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REPRESENTATION OF q-ANALOGUE OF RATIONAL BRAUER ALGEBRAS*

By

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Introduction

Let q and a be indeterminates over a field K of characteristic 0, and let $K(a,q)$ denote the field of rational functions. We define the algebra $H_{m,n}(a,q)$ over $K(a,q)$ by generators and relations. (See the Definition 2.1.) If we replace the indeterminate a with q^{-r} in the definition, we have a *q-analogue of rational Brauer algebra* $H_{m,n}^{r}(q)$, which we have introduced in the previous paper with J. Murakami [8]. (In the paper [8], we called the algebra $H_{m,n}^{r}(q)$ the generalized Hecke algebra.) The algebra $H_{m,n}^{r}(q)$ is semisimple in case $r \geq m+n$, as we already observed in [8]. This observation is extended to the algebra $H_{m,n}(a,q)$. That is to say, $H_{m,n}(a,q)$ is also semisimple.

In this paper, we construct new representations of the algebras $H_{m,n}(a,q)$ and $H_{m,n}^{r}(q)$. These representations are irreducible and they are obtained from the left regular representations of $H_{m,n}(a,q)$ and $H_{m,n}^{r}(q)$ respectively.

Our previous paper was written originally to investigate the centralizer algebra of mixed tensor representations of quantum algebra $\mathcal{U}_q(gl_n(\boldsymbol{C}))$, which was q-analogue version of the work of Benkart et al. [1]. (The existence of their preliminary version of the paper [1] was informed to the author by Professor Okada.) Their original situation was as follows. Let G denote the general linear group $GL(r,\boldsymbol{C})$ of $r\times r$ invertible complex matrices and let V be the vector space on which G acts naturally. Let V^* be the dual space of V. The mixed tensor T of m copies of V and n copies of V^* is defined by $T=(\otimes^m V)\otimes(\otimes^n V^*)$. In this situation, they constructed the irreducible representations of the centralizer algebra $\mathrm{End}_G(T)$, by locating the maximal vectors in the mixed tensor T. Replacing G with $\mathcal{U}_q(gl_n(\boldsymbol{C}))$ and extending the underlying field $\mathbf{C}$ to $\mathbf{C}(q)$, we

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have the q-analogue of their centralizer algebras which we called the generalized Hecke algebra $H^r_{m,n}(q)$. Instead of locating the maximal vectors in T, we used the Bratteli diagram of the inclusions, $C(q) \subset H^r_{1,0} \subset H^r_{2,0} \subset \cdots \subset H^r_{m,0} \subset H^r_{m,1} \subset \cdots \subset H^r_{m,n}$, to construct the irreducible representations of $H^r_{m,n}(q)$. However, the use of Bratteli diagram forced us to use q-rational functions as the matrix elements.

It turns out that if we define $H^r_{m,n}(q)$ over $\boldsymbol{Q}(q)$, the trace of the representing matrix of each generator is in $\boldsymbol{Z}[q, q^{-1}]$. So it is natural to conjecture that if we take a suitable basis in each irreducible representation, the matrix elements are in $\boldsymbol{Z}[q, q^{-1}]$.

Let us recall that as for the (classical) Hecke algebra $H_n(q)$ of type A, all the irreducible representations are afforded by cell representations [6]. For these irreducible representations the integrality holds. Namely, each generator of $H_n(q)$ maps to the matrix over $\boldsymbol{Z}[q, q^{-1}]$ on these representations.

The main purpose of this paper is to show that the conjecture for the integrality of irreducible representations of $H^r_{m,n}(q)$ holds true. For this purpose, we will define a new basis of $H_{m,n}(a, q)$. This paper is organized as follows. Section 1 presents the general results about the Hecke algebra of type A and W-graphs. In Section 2, we define the algebra $H_{m,n}(a, q)$ and define (left and right) k-contractions in $H_{m,n}(a, q)$. Then we show some properties of k-contractions. These k-contractions are originally defined in their paper [1] in the case $q = 1$. They help us to construct all the irreducible representations of the algebra $H_{m,n}(a, q)$ by taking subquotients of the left regular representation. In Section 3 we give the new basis of $H_{m,n}(a, q)$. Taking suitable subquotients of the regular representation of $H_{m,n}(a, q)$ with respect to the new basis, we obtain the irreducible representations of $H_{m,n}(a, q)$. If we define $H^r_{m,n}(q)$ by replacing the indeterminate a with q^{-r} in the definition of $H_{m,n}(a, q)$ and construct the corresponding representations of $H^r_{m,n}(q)$ replacing a with q^{-r} in the procedure, then we obtain the desired representations of $H^r_{m,n}(q)$.

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1. Hecke algebras and W-graphs

First we review general results about Iwahori-Hecke algebra (of type A) and their irreducible representations without proofs. The following results are from Kazhdan-Lusztig [6] or Shi's book [9].

Let (W, S) be a Coxeter system and let A be the ring $\boldsymbol{Z}[q, q^{-1}]$ of Laurent polynomials over $\boldsymbol{Z}$ in the indeterminate q. The Hecke algebra $\mathscr{H}$ is by definition the associative A-algebra with a free A-basis $\{T_w\}_{w \in W}$ over the ring A, obeying the relations:

$$T_w T_{w'} = T_{ww'}, \qquad \text{if } \ell(ww') = \ell(w) + \ell(w'),$$
$$(T_s - q)(T_s + q^{-1}) = 0 \quad \text{if } s \in S.$$

Here $\ell(w)$ denotes the length of w.

In this paper we consider only the case W is the symmetric group. So, $\mathscr{H} = H_n(q)$ can also be defined by

generators:

$$T_1, T_2, \ldots, T_{n-1}$$

and

relations:

$$(T_i - q)(T_i + q^{-1}) = 0 \quad (1 \le i \le n-1),$$
$$T_i T_j = T_j T_i \qquad (1 \le i, j \le n-1, |i-j| \ge 2),$$
$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \quad (1 \le i \le n-2).$$

As they showed [6], a complete set of irreducible representations for the Hecke algebra $H_n(q)$ can be afforded with some multiplicities by dividing W-graphs into left cells. We shall construct some W-graphs for $H_n(q)$ as in [6].

Definition 1.1. A W-graph is, by definition, a set of vertices X, with a set Y of edges (each edge consists of two elements of X) together with two additional data: for each vertex $x \in X$, we are given a subset I_x of S and, for each ordered pair of vertices y, x such that $\{y, x\} \in Y$, we are given an integer $\mu(y, x) \neq 0$. These data are subject to the following requirements: Let E be the free A-module with basis X. Then for any $s \in S$,

$$\tau_s(x) = \begin{cases} -q^{-1}x & s \in I_x \\ qx + \sum \mu(z, x)z & s \notin I_x \end{cases}$$

defines an endomorphism of E and there is a unique representation $\phi\colon \mathscr{H} \to \mathrm{End}(E)$ such that $\phi(T_s) = \tau_s$ for each $s \in S$.

To construct W-graphs, we shall first introduce Kazhdan-Lusztig polynomials and define the relation $\prec$.

Let $a \to \bar{a}$ be the involution of the ring $A = \boldsymbol{Z}[q, q^{-1}]$ defined by $\bar{q} = q^{-1}$. This extends to an involution of $h \to \bar{h}$ of the ring $\mathscr{H}$, defined by $\overline{\sum a_w T_w} = \sum \overline{a_w} T_{w^{-1}}^{-1}$. (Note that T_w is an invertible element of $\mathscr{H}$ for any $w \in W$). Let $\leq$ be the Bruhat order relation on W. The following basic theorem of Kazhdan-Lusztig [6] provides a basis of the algebra $\mathscr{H}$.

THEOREM 1.2. *For any $w \in W$, there is a unique element $C_w \in \mathscr{H}$, such that*
(1) $\overline{C_w} = C_w$,
(2) $C_w = \sum \varepsilon_y \varepsilon_w q_w q_y^{-1} \overline{P_{y,w}(q^2)} T_y$,
where $P_{y,w}(q) \in A$ is a polynomial in q of degree less than or equal to $(1/2)(\ell(w) - \ell(y) - 1)$ for $y < w$, and $P_{w,w} = 1$.

The polynomials $P_{y,w}$ in the above theorem are called Kazhdan-Lusztig polynomials. The proof of the theorem is in their original paper [6].

Next, we define the relation $\prec$.

DEFINITION 1.3. Given $y, w \in W$ we say that $y \prec w$ if the following conditions are satisfied: $y < w$, $\varepsilon_w = -\varepsilon_y$ and $P_{y,w}(q)$ is a polynomial in q of degree exactly $(1/2)(\ell(w) - \ell(y) - 1)$. In this case, the leading coefficient of $P_{y,w}(q)$ is denoted by $\mu(y, w)$. It is a non-zero integer. If $w \prec y$, we set $\mu(w, y) = \mu(y, w)$.

PROPOSITION 1.4. *Let $s \in S, w \in W$.*
(1) *If $sw < w$, then $T_s C_w = -q^{-1} C_w$.*
(2) *If $w < sw$, then $T_s C_w = q C_w + C_{sw} + \Sigma \mu(z, w) C_z$,*
where the sum is taken over all $z \prec w$ for which $sz < z$.

Let Γ_L be the graph whose vertices are the elements of W and whose edges are the subsets of W of the form $\{y, w\}$ with $y \prec w$. For each $w \in W$, let $I_w = \mathscr{L}(w) = \{s \in S | sw < w\}$. Then Proposition 1.4 implies that Γ_L, together with the assignment $w \to I_w$ and with the function μ defined in 1.3 is a W-graph.

We will next decompose W-graphs into 'cells' which will give irreducible representations of $\mathscr{H}$ in case $W = S_n$ (accordingly $\mathscr{H} = H_n(q)$). We shall define, following Kazhdan and Lusztig [6], cells of any Coxeter group (W, S).

