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REALIZATION OF MAXIMAL SUBGROUPS OF RANK 8 OF THE SIMPLY CONNECTED COMPACT SIMPLE LIE GROUP OF TYPE E_8

By

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Introduction

Borel and de Siebenthal ([1]) classified the maximal subgroups of maximal rank of a simply connected compact simple Lie group G and showed that anyone of these subgroups could be realized as the fixed subgroup

$$G^{\sigma} = \{a \in G | \sigma(a) = a\}$$

of a certain automorphism σ of order p $(p = 2, 3, 5)$. The problem of realizing explicitly these automorphisms σ and subgroups G^{σ} is important. In case G is of classical type, this problem is very easy. In [10], [11] and [12], Yokota and some members of his school realized all σ and G^{σ} explicitly in cases G were of type G_2, F_4, E_6 and E_7. In case G is of type E_8, this problem has not been solved completely. In this case, it is known ([1]) that the type of these subgroups G^{σ} and the order of σ are as follows:

type of G^{σ}	:	$A_1 \times E_7$	D_8	A_8	$A_4 \times A_4$	$A_2 \times E_6$
order of σ	:	2	2	3	3	5

The subgroups of type $A_1 \times E_7$ and D_8 have already been realized explicitly in [3], [9] and [10]. One the other hand, Wolf and Gray ([8]) classified the automorphisms of order 3 of the simply connected compact simple Lie group of type E_8 and showed that the subgroups of type A_8 and $A_2 \times E_6$ were isomorphic to $SU(9)/\boldsymbol{Z}_3$ and $(SU(3) \times E_6)/\boldsymbol{Z}_3$, respectively. But the isomorphisms were not completely obtained. In this paper, we shall explicitly give two automorphisms of order 3 such that their fixed subgroups are isomorphic to $SU(9)/\boldsymbol{Z}_3$ and $(SU(3) \times E_6)/\boldsymbol{Z}_3$ respectively, and an automorphism of order 5 whose fixed

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subgroup is of type $A_4 \times A_4$. The last fixed subgroup is realized as $(SU(5) \times SU(5))/\boldsymbol{Z}_5$.

Finally we remark that three new realizations of the complex simple Lie algebra of type E_8 are obtained in this paper.

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§ 1. Preliminaries

1.1 Let $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ be the canonical basis of $\boldsymbol{C}^n$ and $(\boldsymbol{x}, \boldsymbol{y})$ the symmetric bilinear inner product in $\boldsymbol{C}^n$ defined by $(\boldsymbol{e}_i, \boldsymbol{e}_j) = \delta_{ij}$, where δ_{ij} means Kronecker's delta. Let us define a bilinear symmetric inner product in the k-th exterior power $\bigwedge^k(\boldsymbol{C}^n)$ $(0 \leq k \leq n)$ by

$$(\boldsymbol{x}_1 \wedge \cdots \wedge \boldsymbol{x}_k, \boldsymbol{y}_1 \wedge \cdots \wedge \boldsymbol{y}_k) = \det((\boldsymbol{x}_i, \boldsymbol{y}_j)), \quad k \geq 1,$$

$$(a, b) = ab, \qquad a, b \in \bigwedge^0(\boldsymbol{C}^n) = \boldsymbol{C}.$$

Then $\boldsymbol{e}_{i_1} \wedge \cdots \wedge \boldsymbol{e}_{i_k}$ $(i_1 < \cdots < i_k)$ forms an orthonormal basis of $\bigwedge^k(\boldsymbol{C}^n)$. For any $\boldsymbol{u} \in \bigwedge^k(\boldsymbol{C}^n)$, there exists the unique element $*(\boldsymbol{u}) \in \bigwedge^{n-k}(\boldsymbol{C}^n)$ such that

$$(1.1) \qquad (*(\boldsymbol{u}), \boldsymbol{v}) = (\boldsymbol{u} \wedge \boldsymbol{v}, \boldsymbol{e}_1 \wedge \cdots \wedge \boldsymbol{e}_n) \quad \text{for } \boldsymbol{v} \in \bigwedge^{n-k}(\boldsymbol{C}^n).$$

