} \|U_n(w)\|\, dw \right| - \left| \int_0^{T_s^u} (\|U(w)\| - \|U_n(w)\|)\, dw \right| \\
&\geq M|T_s^u - T_{s_n}^{u_n}| - \left| \int_0^{T_s^u} (\|U(w)\| - \|U_n(w)\|)\, dw \right|.
\end{aligned}$$

Since $s_n$ converges to $s$ and since the second part of the right hand of the above last equation equal 0 as $n$ approaches infinity, $\{T_{s_n}^{u_n}\}_{n=1}^{\infty}$ converges to $T_s^u$.

Finally, we note that the map $\Phi$ is the composition of $\Psi$ and $\Gamma$, where the map

$$\Gamma\colon \mathscr{U}_r(T) \times [0, T] \to G$$

is defined by $\Gamma((U, t)) = \pi(g, t, U)$. The continuity of $\Gamma$ is verified in [2, 5] for the case that the set of bounded measurable functions from $[0, T]$ to $L(G)$ with the topology of weak convergence. But which is also satisfied in our case. This completes the proof. □

LEMMA 4. *If $T_\delta < \infty$ for given $\delta > 0$, then $R_r(e, \delta, \Omega) \subset \mathbf{A}(e, T\delta, \Omega)$.*

PROOF. Since $T_\delta = \sup\{T_\delta^u\colon U \in \mathscr{U}_r(\delta)\} < \infty$, $T_\delta^u \geq T_\delta$ for all $U \in \mathscr{U}_r(\delta)$, and by (**), if $g \in R_r(e, \delta, \Omega)$, then $g = \pi(e, T_\delta^u, U)$ for some $U$. It follows that $g \in \mathbf{A}(e, T_\delta, \Omega)$ for all $g \in R_r(e, \delta, \Omega)$. The proof is complete. □

THEOREM 5. *Let $\Omega$ be a non-empty subset of $L(G)$. Then the reachability set $\mathbf{R}(g, \Omega)$ from $g$ is a semigroup. In particular, $R(e, \delta, \Omega)R(e, s, \Omega) = R(e, \delta + s, \Omega)$.*

PROOF. Since $\mathbf{R}(g, \Omega) = \mathbf{R}(e, \Omega)g$, it is sufficent to show $\mathbf{R}(e, \Omega)$ is a smeigroup. Let $g_1 = \pi(e, t_1, U_1)$ with norm cost $\delta$ for $U_1 \in \mathscr{U}(\Omega)$ and let

$g_2 = \pi(e, t_2, U_2)$ with norm cost $s$ for $U_2 \in \mathscr{U}(\Omega)$. We define $V: \boldsymbol{R}^+ \to \Omega$ by

$$V(\tau) = \begin{cases} U_1(\tau), & \text{for } 0 \le \tau \le t_1 \\ U_2(\tau - t_1), & \text{for } \tau > t_1. \end{cases}$$

Then $g_2 g_1 = \pi(e, t_1 + t_2, V)$ and

$$\int_0^{t_1+t_2} \|V(\tau)\|\, d\tau = \int_0^{t_1} \|U_1(\tau)\|\, d\tau + \int_{t_1}^{t_1+t_2} \|U_2(\tau - t_1)\|\, d\tau = \delta + s.$$

Conversely, if $g \in R(e, \delta + s, \Omega)$, then $g = \pi(e, t, V)$ with

$$\int_0^t \|V(\tau)\|\, d\tau = \delta + s$$

for some $U \in \mathscr{U}(\Omega)$ and some $t > 0$. By Intermediate Value Theorem, there exists $t_1 (0 < t_1 < t)$ such that

$$\int_0^{t_1} \|V(\tau)\|\, d\tau = \delta.$$

Now we choose $U_1 \in \mathscr{U}(\Omega)$ such that $U_1(\tau) = V(\tau)$ for $0 < \tau < t_1$ and define $U_2$ by

$$U_2(\tau) = \begin{cases} V(\tau + t_1), & \text{for } 0 \le \tau \le t - t_1 \\ V(\tau), & \text{for } \tau > t - t_1. \end{cases}$$

Then

$$\int_0^{t_1} \|U_1(\tau)\|\, d\tau = \delta$$

and

$$\int_0^{t-t_1} \|U_2(\tau)\|\, d\tau = \int_0^{t-t_1} \|V(\tau + t_1)\|\, d\tau = \int_{t_1}^{t} \|V(\tau)\|\, d\tau = s.$$

And we note that

$$V(\tau) = \begin{cases} U_1(\tau), & \text{for } 0 \le \tau \le t_1 \\ U_2(\tau - t_1), & \text{for } \tau > t_1. \end{cases}$$