For $x, y \in W$, we denote $x - y$ if either $x \prec y$ or $y \prec x$ holds. We define a preorder relation $w \leq_L w'$ on W if there exist elements $w = x_1, x_2, \dots, x_t = w'$ in W such that for each i we have $x_{i-1} - x_i$ and $\mathscr{L}(x_{i-1}) \not\subset \mathscr{L}(x_i)$. We may then define an equivalence relation $w \sim_L w'$ to be $w \leq_L w'$ and $w' \leq_L w$. The equivalence classes with respect to the relation $\sim_L$ are called *left cells*. With the language of cells, we shall consider formulas in the above proposition. In case (2) of the proposition, we have $w < sw$, so that $w \prec sw$ with $\mathscr{L}(sw) \not\subset \mathscr{L}(w)$, implying $sw \leq_L w$. On the other hand, any element $z \prec w$ in the sum satisfies $sz < z$ for the given s, so $\mathscr{L}(z) \not\subset \mathscr{L}(w)$ (because $sw > w$). Thus $z \leq_L w$. In either case of the proposition, it follows that left multiplication by T_s takes C_w into the A-span of itself and various C_x for which $x \leq_L w$.

Now fix a left cell $\lambda \subset W$, and define $\mathscr{I}_\lambda$ to be the A-span of all $C_w (w \in \lambda)$ together with all C_x for which $x \leq_L w$ $(w \in \lambda)$. The preceding discussion shows that $\mathscr{I}_\lambda$ is a left ideal in $\mathscr{H}$. Let $\mathscr{I}_\lambda'$ be the span of those C_x for which $x \leq_L w$ for some $w \in \lambda$ but $x \notin \lambda$. Since $\leq_L$ is transitive, the definition of left cells implies that $\mathscr{I}_\lambda'$ is also a left ideal in $\mathscr{H}$, so the quotient $\mathscr{M}_\lambda := \mathscr{I}_\lambda / \mathscr{I}_\lambda'$ affords a representation of $\mathscr{H}$. In other words, for each left cell, regarded as a full subgraph of Γ_L with the same sets I_x and the same function μ is itself a W-graph. One can similarly define *right cells* by replacing $I_w = \mathscr{L}(w)$ and $\mathscr{L}(x_{i-1}) \not\subset \mathscr{L}(x_i)$ with $I_w = \mathscr{R}(w)$ and $\mathscr{R}(x_{i-1}) \not\subset \mathscr{R}(x_i)$ respectively, where $\mathscr{R}(w) = \{s \in S | ws < w\}$. One can also define *two-sided cells* of W by replacing $\mathscr{L}(x_{i-1}) \not\subset \mathscr{L}(x_i)$ with the condition that $\mathscr{L}(x_{i-1}) \not\subset \mathscr{L}(x_i)$ or $\mathscr{R}(x_{i-1}) \not\subset \mathscr{R}(x_i)$ and replacing $I_w = \mathscr{L}(w)$ with $I_w = \mathscr{L}(w) \sqcup \mathscr{R}(w)$. The notation $x \sim_R y$ (resp. $x \sim_\Gamma y$) means that x, y are in the same right (resp. two-sided) cell of W.

Let W be the symmetric group S_n. Then the cells of W can be classified by the Robinson-Schensted map.

Let $P(n)$ be the set of partitions $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_r)$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$ and $\sum_{i=1}^r \lambda_i = n$. *Standard Young tableau* of shape λ is by definition numbering of cells of λ in such a way that it is increasing from left to right in each row and from top to bottom in each column. The following is an example of a standard Young tableau of shape $\lambda = (3, 2, 2, 1)$.

1	3	4
2	5	
6	8	
7		

In this paper, we adopt the bijection between W and permutations in the following way. Let $i_1 \cdots i_n$ be a permutation of $1, \ldots, n$. Each generator $s_i \in W$ acts from the right as the transposition of letters i and $i+1$, which we denote $i_1 \cdots i_n \cdot s_i$. Then the bijection is given by $w \mapsto 12 \cdots n \cdot w$. For example (1, 2)(2, 3) corresponds to 312. The Robinson-Schensted map $\theta \colon w \to (P(w), Q(w))$ gives a bijection from W to the pairs of standard Young tableaux on $\{1, 2, \ldots, n\}$ having the same shape (cf. [7]).

The following result is due to [2].

THEOREM 1.5. *For $y, w \in S_n$, we have*
(1) $y \sim_L w \Leftrightarrow P(y) = P(w)$,
(2) $y \sim_R w \Leftrightarrow Q(y) = Q(w)$,
(3) $y \sim_\Gamma w \Leftrightarrow P(y)$ *and* $P(w)$ *have the same shape.*

Kazhdan and Lusztig [6] showed the following result on the representations of S_n afforded by the left cells of S_n.

THEOREM 1.6. *Let X be a left cell of $W = S_n$, let Γ be the W-graph associated to X and let ϕ be the representation of $H_n(q)$ (over the quotient field of A) corresponding to Γ. Then ϕ is irreducible and the isomorphism classes of the W-graph Γ depends only on the isomorphism class of ϕ and not on X.*

The above theorem shows that two distinct left cells of $W = S_n$ may produce the same irreducible representations (up to isomorphism). The proof of the theorem, however, shows that if y and y' are distinct elements of X^{-1}, then the $\sim_L$ equivalence classes X_y and $X_{y'}$ which contain y and y' respectively produce the isomorphic left cells. Here the isomorphism between two left cells means the isomorphism between corresponding graphs which preserves μ and I_w. (See [6]). Combining the results of Theorem 1.5, we can see that the set of non-isomorphic irreducible representations of $H_n(q)$ are given by non-isomorphic left cells of S_n. Moreover each non-isomorphic left cell is indexed by the partition $\lambda \in P(n)$.

2. Algebra $H_{m,n}(a, q)$ and k-contractions in $H_{m,n}(a, q)$

In this section we define the $K(a, q)$-algebras $H_{m,n}(a, q)$. Then we define the k-contractions in $H_{m,n}(a, q)$. These k-contractions correspond to the q-analogue version of the ones which they defined in their paper [1].

DEFINITION 2.1. Let K be a field of characteristic 0. Let q and a be indeterminates over K. For integers $m, n \geq 0$, we define $H_{m,n}(a, q)$ to be the associative $K(a, q)$-algebra with the unit generated by

$$T_{m-1}, T_{m-2}, \ldots, T_2, T_1, E, T_1^*, T_2^*, \ldots, T_{n-2}^*, T_{n-1}^*$$

subject to the relations:

(B1) $T_iT_j = T_jT_i$ $(1 \le i, j \le m-1, |i-j| \ge 2)$,

(B2) $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$ $(1 \le i \le m-2)$,

(IH) $(T_i - q)(T_i + q^{-1}) = 0$ $(1 \le i \le m-1)$,

(B1*) $T_i^*T_j^* = T_j^*T_i^*$ $(1 \le i, j \le n-1, |i-j| \ge 2)$,

(B2*) $T_i^*T_{i+1}^*T_i^* = T_{i+1}^*T_i^*T_{i+1}^*$ $(1 \le i \le n-2)$,

(IH*) $(T_i^* - q)(T_i^* + q^{-1}) = 0$ $(1 \le i \le n-1)$,

(HH) $T_iT_j^* = T_j^*T_i$ $(1 \le i \le m-1, 1 \le j \le n-1)$,

(K1) $ET_i = T_iE$ $(2 \le i \le m-1)$,

(K1*) $ET_i^* = T_i^*E$ $(2 \le i \le n-1)$,

(K2) $ET_1E = a^{-1}E$,

(K2*) $ET_1^* = a^{-1}E$,

(K3) $E^2 = -\dfrac{a - a^{-1}}{q - q^{-1}}E$,

(K4) $ET_1^{-1}T_1^*ET_1 = ET_1^{-1}T_1^*ET_1^*$,

(K4′) $T_1ET_1^{-1}T_1^*E = T_1^*ET_1^{-1}T_1^*E$.

In the previous paper with J. Murakami [8], we defined the generalized Hecke algebra $H_{m,n}^r(q)$ which was $K(q)$-algebra obtained by being replaced one of the indeterminate a by q^{-r} in the above definition. Here we take a positive integer r. In the case of the $K(q)$-algebra $H_{m,n}^r(q)$, the relation (K3) is presented as follows: $E^2 = [r]E$, where $[r] = q^{r-1} + q^{r-3} + \cdots + q^{1-r}$.

The following theorem is one of the main results of [8]. (See loc. cit. Theorem 4.11, Corollary 4.13 and Proposition 2.2)

THEOREM 2.2. *If $r \ge m+n$, the $K(q)$-algebra $H_{m,n}^r(q)$ is semisimple and whose dimension is $(m+n)!$.*

The above theorem will be extended to the $K(a,q)$-algebra $H_{m,n}(a,q)$.

THEOREM 2.3. *The $K(a,q)$-algebra $H_{m,n}(a,q)$ is semisimple and whose dimension is $(m+n)!$.*

For the proof of the above theorem, we have only to follow Section 1–4 of [8] replacing q^{-r} with a.

REMARK 2.4. If we take $q_0 \in K\backslash\{0\}$, instead of taking the indeterminate q and put $a = q_0^r$ in Definition 2.1, then we can define the K-algebra $H_{m,n}^r(q_0)$. Furthermore if we take $a_0 \in K\backslash\{0\}$ instead of a and assume $q_0 - q_0^{-1} \neq 0$, the K-algebra $H_{m,n}(a_0, q_0)$ can be defined. If $[i]_{q_0} \neq 0$ for $i = 1, 2, \ldots, m+n+r$, then $H_{m,n}^r(q_0)$ is semisimple. Here $[i]_{q_0}$ is defined by $q_0^{i-1} + q_0^{i-3} + \cdots + q_0^{1-i}$. If $[i]_{q_0} \neq 0$ for $i = 1, 2, \ldots, \max(m, n)$ and $[a_0; j]_{q_0} \neq 0$ for $j = 1, 2, \ldots, m+n$, then $H_{m,n}(a_0, q_0)$ is also semisimple. Here $[a_0; j]_{q_0}$ is defined by $(a_0^{-1}q_0^j - a_0 q_0^{-j})/(q_0 - q_0^{-1})$.