Then the linear transformation

$$* : \bigwedge^k(\boldsymbol{C}^n) \to \bigwedge^{n-k}(\boldsymbol{C}^n)$$

is bijective and it satisfies the following identity:

$$*^2(\boldsymbol{u}) = (-1)^{k(n-k)}\boldsymbol{u} \quad (\boldsymbol{u} \in \bigwedge^k(\boldsymbol{C}^n)).$$

Let ρ and $d\rho$ be the representations of the complex special linear group $SL(n, \boldsymbol{C})$ and its Lie algebra $\mathfrak{sl}(n, \boldsymbol{C})$ on $\bigwedge^k(\boldsymbol{C}^n)$ $(k \geq 1)$ defined by

$$\rho(A)(\boldsymbol{x}_1 \wedge \cdots \wedge \boldsymbol{x}_k) = A\boldsymbol{x}_1 \wedge \cdots \wedge A\boldsymbol{x}_k,$$

$$d\rho(X)(\boldsymbol{x}_1 \wedge \cdots \wedge \boldsymbol{x}_k) = \sum_{j=1}^{k} \boldsymbol{x}_1 \wedge \cdots \wedge X\boldsymbol{x}_j \wedge \cdots \wedge \boldsymbol{x}_k,$$

respectively. In particular, define the representations ρ of $SL(n, \boldsymbol{C})$ and $d\rho$ of $\mathfrak{sl}(n, \boldsymbol{C})$ on $\bigwedge^0(\boldsymbol{C}^n) = \boldsymbol{C}$ by

$$\rho(A)1 = 1, \quad d\rho(X)1 = 0.$$

Hereafter we shall omit the notations ρ and $d\rho$. We can easily obtain the following:

LEMMA 1.1. *For $A \in SL(n, \boldsymbol{C})$, $X \in \mathfrak{sl}(n, \boldsymbol{C})$ and $\boldsymbol{u}, \boldsymbol{v} \in \bigwedge^k(\boldsymbol{C}^n)$, we have*

(1) $$(A\boldsymbol{u}, {}^t A^{-1}\boldsymbol{v}) = (\boldsymbol{u}, \boldsymbol{v}),$$

(2) $$(X\boldsymbol{u}, \boldsymbol{v}) + (\boldsymbol{u}, -{}^t X\boldsymbol{v}) = 0,$$

(3) $$*(A\boldsymbol{u}) = {}^t A^{-1} * (\boldsymbol{u}),$$

(4) $$*(X\boldsymbol{u}) = -{}^t X * (\boldsymbol{u}).$$

For any $\boldsymbol{u}, \boldsymbol{v} \in \bigwedge^k(\boldsymbol{C}^n)$ $(1 \le k \le n)$, let us define a linear transformation $\boldsymbol{u} \times \boldsymbol{v}$ on $\boldsymbol{C}^n$ by

$$\boldsymbol{u} \times \boldsymbol{v} : x \mapsto *(\boldsymbol{v} \wedge *(\boldsymbol{u} \wedge x)) + (-1)^{n-k}\frac{n-k}{n}(\boldsymbol{u}, \boldsymbol{v})x \quad (x \in \boldsymbol{C}^n).$$

Since $\mathrm{tr}(\boldsymbol{u} \times \boldsymbol{v}) = 0$, $\boldsymbol{u} \times \boldsymbol{v}$ can be considered as an element of $\mathfrak{sl}(n, \boldsymbol{C})$ with respect to the canonical basis of $\boldsymbol{C}^n$. Let $^-$ denote the complex conjugation of $\bigwedge^k(\boldsymbol{C}^n)$ with respect to the real form $\bigwedge^k(\boldsymbol{R}^n)$ except in the §3. Furthermore we can easily obtain the following:

LEMMA 1.2. *For $A \in SL(n, \boldsymbol{C})$, $X \in \mathfrak{sl}(n, \boldsymbol{C})$ and $\boldsymbol{u}, \boldsymbol{v} \in \bigwedge^k(\boldsymbol{C}^n)$, we have*

(1) $$A(\boldsymbol{u} \times \boldsymbol{v})A^{-1} = (A\boldsymbol{u}) \times ({}^t A^{-1}\boldsymbol{v}),$$

(2) $$[X, \boldsymbol{u} \times \boldsymbol{v}] = (X\boldsymbol{u}) \times \boldsymbol{v} + \boldsymbol{u} \times (-{}^t X\boldsymbol{v}),$$