Let $g_1 = \pi(e, t_1, U_1)$ and let $g_2 = \pi(e, t - t_1, U_2)$. Then $g_1 \in R(e, \delta, \Omega)$, $g_2 \in R(e, s, \Omega)$, and $g_2 g_1 = \pi(e, t - t_1, U_2) g_1 = \pi(g_1, t - t_1, U_2) = \pi(e, t_1 + (t - t_1), V) = \pi(e, t, V) = g$. This completes the proof. □

THEOREM 6. *Let G be a Lie group with its Lie algebra L(G) and let Ω be a subset of L(G) such that for all* $X \in \Omega$, $\|X\| \geq M$ *for some* $M > 0$. *Then* $R_{pc}(e,\delta,\Omega)$ *is dense in* $R_r(e,\delta,\Omega)$.

PROOF. If $x \in R_r(e,\delta,\Omega)$, then $x = \pi(e, T_\delta^u, U)$ such that $\int_0^{T_\delta^u} \|U(w)\|\, dw = \delta$, where $U : [0, T_\delta^u] \to \Omega$ is a regulated function. By IV. 5.10. Lemma in [3], $U$ is the uniform limit of a sequence $U_n : [0, T_\delta^u] \to \Omega$ of piecewise constant functions. Furthermore, the function $U_n$ may be chosen so that

$$\int_0^{T_\delta^u} \|U_n(w)\|\, dw = \delta, \quad \text{for each } n.$$

And by Lemma 3, $\pi(e, T_\delta^u, U_n)$ converges to $\pi(e, T_\delta^u, U)$ and $\pi(e, T_\delta^u, U_n)$ contained in the set $R_{pc}(e,\delta,\Omega)$. That completes the proof. □

By Proposition 1 and Theorem 6, we have the following result.

THEOREM 7. *Let G be a Lie group with its Lie algebra L(G) and let Ω be a subset of L(G) such that for all* $X \in \Omega$, $\|X\| \geq M$ *for some* $M > 0$. *Then for* $\delta \geq 0$, *the following sets are all equal*:

(1) *The closure of the set* $R_\delta$, *where*

$$R_\delta = \left\{ \exp(\tau_1 X_1) \cdots \exp(\tau_n X_n) : \sum_{i=1}^{n} \tau_i \|X_i\| = \delta \text{ and } X_1, \ldots, X_n \in \Omega \right\}.$$

(2) $\overline{R_{pc}(e,\delta,\Omega)}$, *the closure of the reachable set from e at norm cost δ for Ω with the piecewise constant control functions.*

(3) $\overline{R_r(e,\delta,\Omega)}$, *the closure of the reachable set from e at norm cost δ for Ω with the regulated functions.*

COROLLARY 8. *Let G be a Lie group with its Lie algebra L(G) and let Ω be a compact and convex subset of L(G) such that for all* $X \in \Omega$, $\|X\| \geq M$ *for some* $M > 0$. *Then the map* $\sigma : \boldsymbol{R}^+ \to K(G)$ *defined by* $\sigma(\delta) = \overline{R_r(e,\delta,\Omega)}$ *is a homomorphism, where K(G) is the topological semigroup of all non-empty compact subsets of G with Vietoris topology.*

PROOF. Since $T_\delta < \infty$ for each $\delta > 0$, and since $\mathbf{A}(e, T_\delta, \Omega)$ is compact (see Corollary 1.5 of [2]), by Lemma 4, $\overline{R_r(e,\delta,\Omega)} \subset \overline{\mathbf{A}(e, T_\delta, \Omega)} = \mathbf{A}(e, T_\delta, \Omega)$. Thus $\overline{R_{pc}(e,\delta,\Omega)} = \overline{R_r(e,\delta,\Omega)}$ is compact. And we also note that

$R_{pc}(e,\delta,\Omega)R_{pc}(e,s,\Omega) = R_{pc}(e,\delta+s,\Omega)$ (see the proof of Theorem 5).

$$\begin{aligned}\sigma(\delta)\sigma(s) &= \overline{R_r(e,\delta,\Omega)R_r(e,s,\Omega)} = \overline{R_{pc}(e,\delta,\Omega)} = \overline{R_{pc}(e,s,\Omega)} \\ &= \overline{R_{pc}(e,\delta,\Omega)R_{pc}(e,s,\Omega)} = \overline{R_{pc}(e,\delta+s,\Omega)} \\ &= \overline{R_r(e,\delta+s,\Omega)} = \sigma(\delta+s).\end{aligned}$$

This completes the proof. □

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