We introduce the *k-contract sets* $(\underline{m}, \underline{n})$, which is defined by

$$(\underline{m}, \underline{n}) = \{(m_1, n_1), \ldots, (m_k, n_k)\}.$$

Here $\underline{m} = (m_1, \ldots, m_k)$ and $\underline{n} = (n_1, \ldots, n_k)$ are ordered subsets of $\{1, 2, \ldots, m\}$ and $\{1, 2, \ldots, n\}$ respectively. We further assume $m_1, m_2, \cdots, m_k$ are in increasing order (i.e. $m_1 < m_2 < \cdots < m_k$).

There are two standard ways in indexing $(\underline{m}, \underline{n})$. One is to index them by the *two line array* L, which is $2 \times k$ matrix whose first row is assigned by $\underline{m}$ and the second row is by $\underline{n}$. The other is to index them by the triple (A, B, σ) with $A \subset \{1, 2, \ldots, m\}, B \subset \{1, 2, \ldots, n\}$ $(|A| = |B| = k)$ and $\sigma \in S_k$, where S_k is the group of permutations of k letters $\{1, 2 \ldots, k\}$. We label the elements of A and B with $a_1, a_2, \ldots, a_k$ and $b_1, b_2, \ldots, b_k$ respectively in increasing order. The correspondence $(m_i, n_i) \leftrightarrow (a_i, b_{\sigma(i)})$ defines the bijection between L and (A, B, σ).

Let $A = \{a_1 < a_2 \cdots < a_k\} \subset \{1, 2, \cdots, m\}$ and $B = \{b_1 < b_2 < \cdots b_k\} \subset \{1, 2, \cdots, n\}$. Define

$$T_A = (T_{a_1-1} T_{a_1-2} \cdots T_1)(T_{a_2-1} T_{a_2-2} \cdots T_2) \cdots (T_{a_k-1} T_{a_k-2} \cdots T_k)$$

and

$$\overline{T_B^*} = (T_{b_1-1}^{*-1} T_{b_1-2}^{*-1} \cdots T_1^{*-1})(T_{b_2-1}^{*-1} T_{b_2-2}^{*-1} \cdots T_2^{*-1}) \cdots (T_{b_k-1}^{*-1} T_{b_k-2}^{*-1} \cdots T_k^{*-1}).$$

We understand $T_{a_l-1} T_{a_l-2} \cdots T_l = 1$ if $l = a_l$. Note that if there exists an l such that $a_l = l$ and $a_{l+1} > l+1$, then $a_1 = 1$, $a_2 = 2, \cdots, a_l = l$. In this case we have

$$T_A = (T_{a_{l+1}-1} T_{a_{l+1}-2} \cdots T_{l+1})(T_{a_{l+2}-1} T_{a_{l+2}-2} \cdots T_{l+2}) \cdots (T_{a_k-1} T_{a_k-2} \cdots T_k).$$

Similarly we define

$$T_A^{\mathrm{op}} = (T_k T_{k+1} \cdots T_{a_k-1})(T_{k-1} T_k \cdots T_{a_{k-1}-1}) \cdots (T_1 T_2 \cdots T_{a_1-1}),$$

and

$$\overline{T_B^{*\mathrm{op}}} = (T_k^{*-1} T_{k+1}^{*-1} \cdots T_{b_k-1}^{*-1})(T_{k-1}^{*-1} T_k^{*-1} \cdots T_{b_{k-1}-1}^{*-1}) \cdots (T_1^{*-1} T_2^{*-1} \cdots T_{b_1-1}^{*-1}).$$

The following lemma follows from the relation (**B1**), (**B2**), (**IH**) and (**B1***), (**B2***), (**IH***) in Definition 2.1.

LEMMA 2.5. *Let A be as above. Take $T_i \in H_m(q) \subset H_{m,n}(a,q)$. Let $A_{i,i+1} = (A \backslash \{i\}) \cup \{i+1\}$ (if $i \in A$) and let $A_{i+1,i} = (A \backslash \{i+1\}) \cup \{i\}$ (if $i+1 \in A$). Then we have the following formulas.*

(1) *If $i = a_l \in A$ and $i+1 = a_{l+1} \in A$ for some $l \leq k-1$, then $T_i T_A = T_A T_l$.*
(2) *If $i \in A$ and $i+1 \notin A$, then $T_i T_A = T_{A_{i,i+1}}$.*
(3) *If $i \notin A$ and $i+1 \in A$, then $T_i T_A = (q - q^{-1}) T_A + T_{A_{i+1,i}}$.*
(4) *If $i \notin A$ and $i+1 \notin A$, then there exists an $l > k$ such that $T_i T_A = T_A T_l$.*

PROOF. (1) This follows from the following calculation.

$$\begin{aligned}
&T_i (T_{a_l-1} T_{a_l-2} T_{a_l-3} \cdots T_{l+1} T_l)(T_{a_{l+1}-1} T_{a_{l+1}-2} T_{a_{l+1}-3} \cdots T_{l+1}) \\
&\quad = (T_i)(T_{i-1} T_{i-2} T_{i-3} \cdots T_{l+1} T_l)\{(T_i)(T_{i-1} T_{i-2} T_{i-3} \cdots T_{l+1})\} \\
&\quad = (T_i T_{i-1} T_i)(T_{i-2} T_{i-3} \cdots T_{l+1} T_l)(T_{i-1} T_{i-2} T_{i-3} \cdots T_{l+1}) \\
&\quad = (T_{i-1})\{(T_i T_{i-1})(T_{i-2} T_{i-3} \cdots T_{l+1} T_l)\}\{(T_{i-1})(T_{i-2} T_{i-3} \cdots T_{l+1})\} \\
&\quad = \vdots \\
&\quad = (T_{i-1} T_{i-2})\{(T_i T_{i-1} T_{i-2})(T_{i-3} \cdots T_{l+1} T_l)\}\{(T_{i-2})(T_{i-3} \cdots T_{l+1})\} \\
&\quad = \vdots \\
&\quad = (T_{i-1} T_{i-2} \cdots T_{l+1})\{(T_i T_{i-1} T_{i-2} T_{i-3} \cdots T_{l+1})(T_l)\}\{(T_{l+1})\} \\
&\quad = (T_{i-1} T_{i-2} \cdots T_l)(T_i T_{i-1} T_{i-2} T_{i-3} \cdots T_{l+1} T_l) \\
&\quad = (T_{a_l-1} T_{a_l-2} \cdots T_l)(T_{a_{l+1}-1} T_{a_{l+1}-2} \cdots T_{l+1}) T_l.
\end{aligned}$$

(2) (3) These are obvious.

(4) Let p be an index such that $p \leq a_p \leq i-1 < i < i+2 \leq a_{p+1}$. Since $a_{p+1} - 1 \geq i+1$ and $p+1 \leq i$, we have

$$T_i (T_{a_{p+1}-1} T_{a_{p+1}-2} \cdots T_{p+1}) = (T_{a_{p+1}-1} T_{a_{p+1}-2} \cdots T_{p+1}) T_{i+1}.$$

Hence we have $T_i T_A = T_A T_{k-p+i}$.

Similarly, we have the following lemmas.

LEMMA 2.6. *Let A, $A_{i,i+1}$ and $A_{i+1,i}$ be as in the previous lemma. Then we have the following formulas.*

(1) *If $i = a_l \in A$ and $i+1 = a_{l+1} \in A$, then $T_A^{\mathrm{op}} T_i = T_l T_A^{\mathrm{op}}$.*

(2) *If $i \in A$ and $i+1 \notin A$, then $T_A^{\mathrm{op}} T_i = T_{A_{i,i+1}}^{\mathrm{op}}$.*

(3) *If $i \notin A$ and $i+1 \in A$, then $T_A^{\mathrm{op}} T_i = (q - q^{-1}) T_A^{\mathrm{op}} + T_{A_{i+1,i}}^{\mathrm{op}}$.*

(4) *If $i \notin A$ and $i+1 \notin A$, then there exists an $l > k$ such that $T_A^{\mathrm{op}} T_i = T_l T_A^{\mathrm{op}}$.*

LEMMA 2.7. *Let B be the one defined before Lemma 2.5. Take $T_i^* \in H_{m,n}(a,q)$. Let $B_{i,i+1} = (B \backslash \{i\}) \cup \{i+1\}$ (if $i \in B$) and let $B_{i+1,i} = (B \backslash \{i+1\}) \cup \{i\}$ (if $i+1 \in B$). Then we have the following formulas.*

(1) *If $i = b_l \in B$ and $i+1 = b_{l+1} \in B$, then $T_i^* \overline{T_B^*} = \overline{T_B^*} T_l^*$.*

(2) *If $i \in B$ and $i+1 \notin B$, then $T_i^* \overline{T_B^*} = (q - q^{-1}) \overline{T_B^*} + \overline{T_{B_{i,i+1}}^*}$.*

(3) *If $i \notin B$ and $i+1 \in B$, then $T_i^* \overline{T_B^*} = \overline{T_{B_{i+1,i}}^*}$.*

(4) *If $i \notin B$ and $i+1 \notin B$, then there exists an $l > k$ such that $T_i^* \overline{T_B^*} = \overline{T_B^*} T_l^*$.*

LEMMA 2.8. *Let B, $B_{i,i+1}$ and $B_{i+1,i}$ be as in the previous lemma. Then we have the following formulas.*

(1) *If $i = b_l \in B$ and $i+1 = b_{l+1} \in B$, then $\overline{T_B^{*\mathrm{op}}} T_i^* = T_l^* \overline{T_B^{*\mathrm{op}}}$.*

(2) *If $i \in B$ and $i+1 \notin B$, then $\overline{T_B^{*\mathrm{op}}} T_i^* = (q - q^{-1}) \overline{T_B^{*\mathrm{op}}} + \overline{T_{B_{i,i+1}}^{*\mathrm{op}}}$.*