(3) $${}^t(\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{v} \times \boldsymbol{u},$$

(4) $$\overline{\boldsymbol{u} \times \boldsymbol{v}} = \bar{\boldsymbol{u}} \times \bar{\boldsymbol{v}},$$

(5) $$\mathrm{tr}\{X(\boldsymbol{u} \times \boldsymbol{v})\} = (-1)^{n-k}(X\boldsymbol{u}, \boldsymbol{v}).$$

1.2 Let $\mathfrak{g}$ be a complex simple Lie algebra of type E_8. Since $\mathfrak{g}$ is simple, the Lie algebra $\mathrm{Der}(\mathfrak{g})$ of all derivations of $\mathfrak{g}$ consists of $\mathrm{ad}(R)$ $(R \in \mathfrak{g})$ and it is isomorphic to the Lie algebra $\mathfrak{g}$. Let $\mathrm{Aut}(\mathfrak{g})$ be an automorphism group of $\mathfrak{g}$ and $\mathrm{Innaut}(\mathfrak{g})$ an inner automorphism group generated by $\{\exp(\mathrm{ad}\, R) | R \in \mathfrak{g}\}$. Since $\mathfrak{g}$ is of type E_8, the group $\mathrm{Aut}(\mathfrak{g})$ coincide the group $\mathrm{Innaut}(\mathfrak{g})$. Hence $\mathrm{Aut}(\mathfrak{g})$ is connected. Let $\mathfrak{g}'$ be a compact real form of $\mathfrak{g}$ and γ a conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}'$. Define an inner product on $\mathfrak{g}$ by

$$\langle R_1, R_2 \rangle = -B_{\mathfrak{g}}(R_1, \gamma R_2).$$

Then it is positive definite Hermitian inner product. Let us define a group E_8 as follows:

$$E_8 = \{\alpha \in \mathrm{Aut}(\mathfrak{g}) | \langle \alpha R_1, \alpha R_2 \rangle = \langle R_1, R_2 \rangle\}.$$

Since this group is a closed subgroup of the unitary group

$$U(\mathfrak{g}) = \{\alpha \in \mathrm{Iso}(\mathfrak{g}) | \langle \alpha R_1, \alpha R_2 \rangle = \langle R_1, R_2 \rangle\},$$

the group E_8 is compact. It is clear that the Lie algebra of this group is isomorphic to $\mathfrak{g}'$. Hence the group E_8 is of type E_8. In order to prove that the group E_8 is connected, we use the following:

LEMMA 1.3. ([2] p. 450). *Let G be an algebraic subgroup of the general linear group $GL(n, \boldsymbol{C})$ such that the condition $A \in G$ implies $A^* \in G$. Then G is homeomorphic to the topological product of $G \cap U(n)$ and a Euclidean space $\boldsymbol{R}^d$:*

$$G \simeq (G \cap U(n)) \times \boldsymbol{R}^d$$

where $U(n)$ is the unitary subgroup of $GL(n, \boldsymbol{C})$.

It is clear that the group $\mathrm{Aut}(\mathfrak{g})$ is the algebraic subgroup of $GL(248, \mathbf{C}) = \mathrm{Iso}(\mathfrak{g})$. Since $\mathrm{Aut}(\mathfrak{g})$ is generated by $\{\exp(\mathrm{ad}\, R) | R \in \mathfrak{g}\}$ and

$$\begin{aligned}
\langle \exp(\mathrm{ad}\, R) R_1, R_2 \rangle &= -B_{\mathfrak{g}}(\exp(\mathrm{ad}\, R) R_1, \gamma R_2) \\
&= B_{\mathfrak{g}}(R_1, \exp(\mathrm{ad}\, R) \gamma R_2) \\
&= -B_{\mathfrak{g}}(R_1, \gamma \exp(\mathrm{ad}(-\gamma R)) R_2) \\
&= \langle R_1, \exp(\mathrm{ad}(-\gamma R)) R_2 \rangle,
\end{aligned}$$