(3) *If $i \notin B$ and $i+1 \in B$, then $\overline{T_B^{*\mathrm{op}}} T_i^* = \overline{T_{B_{i+1,i}}^{*\mathrm{op}}}$.*

(4) *If $i \notin B$ and $i+1 \notin B$, then there exists an $l > k$ such that $\overline{T_B^{*\mathrm{op}}} T_i^* = T_l^* \overline{T_B^{*\mathrm{op}}}$.*

The *i-trivial contraction* E_i $(i = 0, 1, \ldots, k)$ is defined by:

$$E_0 = 1,$$

$$E_1 = E,$$

$$E_i = E(T_1 T_2 \cdots T_{i-1})(T_1^{*-1} T_2^{*-1} \cdots T_{i-1}^{*-1}) E_{i-1} \quad (i = 2, 3, \cdots, k).$$

These trivial contractions $\{E_i\}$ $(i = 2, 3, \ldots, k)$ are also defined by

$$E_i = E_{i-1}(T_{i-1} T_{i-2} \cdots T_1)(T_{i-1}^{*-1} T_{i-2}^{*-1} \cdots T_1^{*-1}) E.$$

It is proved by induction on i. Note that this element is of the form

$$E(T_1 T_1^{*-1}) E(T_2 T_1 T_2^{*-1} T_1^{*-1}) E \cdots E(T_{i-1} T_{i-2} \cdots T_1 T_{i-1}^{*-1} T_{i-2}^{*-1} \cdots T_1^{*-1}) E.$$

If we move T_2, T_2^{*-1} in the second parenthesis to the first, T_3, T_3^{*-1} in the third parenthesis to the first and iterate this procedure, we have that it coincides with $E(T_1 T_2 \cdots T_{i-1} T_1^{*-1} T_2^{*-1} \cdots T_{i-1}^{*-1}) E_{i-1}$ by the induction hypothesis.

As for the trivial k-contraction, the following lemma generalizes the relation $K4$ and $K4'$ in Definition 2.1.

LEMMA 2.9. *Let $\sigma \in S_k$. Then T_σ is defined as in Section 1. Similarly $T^*_{\sigma^{-1}} \in \mathrm{alg}\{T_1^*, T_2^*, \dots, T_{k-1}^*\}$ is defined. For these T_σ and $T^*_{\sigma^{-1}}$, we have*
(1) $T_\sigma E_k = T^*_{\sigma^{-1}} E_k$,
(2) $E_k T_\sigma = E_k T^*_{\sigma^{-1}}$.

PROOF. (1) If $k = 1, 2$, it is easy to see. We assume $T_\sigma E_{k-1} = T^*_{\sigma^{-1}} E_{k-1}$ holds for any $\sigma \in S_{k-1}$. In particular we have $T_{i-1} E_{k-1} = T^*_{i-1} E_{k-1}$. Hence for any $i \geq 2$ we have

$$\begin{aligned}
T_i E_k &= T_i E(T_1 T_2 \cdots T_{k-1})(T_1^{*-1} T_2^{*-1} \cdots T_{k-1}^{*-1}) E_{k-1} \\
&= E(T_1 T_2 \cdots T_{k-1}) T_{i-1} (T_1^{*-1} T_2^{*-1} \cdots T_{k-1}^{*-1}) E_{k-1} \\
&= E(T_1 T_2 \cdots T_{k-1})(T_1^{*-1} T_2^{*-1} \cdots T_{k-1}^{*-1}) T_{i-1} E_{k-1} \\
&= E(T_1 T_2 \cdots T_{k-1})(T_1^{*-1} T_2^{*-1} \cdots T_{k-1}^{*-1}) T^*_{i-1} E_{k-1} \\
&= E(T_1 T_2 \cdots T_{k-1}) T_i^* (T_1^{*-1} T_2^{*-1} \cdots T_{k-1}^{*-1}) E_{k-1} \\
&= T_i^* E(T_1 T_2 \cdots T_{k-1})(T_1^{*-1} T_2^{*-1} \cdots T_{k-1}^{*-1}) E_{k-1} \\
&= T_i^* E_k.
\end{aligned}$$

If we write $\sigma = \sigma' s$ $(\ell(\sigma) > \ell(\sigma'), s \in S)$, then

$$T_\sigma E_k = T_{\sigma'} T_s E_k = T_{\sigma'} T_s^* E_k = T_s^* T_{\sigma'} E_k = T_s^* T^*_{\sigma'^{-1}} E_k = T^*_{\sigma^{-1}} E_k.$$

Hence (1) holds by induction on $\ell(\sigma)$. Similarly, we can show that (2) holds.

Let $L = (A, B, \sigma)$ be a k-contract set and let E_k be the k-trivial contraction. A *left k-contraction* E_L is defined by:

$$E_L = T_A \overline{T_B^*} T_\sigma E_k.$$

As for T_σ, we review the *monomials in normal form* in Hecke algebra, $H_k(q)$. (See for example [4].) Consider the following sets of monomials.

$$U_1 = \{1, T_1, T_2 T_1, \dots, T_{k-1} T_{k-2} \cdots T_1\},$$

$$U_2 = \{1, T_2, T_3 T_2, \dots, T_{k-1} T_{k-2} \cdots T_2\},$$

$$\vdots \quad \vdots \quad \vdots$$

$$U_i = \{1, T_i, T_{i+1}T_i, \dots, T_{k-1}T_{k-2}\cdots T_i\},$$

$$\vdots \quad \vdots \quad \vdots$$

$$U_{k-2} = \{1, T_{k-2}, T_{k-1}T_{k-2}\},$$

$$U_{k-1} = \{1, T_{k-1}\}.$$

We shall say that $V_1V_2\cdots V_{k-1}$ is a *monomial in normal form* in $H_k(q)$, if $V_i \in U_i$ for $i = 1, 2, \dots, k-1$. We assume that T_σ is written in normal form. If $\sigma(1) = 1$ then $V_1 = 1$. On the other hand, if $\sigma(1) \neq 1$ then $V_1 \neq 1$. Similarly, we shall say that $V_1^*V_2^*\cdots V_{k-1}^*$ is a monomial in normal form in $H_k^*(q) = \mathrm{alg}\{1, T_1^*, T_2^*, \dots, T_{k-1}^*\}$, if $V_i^* \in U_i^*$ for $i = 1, 2, \dots, k-1$, where U_i^* is the one defined by taking $\{T_i^*\}$ for $\{T_i\}$ in the definition of U_i. We also assume that T_σ^* is written in normal form.

Then we have the following lemma.

Lemma 2.10. *Let $L = (A, B, \sigma)$ be a k-contract set and let E_L be a left k-contraction defined by L. For a finite set $X = \{x_1 < x_2 < \cdots < x_k\}$ of positive integers, let $R_l(X)$ denote the set $\{1, x_1+1, x_2+1, \dots, x_{l-1}+1, x_{l+1}, x_{l+2}, \dots, x_k\}$ and let $R_l^*(X)$ denote the set $\{1, 2, \dots, l, x_{l+1}, x_{l+2}, \dots, x_k\}$. Then we have the following formulas.*

(1) *If $1 \in A$ and $1 \in B$ and $\sigma(1) = 1$ then*

$$EE_L = -\frac{a - a^{-1}}{q - q^{-1}} E_L.$$

(2) *If $1 \in A$, $1 \in B$ and $\sigma(1) \neq 1$, then for the l such that $\sigma(l) = 1$.*

$$EE_L = \begin{cases} a^{-1}T_{a_l-1}T_{a_l-2}\cdots T_2E_{(R_l(A),B,(l,l-1,\dots,1)\sigma)} & \text{if } a_l > 2, \\ a^{-1}E_{(R_l(A),B,(l,l-1,\dots,1)\sigma)} & \text{if } a_l = 2. \end{cases}$$

(3) *If $1 \notin A$ and $1 \in B$, then for the l such that $\sigma(l) = 1$, we have*

$$EE_L = \begin{cases} a^{-1}T_{a_l-1}T_{a_l-2}\cdots T_2E_{(R_l(A),B,(l,l-1,\dots,1)\sigma)} & \text{if } a_l > 2, \\ a^{-1}E_{(R_l(A),B,(l,l-1,\dots,1)\sigma)} & \text{if } a_l = 2. \end{cases}$$

(4) *If $1 \in A$ and $1 \notin B$, then putting $\sigma(1) = l$, we have*

$$EE_L = \begin{cases} a(T_{b_1-1}^{*-1}T_{b_1-2}^{*-1}\cdots T_2^{*-1})(T_{b_2-1}^{*-1}T_{b_2-2}^{*-1}\cdots T_3^{*-1}) & \\ \quad\cdots(T_{b_l-1}^{*-1}T_{b_l-2}^{*-1}\cdots T_{l+1}^{*-1})E_{(A,R_l^*(B),\sigma(1,2,\dots,l))} & \text{if } b_l > 2, \\ aE_{(A,R_l^*(B),\sigma(1,2,\dots,l))} & \text{if } b_l = 2. \end{cases}$$

PROOF. (1) This is obvious.

(2) Since T_A does not involve T_1 and $\overline{T_B^*}$ does not involve T_1^{*-1} either, they commute with E. In addition, we note that $T_\sigma = (T_{l-1}T_{l-2}\cdots T_1)T_{(l,l-1,\dots 1)\sigma}$ for the l such that $\sigma(l)=1$ and $T_{(l,l-1,\dots 1)\sigma}$ does not involve T_1. So we have

$$\begin{aligned} EE_L &= T_A\overline{T_B^*}E(T_{l-1}T_{l-2}\cdots T_1)T_{(l,l-1,\dots 1)\sigma}ET_1T_1^{*-1}E\cdots \\ &= T_A\overline{T_B^*}(T_{l-1}T_{l-2}\cdots T_2)ET_1ET_{(l,l-1,\dots 1)\sigma}T_1T_1^{*-1}E\cdots \\ &= a^{-1}T_A\overline{T_B^*}(T_{l-1}T_{l-2}\cdots T_2)T_{(l,l-1,\dots 1)\sigma}ET_1T_1^{*-1}E\cdots \\ &= a^{-1}T_A(T_{l-1}T_{l-2}\cdots T_2)\overline{T_B^*}T_{(l,l-1,\dots 1)\sigma}E_k. \end{aligned}$$

Here we have

$$\begin{aligned} &T_A(T_{l-1}T_{l-2}\cdots T_2) \\ &\quad= \{(T_{a_p-1}\cdots T_p)\cdots(T_{a_l-1}\cdots T_l) \\ &\qquad\cdot(T_{a_{l+1}-1}\cdots T_{l+1})\cdots(T_{a_k-1}\cdots T_k)\}(T_{l-1}T_{l-2}\cdots T_2)\quad (a_p>p>1) \\ &\quad= (T_{a_p-1}\cdots T_p)\cdots(T_{a_l-1}\cdots T_l)(T_{l-1}T_{l-2}\cdots T_2) \\ &\qquad\cdot(T_{a_{l+1}-1}\cdots T_{l+1})\cdots(T_{a_k-1}\cdots T_k). \end{aligned}$$

Since for $a_l-1>j\geq 2$

$$T_j(T_{a_l-1}\cdots T_lT_{l-1}T_{l-2}\cdots T_2) = (T_{a_l-1}\cdots T_lT_{l-1}T_{l-2}\cdots T_2)T_{j+1},$$

we obtain

$$T_A(T_{l-1}T_{l-2}\cdots T_2) = (T_{a_l-1}T_{a_l-2}\cdots T_2)T_{R_l(A)}.$$

Hence we obtain the formula.

(3) In this case T_A involves T_1. So we have

$$\begin{aligned} EE_L &= \overline{T_B^*}E(T_{a_1-1}\cdots T_2)(T_1)\cdots(T_{a_k-1}\cdots T_{k+1})(T_k)(T_{l-1}T_{l-2}\cdots T_1)T_{(l,l-1,\dots 1)\sigma}E_k \\ &= \overline{T_B^*}(T_{a_1-1}\cdots T_2)\cdots(T_{a_k-1}\cdots T_{k+1})E(T_1\cdots T_k)(T_{l-1}T_{l-2}\cdots T_1)T_{(l,l-1,\dots 1)\sigma}E_k \\ &= \overline{T_B^*}(T_{a_1-1}\cdots T_2)\cdots(T_{a_k-1}\cdots T_{k+1})E(T_lT_{l-1}\cdots T_2)(T_1\cdots T_k)T_{(l,l-1,\dots 1)\sigma}E_k \\ &= a^{-1}\overline{T_B^*}(T_{a_1-1}\cdots T_2)\cdots(T_{a_k-1}\cdots T_{k+1})(T_lT_{l-1}\cdots T_2)(T_2\cdots T_k)T_{(l,l-1,\dots 1)\sigma}E_k. \end{aligned}$$

Here we have

$$
\begin{aligned}
&(T_{a_1-1}\cdots T_2)\cdots(T_{a_k-1}\cdots T_{k+1})(T_lT_{l-1}\cdots T_2)(T_2\cdots T_k)\\
&\quad=(T_{a_1-1}\cdots T_2)\cdots(T_{a_{l-1}-1}\cdots T_l)(T_{a_l-1}\cdots T_{l+1})(T_lT_{l-1}\cdots T_2)\\
&\qquad\cdot(T_{a_{l+1}-1}\cdots T_{l+2})\cdots(T_{a_k-1}\cdots T_{k+1})(T_2\cdots T_k)\\
&\quad=(T_{a_l-1}\cdots T_{l+1}T_lT_{l-1}\cdots T_2)(T_{a_1}\cdots T_3)\cdots(T_{a_{l-1}}\cdots T_{l+1})\\
&\qquad\cdot(T_{a_{l+1}-1}\cdots T_{l+2})\cdots(T_{a_k-1}\cdots T_{k+1})(T_2\cdots T_k)\\
&\quad=(T_{a_l-1}\cdots T_{l+1}T_lT_{l-1}\cdots T_2)(T_{a_1}\cdots T_3T_2)\cdots(T_{a_{l-1}}\cdots T_{l+1}T_l)\\
&\qquad\cdot(T_{a_{l+1}-1}\cdots T_{l+2}T_{l+1})\cdots(T_{a_k-1}\cdots T_{k+1}T_k)\\
&\quad=(T_{a_l-1}\cdots T_{l+1}T_lT_{l-1}\cdots T_2)T_{R_l(A)}.
\end{aligned}
$$

Hence we obtain the formula.

(4) We note that $T^*_{\sigma^{-1}}=(T^*_{l-1}T^*_{l-2}\cdots T^*_1)T^*_{(l,l-1,\dots,1)\sigma^{-1}}$ and $T^*_{(l,l-1,\cdots,1)\sigma^{-1}}$ does not involve T^*_1. Hence we have

$$
\begin{aligned}
ET_A\overline{T^*_B}T_\sigma E_k &= T_AE\overline{T^*_B}T^*_{\sigma^{-1}}E_k\\
&= T_AE\overline{T^*_B}(T^*_{l-1}T^*_{l-2}\cdots T^*_1)T^*_{(l,l-1,\dots,1)\sigma^{-1}}ET_1T_1^{*-1}E\cdots\\
&= T_AE\overline{T^*_B}(T^*_{l-1}T^*_{l-2}\cdots T^*_1)ET^*_{(l,l-1,\dots,1)\sigma^{-1}}T_1T_1^{*-1}E\cdots.
\end{aligned}
$$

Here we have

$$
\begin{aligned}
&E\overline{T^*_B}(T^*_{l-1}T^*_{l-2}\cdots T^*_1)E\\
&\quad=E(T^{*-1}_{b_1-1}\cdots T_2^{*-1})(T_1^{*-1})(T^{*-1}_{b_2-1}\cdots T_3^{*-1})(T_2^{*-1})\cdots(T^{*-1}_{b_l-1}\cdots T^{*-1}_{l+1})(T_l^{*-1})\\
&\qquad\cdot(T^{*-1}_{b_{l+1}-1}\cdots T^{*-1}_{l+1})\cdots(T^{*-1}_{b_k-1}\cdots T_k^{*-1})(T^*_{l-1}T^*_{l-2}\cdots T^*_1)E\\
&\quad=E(T^{*-1}_{b_1-1}\cdots T_2^{*-1})(T^{*-1}_{b_2-1}\cdots T_3^{*-1})\cdots(T^{*-1}_{b_l-1}\cdots T^{*-1}_{l+1})\\
&\qquad\cdot(T_1^{*-1}T_2^{*-1}\cdots T_l^{*-1})(T^*_{l-1}T^*_{l-2}\cdots T^*_1)(T^{*-1}_{b_{l+1}-1}\cdots T^{*-1}_{l+1})\cdots(T^{*-1}_{b_k-1}\cdots T_k^{*-1})E\\
&\quad=E(T^{*-1}_{b_1-1}\cdots T_2^{*-1})(T^{*-1}_{b_2-1}\cdots T_3^{*-1})\cdots(T^{*-1}_{b_l-1}\cdots T^{*-1}_{l+1})\\
&\qquad\cdot(T^*_lT^*_{l-1}\cdots T^*_2)(T_1^{*-1}T_2^{*-1}\cdots T_l^{*-1})(T^{*-1}_{b_{l+1}-1}\cdots T^{*-1}_{l+1})\cdots(T^{*-1}_{b_k-1}\cdots T_k^{*-1})E\\
&\quad=a(T^{*-1}_{b_1-1}\cdots T_2^{*-1})(T^{*-1}_{b_2-1}\cdots T_3^{*-1})\cdots(T^{*-1}_{b_l-1}\cdots T^{*-1}_{l+1})\\
&\qquad\cdot(T^*_lT^*_{l-1}\cdots T^*_2)(T_2^{*-1}\cdots T_l^{*-1})(T^{*-1}_{b_{l+1}-1}\cdots T^{*-1}_{l+1})\cdots(T^{*-1}_{b_k-1}\cdots T_k^{*-1})E
\end{aligned}
$$

$$= a(T^{*-1}_{b_1-1}\cdots T^{*-1}_2)(T^{*-1}_{b_2-1}\cdots T^{*-1}_3)\cdots(T^{*-1}_{b_l-1}\cdots T^{*-1}_{l+1})$$

$$\cdot(T^{*-1}_{b_{l+1}-1}\cdots T^{*-1}_{l+1})\cdots(T^{*-1}_{b_k-1}\cdots T^{*-1}_k)E$$

$$= a(T^{*-1}_{b_1-1}\cdots T^{*-1}_2)(T^{*-1}_{b_2-1}\cdots T^{*-1}_3)\cdots(T^{*-1}_{b_l-1}\cdots T^{*-1}_{l+1})\overline{T_{R^*_l(B)}}E.$$

Hence we obtain the formula.

REMARK 2.11. If $1\notin A$ and $1\notin B$, then T_A involves T_1 and $\overline{T^*_B}$ involves T^{*-1}_1. This case will be treated in Lemma 3.1.

REMARK 2.12. Let $L'=(A',B',\sigma')$ be a k-contract set and let E_k be the k-trivial contraction. If we define the *right k-contraction* by

$$E^{\mathrm{op}}_{L'}=E_kT_{\sigma'}T^{\mathrm{op}}_A\overline{T^{*\mathrm{op}}_B}$$

and write $T_{\sigma'}$ in suitable form, then we have the similar formulas for $E^{\mathrm{op}}_{L'}$ corresponding to Lemma 2.10.