$\alpha \in \mathrm{Aut}(\mathfrak{g})$ implies $\alpha^* \in \mathrm{Aut}(\mathfrak{g})$, where α^* is the transpose of α with respect to $\langle R_1, R_2 \rangle : \langle \alpha R_1, R_2 \rangle = \langle R_1, \alpha^* R_2 \rangle$. It is clear that $\mathrm{Aut}(\mathfrak{g}) \cap U(\mathfrak{g}) = E_8$ and $\dim_{\boldsymbol{R}} E_8^{\mathbf{C}} - \dim_{\boldsymbol{R}} E_8 = 248$. Hence we have

$$\mathrm{Aut}(\mathfrak{g}) \simeq E_8 \times \boldsymbol{R}^{248}.$$

Since $\mathrm{Aut}(\mathfrak{g})$ is connected, the group E_8 is also connected. From the general theory of Lie groups ([7]), the connected compact simple Lie group of type E_8 is simply connected. Hence we have

PROPOSITION 1.4. *Let $\mathfrak{g}$ be a complex simple Lie algebra of type E_8 and γ a conjugation of $\mathfrak{g}$ with respect to a compact real form of $\mathfrak{g}$. Then the group*

$E_8 = \{\alpha \in \mathrm{Aut}(\mathfrak{g}) | \langle \alpha R_1, \alpha R_2 \rangle = \langle R_1, R_2 \rangle\}$ *is simply connected compact Lie group of type* E_8.

§2. The subgroup of type A_8

In this section, let us consider a complex vector space direct sum

$$\mathfrak{g} = \mathfrak{sl}(9, \boldsymbol{C}) \oplus \textstyle\bigwedge^3(\boldsymbol{C}^9) \oplus \bigwedge^3(\boldsymbol{C}^9).$$

We define an anti-symmetric bilinear product on $\mathfrak{g}$ by

$$(X, \boldsymbol{u}, \boldsymbol{v}) = [(X_1, \boldsymbol{u}_1, \boldsymbol{v}_1), (X_2, \boldsymbol{u}_2, \boldsymbol{v}_2)]$$

where

$$\begin{cases} X = [X_1, X_2] + \boldsymbol{u}_1 \times \boldsymbol{v}_2 - \boldsymbol{u}_2 \times \boldsymbol{v}_1, \\ \boldsymbol{u} = X_1 \boldsymbol{u}_2 - X_2 \boldsymbol{u}_1 + *(\boldsymbol{v}_1 \wedge \boldsymbol{v}_2), \\ \boldsymbol{v} = -{}^t X_1 \boldsymbol{v}_2 + {}^t X_2 \boldsymbol{v}_1 - *(\boldsymbol{u}_1 \wedge \boldsymbol{u}_2). \end{cases}$$

In order to prove the Jacobi identity, we show the following:

LEMMA 2.1. *For* $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \bigwedge^3(\boldsymbol{C}^9)$, *we have*

$$\boldsymbol{u} \times *(\boldsymbol{v} \wedge \boldsymbol{w}) + \boldsymbol{v} \times *(\boldsymbol{w} \wedge \boldsymbol{u}) + \boldsymbol{w} \times *(\boldsymbol{u} \wedge \boldsymbol{v}) = 0, \tag{1}$$

$$(\boldsymbol{u} \times \boldsymbol{w})\boldsymbol{v} - (\boldsymbol{v} \times \boldsymbol{w})\boldsymbol{u} + *(*(\boldsymbol{u} \wedge \boldsymbol{v}) \wedge \boldsymbol{w}) = 0. \tag{2}$$

PROOF. Put $\boldsymbol{u} = \boldsymbol{u}_1 \wedge \boldsymbol{u}_2 \wedge \boldsymbol{u}_3$, $\boldsymbol{v} = \boldsymbol{u}_4 \wedge \boldsymbol{u}_5 \wedge \boldsymbol{u}_6$ and $\boldsymbol{w} = \boldsymbol{u}_7 \wedge \boldsymbol{u}_8 \wedge \boldsymbol{u}_9$. For $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{C}^9$, we see