3. Irreducible representations of $H_{m,n}(a,q)$

As we mentioned before, the main purpose of this paper is to construct irreducible representations of $H^r_{m,n}(q)$ so that they keep the integrality. To make use of the results in Section 1, we take the field of rational functions $\boldsymbol{Q}(a,q)$ (resp. $\boldsymbol{Q}(q)$) for the underlying field of $H_{m,n}(a,q)$ (resp. $H^r_{m,n}(q)$). First we define two sided ideals $\mathscr{H}_k$ of $H_{m,n}(a,q)$. Then we define irreducible representations of $H_{m,n}(a,q)$ by taking quotients of $\mathscr{H}_k$. If we define $H^r_{m,n}(q)$ by replacing one of the parameter a with q^{-r} in the definition of $H_{m,n}(a,q)$ and define the corresponding quotients, we obtain the desired irreducible representations of $H^r_{m,n}(q)$. These representations are also irreducible in case $r\geq m+n$. Similar arguments are also valid for the $\boldsymbol{Q}$-algebra $H_{m,n}(a_0,q_0)$ and $H^r_{m,n}(q_0)$, if $q_0,a_0\in\boldsymbol{Q}\backslash\{0\}$ satisfy the conditions in Remark 2.4.

In the following, we identify $\mathrm{alg}\{T_{m-1},T_{m-2},\ldots T_{k+1}\}$ with the Hecke algebra $H_{m-k}(q)$. The isomorphism is given by $T_i\mapsto T_{m-i}$. It follows from Section 1 that we can define the basis $\{C_x\}_{x\in W_1}$ of $\mathrm{alg}\{T_{m-1},T_{m-2},\cdots T_{k+1}\}$, where $W_1=S_{m-k}$. Similarly, we identify $\mathrm{alg}\{T^*_{n-1},T^*_{n-2},\ldots T^*_{k+1}\}$ with the Hecke algebra $H_{n-k}(q)$ by $T_i\mapsto T^*_{n-i}$. The basis is given by $\{C^*_w\}_{w\in W_2}$, where $W_2=S_{n-k}$. (We added asterisks to indicate that they are in $\mathrm{alg}\{T^*_{n-1},T^*_{n-2},\ldots T^*_k\}$.)

Let $W_1=S_{m-k}$, $W_2=S_{n-k}$ be symmetric groups and let $L=(A,B,\sigma)$ and $L'=(A',B',\sigma')$ be a pair of k-contract sets. ($k\leq\min(m,n)$.) Let $\tilde{\mathscr{H}}_k$ be the vector space over $\boldsymbol{Q}(a,q)$ spanned by

$$C_k = \{C_{(L,L',x,y)} = T_A \overline{T_B^*} T_\sigma E_k T_{\sigma'^{-1}} C_x C_y^* \overline{T_{B'}^{*\mathrm{op}}} T_{A'}^{\mathrm{op}} \quad | x \in W_1, \quad y \in W_2, \quad L, L' : k\text{-contract set}\}.$$

We note that $\tilde{\mathscr{H}}_k$ is also spanned by

$$B_k = \{T_{(L,L',x,y)} = T_A \overline{T_B^*} T_\sigma E_k T_{\sigma'^{-1}} T_x T_y^* \overline{T_{B'}^{*\mathrm{op}}} T_{A'}^{\mathrm{op}} \quad | x \in W_1, \quad y \in W_2, \quad L, L' : k\text{-contract set}\}.$$

Let R be the ring of polynomials $\boldsymbol{Z}[q, q^{-1}, a, a^{-1}, (a - a^{-1})/(q - q^{-1})]$ over the rational integers $\boldsymbol{Z}$. We denote $\tilde{\mathscr{H}}_{R,k}$ to be the R span of the elements of B_k. Take a k-contract set $L_0 = (A_0, B_0, \sigma_0)$. If we fix the index L' to be L_0 in the above definitions, then we have the subspace $\tilde{\mathscr{H}}_{g,k}(L_0)$ of $\tilde{\mathscr{H}}_k$ spanned by

$$C_{g,k}(L_0) = \{C_{(L,L_0,x,y)} = T_A \overline{T_B^*} T_\sigma E_k T_{\sigma_0^{-1}} C_x C_y^* \overline{T_{B_0}^{*\mathrm{op}}} T_{A_0}^{\mathrm{op}} \quad | x \in W_1, \quad y \in W_2, \quad L : k\text{-contract set}\},$$

which is also spanned by

$$B_{g,k}(L_0) = \{T_{(L,L_0,x,y)} = T_A \overline{T_B^*} T_\sigma E_k T_{\sigma_0^{-1}} T_x T_y^* \overline{T_{B_0}^{*\mathrm{op}}} T_{A_0}^{\mathrm{op}} \quad | x \in W_1, \quad w \in W_2, \quad L : k\text{-contract set}\}.$$

Note that T_x and T_y^* both commute with T_σ, E_k and $T_{\sigma'}$ in the definition of B_k. Hence, C_x and C_y^* both commute with T_σ, E_k and $T_{\sigma'}$.

We denote $\tilde{\mathscr{H}}_{R,g,k}(L_0)$ to be the R span of the elements of $B_{y,k}(L_0)$ as before. By Lemma 2.5 and Lemma 2.7, we find T_i and T_j^* act on $\tilde{\mathscr{H}}_{g,k}(L_0)$ and $\tilde{\mathscr{H}}_{R,g,k}(L_0)$ from the left. The following lemma shows that we can construct left $H_{m,n}(a,q)$-modules.

LEMMA 3.1. *Take $T_{(L,L_0,x,y)} \in B_{g,k}(L_0)$. Then $ET_{(L,L_0,x,y)}$ is in $\tilde{\mathscr{H}}_{g,k}(L_0)$ or in $\tilde{\mathscr{H}}_{k+1}$.*

PROOF. If A or B involves 1, then $ET_{(L,L_0,x,y)} \in \tilde{\mathscr{H}}_{g,k}(L_0)$ by Lemma 2.10, Lemma 2.5 and Lemma 2.7. In these cases, T_A does not involve T_1 or $\overline{T_B^*}$ does not involve T_1^{*-1}. We assume that $1 \notin A$ and $1 \notin B$. In this case T_A involves T_1

and $\overline{T_B^*}$ involves T_1^{*-1} as we mentioned in Remark 2.11. Then we have

$$\begin{aligned} ET_{(L,L_0,x,y)} &= ET_A\overline{T_B^*}T_\sigma E_k T_{\sigma_0^{-1}}T_xT_y^*\overline{T_{B_0}^{*\mathrm{op}}}T_{A_0}^{\mathrm{op}} \\ &= T_{A^+}\overline{T_{B^+}^*}E(T_1T_2\cdots T_k)(T_1^{*-1}T_2^{*-1}\cdots T_k^{*-1})T_\sigma E_k T_{\sigma_0^{-1}}T_xT_y^*\overline{T_{B_0}^{*\mathrm{op}}}T_{A_0}^{\mathrm{op}} \\ &= T_{A^+}\overline{T_{B^+}^*}E(T_1T_2\cdots T_k)T_\sigma(T_1^{*-1}T_2^{*-1}\cdots T_k^{*-1})E_k T_{\sigma_0^{-1}}T_xT_y^*\overline{T_{B_0}^{*\mathrm{op}}}T_{A_0}^{\mathrm{op}} \\ &= T_{A^+}\overline{T_{B^+}^*}ET_{\sigma^+}(T_1T_2\cdots T_k)(T_1^{*-1}T_2^{*-1}\cdots T_k^{*-1})E_k T_{\sigma_0^{-1}}T_xT_y^*\overline{T_{B_0}^{*\mathrm{op}}}T_{A_0}^{\mathrm{op}} \\ &= T_{A^+}\overline{T_{B^+}^*}T_{\sigma^+}E(T_1T_2\cdots T_k)(T_1^{*-1}T_2^{*-1}\cdots T_k^{*-1})E_k T_{\sigma_0^{-1}}T_xT_y^*\overline{T_{B_0}^{*\mathrm{op}}}T_{A_0}^{\mathrm{op}} \\ &= T_{A^+}\overline{T_{B^+}^*}T_{\sigma^+}E_{k+1}(T_{\sigma_0^{-1}}T_xT_y^*\overline{T_{B_0}^{*\mathrm{op}}}T_{A_0}^{\mathrm{op}}), \end{aligned}$$

where $\sigma^+ = (k+1,k,\cdots,1)\sigma(1,2,\cdots,k+1) \in S_{k+1}$, $A^+ = A \cup \{1\}$ and $B^+ = B \cup \{1\}$. Since the triple (A^+, B^+, σ^+) makes a $(k+1)$-contract set, and by Lemma 2.5, 2.6, 2.7, 2.8, $\tilde{\mathscr{H}}_k$ coincides with the $(H_m(q)\otimes H_n(q), H_m(q)\otimes H_n(q))$-bimodule generated by E_k, the last term is in $\tilde{\mathscr{H}}_{k+1}$.

By Lemma 2.5 and Lemma 2.7 and the previous lemma we have the following proposition.

Proposition 3.2. *For k $(0 \le k \le \min(m,n))$, let*

$$\mathscr{H}_{g,k}(L_0) = \tilde{\mathscr{H}}_{g,k}(L_0) + \tilde{\mathscr{H}}_{k+1} + \tilde{\mathscr{H}}_{k+2} + \cdots + \tilde{\mathscr{H}}_{\min(m,n)}.$$

Then $\mathscr{H}_{g,k}(L_0)$ is a left ideal of $H_{m,n}(a,q)$.

If we fix a left k-contract set L to be L_0 instead of L' in the definition of B_k and C_k, then we have a subspace $\tilde{\mathscr{H}}_{d,k}(L_0)$ of $\tilde{\mathscr{H}}_k$. Hence, we have the following proposition.

Proposition 3.3. *For k $(0 \le k \le \min(m,n))$, let*

$$\mathscr{H}_{d,k}(L_0) = \tilde{\mathscr{H}}_{d,k}(L_0) + \tilde{\mathscr{H}}_{k+1} + \tilde{\mathscr{H}}_{k+2} + \cdots + \tilde{\mathscr{H}}_{\min(m,n)}.$$

Then $\mathscr{H}_{d,k}$ is a right ideal of $H_{m,n}(a,q)$.

If we denote $H_{m,n}(R)$ to be the algebra over R defined by the generators and relations in Definition 2.1, then the R-linear combination $\mathscr{H}_{R,g,k}(L_0)$ (resp. $\mathscr{H}_{R,d,k}(L_0)$) of $\tilde{\mathscr{H}}_{R,g,k}(L_0)$ (resp. $\tilde{\mathscr{H}}_{R,d,k}(L_0)$), $\tilde{\mathscr{H}}_{R,k+1}, \tilde{\mathscr{H}}_{R,k+2}, \cdots, \tilde{\mathscr{H}}_{R,\min(m,n)}$ is a left (resp. right) ideal of $H_{m,n}(R)$.

Since $\tilde{\mathscr{H}}_k = \sum_L \tilde{\mathscr{H}}_{d,k}(L) = \sum_{L'} \tilde{\mathscr{H}}_{g,k}(L')$, we have further the following proposition.

Proposition 3.4. *For k $(0 \leq k \leq \min(m,n))$, let*

$$\mathscr{H}_k = \tilde{\mathscr{H}}_k + \tilde{\mathscr{H}}_{k+1} + \cdots + \tilde{\mathscr{H}}_{\min(m,n)}.$$

Then $\mathscr{H}_k$ is a two sided ideal of $H_{m,n}(a,q)$.

Corollary 3.5.

$$\mathscr{H}_0 = H_{m,n}(a,q).$$

Proof. This follows from $1 \in \mathscr{H}_0$ and the previous proposition.

Similarly, if we define $\mathscr{H}_{R,k}$ to be the R-linear combination of $\tilde{\mathscr{H}}_{R,k}$, $\tilde{\mathscr{H}}_{R,k+1}, \ldots, \tilde{\mathscr{H}}_{R,\min(m,n)}$, then we find $\mathscr{H}_{R,k}$ is a two sided ideal of $H_{m,n}(R)$ and $\mathscr{H}_{R,0} = H_{m,n}(R)$.

We can see $\bigsqcup_{k=0}^{\min(m,n)} B_k$ forms a basis of $H_{m,n}(a,q)$ as follows. Let B_k be the one just defined. Then

$$\sum_{k=0}^{\min(m,n)} |B_k| = \sum_{k=0}^{\min(m,n)} \binom{m}{k}^2 \binom{n}{k}^2 (k!)^2 (m-k)!(n-k)!,$$

which is equal to $(m+n)!$. (See Lemma 1.7 in [8].) Hence $\dim H_{m,n}(a,q) \leq (m+n)!$. On the other hand we already know that $\dim H_{m,n}(a,q) = (m+n)!$ (Theorem 2.3). Since the above corollary implies $\bigsqcup_{k=0}^{\min(m,n)} B_k$ generates $H_{m,n}(a,q)$ as vector space, we find $\bigsqcup_{k=0}^{\min(m,n)} B_k$ forms a basis of $H_{m,n}(a,q)$. Similarly we can see $\bigsqcup_{k=0}^{\min(m,n)} C_k$ forms a basis of $H_{m,n}(a,q)$.

Let $J_k = \mathscr{H}_k / \mathscr{H}_{k+1}$ $(k = 0, 1, 2, \cdots, \min(m,n) - 1)$ be quotient modules and $J_{\min(m,n)} = \mathscr{H}_{\min(m,n)}$. Note that the modules $\mathscr{H}_{R,k}/\mathscr{H}_{R,k+1}$ are R-free and the same holds for all modules constructed below. This fact will be used in the proof of Theorem 3.9. Since we already know $H_{m,n}(a,q)$ is semisimple, the canonical projection $\mathscr{H}_k \mapsto J_k$ splits. Similarly, we define $J_{g,k}(L_0) = \mathscr{H}_{g,k}(L_0)/\mathscr{H}_{k+1}$ (resp. $J_{d,k}(L_0) = \mathscr{H}_{d,k}(L_0)/\mathscr{H}_{k+1}$) $(k = 0, 1, 2, \ldots, \min(m,n) - 1)$ and $J_{g,\min(m,n)}(L_0) = \mathscr{H}_{g,\min(m,n)}(L_0)$ (resp. $J_{d,\min(m,n)}(L_0) = \mathscr{H}_{d,\min(m,n)}(L_0)$). Since it is easily checked that the left (resp. right) module structures of $J_{g,k}(L_0)$ (resp. $J_{d,k}(L_0)$) do not depend on the choice of L_0, we write $J_{g,k} = J_{g,k}(L_0)$ (resp. $J_{d,k} = J_{d,k}(L_0)$).

Although the quotients $J_{g,k}$ $(k = 0, 1, 2, \ldots, \min(m,n))$ define the representations of $H_{m,n}(a,q)$, they are still very large modules. So we divide them into smaller submodules or subquotients. Let $L=(A,B,\sigma)$ be a k-contract set and let

$$[L,x,y]=T_A\overline{T_B^*}T_\sigma E_k C_x C_y^* + \mathscr{H}_{k+1}$$

be a representative of $J_{g,k}$. We consider a subspace $J_{g,k}(I_1,I_2)$ of $J_{g,k}$ spanned by

$$\{[L,x,y]\in J_{g,k}\,|\,C_x\in I_1,\, C_y^*\in I_2\},$$

where I_1 and I_2 are left ideals of $H_{m-k}(q)$ and $H_{n-k}(q)$ respectively. By Lemma 2.5, Lemma 2.7 and Lemma 2.10, we find $J_{g,k}(I_1,I_2)$ is a left $H_{m,n}(a,q)$-module.

Let $\mathscr{I}_\lambda$ and $\mathscr{I}_\mu$ be ideals of $H_{m-k}(q)$ and $H_{n-k}(q)$ indexed by left cells $\lambda\subset W_1$ and $\mu\subset W_2$. Let $\mathscr{I}_\lambda'$ and $\mathscr{I}_\mu'$ be the maximal ideals. (Recall the definitions in Section 1.) We shall say the following theorem.

THEOREM 3.6. *Let $\mathscr{I}_\lambda\supset\mathscr{I}_\lambda'$ and $\mathscr{I}_\mu\supset\mathscr{I}_\mu'$ be as above. Let*

$$J_{g,k}(\lambda,\mu)=J_{g,k}(\mathscr{I}_\lambda,\mathscr{I}_\mu)/[J_{g,k}(\mathscr{I}_\lambda,\mathscr{I}_\mu')+J_{g,k}(\mathscr{I}_\lambda',\mathscr{I}_\mu)].$$

Then $J_{g,k}(\lambda,\mu)$ is an irreducible $H_{m,n}(a,q)$-module.

Before proving the above theorem, we prove the following lemma.

LEMMA 3.7. *If we take $0\neq\bar{v}\in J_{g,k}(\lambda,\mu)$ then there exists a right k-contraction E_L^{op} such that $E_L^{\mathrm{op}}\bar{v}\neq 0$.*

PROOF. There exists a $v\in J_{g,k}(\mathscr{I}_\lambda,\mathscr{I}_\mu)$ such that $\bar{v}$ (natural surjection of v) $\in J_{g,k}(\lambda,\mu)$. Note that

$$v\in J_{g,k}(\mathscr{I}_\lambda,\mathscr{I}_\mu)\subset J_{g,k}\subset J_k=\mathscr{H}_k/\mathscr{H}_{k+1}$$

and hence $\mathscr{H}_{k+1}\bar{v}=0$. If we have $E_L^{\mathrm{op}}\bar{v}=0$ for all right k-contractions, then $\tilde{\mathscr{H}}_k\bar{v}=0$. Hence $\mathscr{H}_k\bar{v}=0$. Since $H_{m,n}(a,q)$ is semisimple, $\mathscr{H}_k$ and $\mathscr{H}_{k+1}$ are direct sums of matrix algebras, and hence $\bar{v}\in\mathscr{H}_k/\mathscr{H}_{k+1}$ and $\mathscr{H}_k\bar{v}=0$ imply $\bar{v}=0$. (Note that there is the canonical projection in $\mathscr{H}_k$.) This contradicts $\bar{v}\neq 0$.

PROOF OF THE THEOREM. Suppose $0\neq\bar{v}\in J_{g,k}(\lambda,\mu)$. We claim that $H_{m,n}(a,q)\bar{v}=J_{g,k}(\lambda,\mu)$. For a $v\in J_{g,k}\subset J_k$ such that $\bar{v}\in J_{g,k}(\lambda,\mu)$, we can write

$$v=\sum a_{L,x,y}[L,x,y],$$

where $a_{L,x,y}\in\boldsymbol{Q}(a,q)$. By the above lemma, there exists a right k-contraction $E_{L_1}^{\mathrm{op}}$

so that $E_{L_1}^{\mathrm{op}}\bar{v} \neq 0$. Then we can write

$$0 \neq E_{L_1}^{\mathrm{op}} v = \sum \tilde{a}_{L,x,y}[L,x,y].$$

Recall that $E_{L_1}^{\mathrm{op}} H_{m,n}(a,q)$ is contained in the span of $\{E_L^{\mathrm{op}}\}$, and $\{T_A \overline{T_B^*} T_\sigma T_x T_y^* E_L^{\mathrm{op}}\}$ is a basis of $\tilde{\mathscr{H}}_k$. Hence $\tilde{a}_{L,x,y} = 0$, unless $E_L = E_k$. Since $\mathscr{I}_\lambda/\mathscr{I}_\lambda'$ and $\mathscr{I}_\mu/\mathscr{I}_\mu'$ are irreducible, we have that $H_{m,n}(a,q)\bar{v}$ contains the span of $\{E_k C_x C_y^*\}$. By multiplying $T_A \overline{T_B^*} T_\sigma$ for various $L = (A,B,\sigma)$, we find $H_{m,n}(a,q)\bar{v} = J_{g,k}(\lambda,\mu)$.

Next, we prove that $J_{g,k}(\lambda,\mu)$ and $J_{g,k}(\lambda',\mu')$ are non-isomorphic for the distinct pairs (λ,μ) and (λ',μ'). Let $\Lambda_{m,n}^k$ be a set of pairs of partitions defined by

$$\Lambda_{m,n}^k = \{(\lambda,\mu) \mid \lambda \in P(m-k), \mu \in P(n-k)\}.$$

THEOREM 3.8. *Suppose $(\lambda,\mu) \in \Lambda_{m,n}^k$ and $(\lambda',\mu') \in \Lambda_{m,n}^{k'}$ are pairs of partitions for k, $k' \in \{0,1,2,\cdots,\min(m,n)\}$. Then $J_{g,k}(\lambda,\mu) \cong J_{g,k'}(\lambda',\mu')$ as $H_{m,n}(a,q)$-modules if and only if $\lambda = \lambda', \mu = \mu'$ and $k = k'$.*

PROOF. Assume that $J_{g,k}(\lambda,\mu) \cong J_{g,k'}(\lambda',\mu')$. Let $\phi\colon J_{g,k}(\lambda,\mu) \mapsto J_{g,k'}(\lambda',\mu')$ be an $H_{m,n}(a,q)$-module isomorphism. Suppose that $k' \neq k$. Without loss of generality, we can assume that $k < k'$. Then by the definition of $J_{g,k}(\lambda,\mu)$, we have

$$0 = \phi(E_{L'}^{\mathrm{op}} J_{g,k}(\lambda,\mu)) = E_{L'}^{\mathrm{op}} \phi(J_{g,k}(\lambda,\mu)) = E_{L'}^{\mathrm{op}} J_{g,k'}(\lambda',\mu'),$$

for any k'-contract set L'. By Lemma 3.7, however, there is a k'-contract set L_1 such that $E_{L_1}^{\mathrm{op}} J_{g,k'}(\lambda',\mu') \neq 0$. This gives a contradiction.

Thus, we can reduce to the case where $k' = k \geq 1$. Let $p_\lambda, p_{\lambda'}$ (resp. $p_\mu^*, p_{\mu'}^*$) be the central idempotents in $H_{m-k}(q)$ (resp. $H_{n-k}^*(q)$) corresponding to the irreducible modules $\mathscr{I}_\lambda/\mathscr{I}_\lambda'$ and $\mathscr{I}_{\lambda'}/\mathscr{I}_{\lambda'}'$ (resp. $\mathscr{I}_\mu^*/\mathscr{I}_\mu^{*\prime}$ and $\mathscr{I}_{\mu'}^*/\mathscr{I}_{\mu'}^{*\prime}$). If $(\lambda,\mu) \neq (\lambda',\mu')$, then $p_{\lambda'} p_{\mu'}^* p_\lambda p_\mu^* = 0$. We regard these central idempotents as elements of $H_{m,n}(a,q)$. We note that these elements still commute with the trivial k-contraction E_k. Since we have proved that $E_L^{\mathrm{op}} J_{g,k}(\lambda,\mu)$ is the span of $\{E_i C_x C_y^*\}$ in the proof of Theorem 3.6, we find

$$p_\lambda p_\mu^* E_L^{\mathrm{op}} J_{g,k}(\lambda,\mu) = E_L^{\mathrm{op}} J_{g,k}(\lambda,\mu)$$

and

$$p_{\lambda'} p_{\mu'}^* E_L^{\mathrm{op}} J_{g,k}(\lambda',\mu') = E_L^{\mathrm{op}} J_{g,k}(\lambda',\mu').$$

Hence we have

$$p_{\lambda'} p_{\mu'}^* E_L^{\mathrm{op}} J_{g,k}(\lambda,\mu) = p_{\lambda'} p_{\mu'}^* p_\lambda p_\mu^* E_L^{\mathrm{op}} J_{g,k}(\lambda,\mu) = 0.$$

Hence

$$\begin{aligned} 0 &= \phi(p_{\lambda'}p^*_{\mu'}E^{\mathrm{op}}_L J_{g,k}(\lambda,\mu)) \\ &= p_{\lambda'}p^*_{\mu'}E^{\mathrm{op}}_L \phi(J_{g,k}(\lambda,\mu)) \\ &= p_{\lambda'}p^*_{\mu'}E^{\mathrm{op}}_L J_{g,k}(\lambda',\mu') \\ &= E^{\mathrm{op}}_L J_{g,k}(\lambda',\mu'). \end{aligned}$$

Again by Lemma 3.7 we have a contradiction. So we have $\lambda = \lambda'$ and $\mu = \mu'$.

Let f^λ and f^μ be the dimensions of the irreducible characters χ^λ and χ^μ of the symmetric group S_{m-k} and S_{n-k} respectively. Then the degree of the representation $J_{g,k}(\lambda,\mu)$ is

$$\binom{m}{k}\binom{n}{k}(k!)f^\lambda f^\mu.$$

From this, we obtain the following conclusion.

THEOREM 3.9. *The set* $\{J_{g,k}(\lambda,\mu) \mid (\lambda,\mu) \in \Lambda^k_{m,n}, k = 0,1,2,\cdots,\min(m,n)\}$ *is a complete set of representatives for the isomorphism classes of irreducible modules of* $H_{m,n}(a,q)$. *Moreover the generators of* $H_{m,n}(a,q)$ *in Definition 2.1 will be mapped to the matrices over* $R = \boldsymbol{Z}[q,q^{-1},a,a^{-1},(a-a^{-1})/(q-q^{-1})]$ *by these modules.*

PROOF. The first statement follows from the fact that

$$\sum_{\lambda,\mu}\sum_{k=0}^{\min(m,n)}\binom{m}{k}^2\binom{n}{k}^2(k!)^2(f^\lambda)^2(f^\mu)^2 = (m+n)! = \dim H_{m,n}(a,q).$$

See (5.4) in [1] for details. By the comments below Proposition 3.3, $\tilde{\mathscr{H}}_{R,g,k}(L_0)$ is a left ideal of $H_{m,n}(R)$. If we define R-modules $J_{R,g,k}(\lambda,\mu)$ in the course of our construction of $J_{g,k}(\lambda,\mu)$, then they are $H_{m,n}(R)$-modules. This proves the second statement.

We finally obtain the following theorem.

THEOREM 3.10. *If we construct* $J_{g,k}(\lambda,\mu)$ *as* $H^r_{m,n}(q)$*-modules replacing one of the indeterminate* a *with* q^{-r} $(r \geq m+n)$ *in the course of the construction of* $H_{m,n}(a,q)$*-modules* $J_{g,k}(\lambda,\mu)$, *then the set* $\{J_{g,k}(\lambda,\mu)\}$ *is a complete set of representatives for the isomorphism classes of irreducible modules of* $H^r_{m,n}(q)$. *Moreover the generators of* $H^r_{m,n}(q)$ *in Definition 2.1 will be mapped to the matrices over* $\boldsymbol{Z}[q,q^{-1}]$ *by these modules.*

PROOF. First we note that even if we replace a with q^{-r} in Lemma 2.5–2.10, those identities are still valid for the $\boldsymbol{Q}(q)$-algebra $H_{m,n}^{r}(q)$. Similarly, Proposition 3.2, 3.3 and 3.4 hold for $H_{m,n}^{r}(q)$. Since we assume $r \geq m+n$, $H_{m,n}^{r}(q)$ is semisimple and its dimension is $(m+n)!$. So we can construct $\{J_{g,k}(\lambda,\mu)\}$ as $H_{m,n}^{r}(q)$-modules. Lemma 3.7 also holds for $H_{m,n}^{r}(q)$ since $H_{m,n}^{r}(q)$ is semisimple. Accordingly, even if we replace $H_{m,n}(a,q)$ with $H_{m,n}^{r}(q)$ in Theorem 3.6, 3.8 and 3.9, those theorems are still valid for $\boldsymbol{Q}(q)$-algebra $H_{m,n}^{r}(q)$ and the proof completes.

REMARK 3.11. As we mentioned in Remark 2.4, we can define the algebras $H_{m,n}^{r}(q_0)$ and $H_{m,n}(a_0,q_0)$ over $\boldsymbol{Q}$, taking special values $q_0, a_0 \in \boldsymbol{Q}\backslash\{0\}$. For these $\boldsymbol{Q}$-algebras, we can also construct $H_{m,n}^{r}(q_0)$-modules and $H_{m,n}(a_0,q_0)$-modules in the same way, In case $H_{m,n}^{r}(q_0)$ (resp. $H_{m,n}(a_0,q_0)$) is semisimple (see Remark 2.4), these modules are complete set of representatives for the isomorphism classes of irreducible modules of $H_{m,n}^{r}(q_0)$ (resp. $H_{m,n}(a_0,q_0)$). If the algebra is not semisimple, these modules are not necessarily irreducible nor mutually non-isomorphic.

References

[1] G. Benkart, M. Chakrabarti, T. Halverson, R. Leduc, C. Lee and J. Stroomer, Tensor product representations of general linear groups and their connections with Brauer algebras, J. Algebra. **166** (1994), 529–567.

[2] D. Barbasch and D. Vogan, Primitive ideals and orbital integrals in complex classical groups, Math. Ann. **259** (1982), 153–199.

[3] T. Halverson, Characters of the centralizer algebras of mixed tensor representations of $GL(r, \boldsymbol{C})$ and the quantum group $\mathcal{U}_q(gl(r, \mathbf{C}))$, preprint.

[4] P. de la Harpe, M. Kervaire and C. Weber, On the Jones polynomial, l'Enseignement math. **32** (1986), 271–335.

[5] J. E. Humphreys, Reflection Groups and Coxeter Groups, Cambridge University Press, Cambridge, 1992.

[6] D. Kazhdan and G. Lusztig, Representations of Coxeter groups and Hecke algebras, Invent. Math. **53** (1979), 165–184.

[7] D. E. Knuth, The art of computer programming, Addison-Wesley, 1973.

[8] M. Kosuda and J. Murakami, Centralizer algebras of the mixed tensor representations of quantum group $U_q(gl(n, \boldsymbol{C}))$, Osaka J. Math. **30** (1993), 475–507.

[9] J. Y. Shi, The Kazhdan-Lusztig Cells in Certain Affine Weyl Groups, Lect. Notes in Math. 1179, Springer-Verlag, Berlin, 1986.

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