$$\begin{aligned} ((\boldsymbol{u} \times \boldsymbol{v})\boldsymbol{x}, \boldsymbol{y}) &= (*(\boldsymbol{v} \wedge *(\boldsymbol{u} \wedge \boldsymbol{x})), \boldsymbol{y}) + 2/3(\boldsymbol{u}, \boldsymbol{v})(\boldsymbol{x}, \boldsymbol{y}) \\ &= -(\boldsymbol{x} \wedge \boldsymbol{u}, \boldsymbol{y} \wedge \boldsymbol{v}) + 2/3(\boldsymbol{u}, \boldsymbol{v})(\boldsymbol{x}, \boldsymbol{y}) \\ &= (\boldsymbol{x} \wedge \boldsymbol{u}_2 \wedge \boldsymbol{u}_3, \boldsymbol{v})(\boldsymbol{u}_1, \boldsymbol{y}) - (\boldsymbol{x} \wedge \boldsymbol{u}_1 \wedge \boldsymbol{u}_3, \boldsymbol{v})(\boldsymbol{u}_2, \boldsymbol{y}) \\ &\quad + (\boldsymbol{x} \wedge \boldsymbol{u}_1 \wedge \boldsymbol{u}_2, \boldsymbol{v})(\boldsymbol{u}_3, \boldsymbol{y}) - 1/3(\boldsymbol{u}, \boldsymbol{v})(\boldsymbol{x}, \boldsymbol{y}). \end{aligned}$$

Hence we have

$$\begin{aligned} (\boldsymbol{u} \times \boldsymbol{v})\boldsymbol{x} &= (\boldsymbol{x} \wedge \boldsymbol{u}_2 \wedge \boldsymbol{u}_3, \boldsymbol{v})\boldsymbol{u}_1 + (\boldsymbol{u}_1 \wedge \boldsymbol{x} \wedge \boldsymbol{u}_3, \boldsymbol{v})\boldsymbol{u}_2 \\ &\quad + (\boldsymbol{u}_1 \wedge \boldsymbol{u}_2 \wedge \boldsymbol{x}, \boldsymbol{v})\boldsymbol{u}_3 - 1/3(\boldsymbol{u}, \boldsymbol{v})\boldsymbol{x}. \end{aligned} \tag{3}$$

Using this identity, we have

$$\{\boldsymbol{u} \times *(\boldsymbol{v} \wedge \boldsymbol{w}) + \boldsymbol{v} \times *(\boldsymbol{w} \wedge \boldsymbol{u}) + \boldsymbol{w} \times *(\boldsymbol{u} \wedge \boldsymbol{v})\}\boldsymbol{x}$$

$$= \sum_{j=1}^{9} (\boldsymbol{u}_1 \wedge \cdots \wedge \boldsymbol{u}_{j-1} \wedge \boldsymbol{x} \wedge \boldsymbol{u}_{j+1} \wedge \cdots \wedge \boldsymbol{u}_9, \boldsymbol{e}_1 \wedge \cdots \wedge \boldsymbol{e}_9)\boldsymbol{u}_j$$

$$- (\boldsymbol{u}_1 \wedge \cdots \wedge \boldsymbol{u}_9, \boldsymbol{e}_1 \wedge \cdots \wedge \boldsymbol{e}_9)\boldsymbol{x} = \text{(i)}.$$

Let us put $\boldsymbol{x} = \sum_{i=1}^{9} x_i \boldsymbol{e}_i$, $\boldsymbol{u}_j = \sum_{k=1}^{9} u_{jk} \boldsymbol{e}_k$ and $U = (u_{jk}) \in M(9, \boldsymbol{C})$. Hence we see

$$(\boldsymbol{u}_1 \wedge \cdots \wedge \boldsymbol{u}_{j-1} \wedge \boldsymbol{x} \wedge \boldsymbol{u}_{j+1} \wedge \cdots \wedge \boldsymbol{u}_9, \boldsymbol{e}_1 \wedge \cdots \wedge \boldsymbol{e}_9) = \sum_{k=1}^{9} \tilde{U}_{jk} x_k,$$

$$(\boldsymbol{u}_1 \wedge \cdots \wedge \boldsymbol{u}_9, \boldsymbol{e}_1 \wedge \cdots \wedge \boldsymbol{e}_9) = \det U,$$

where $\tilde{U}_{jk}$ is the factor of u_{jk} of the matrix U. Therefore we have

$$\text{(i)} = \sum_{j,k} x_k \tilde{U}_{jk} \boldsymbol{u}_j - (\det U)\boldsymbol{x} = \sum_{i,j,k} x_k \tilde{U}_{jk} u_{ji} \boldsymbol{e}_i - (\det U)\boldsymbol{x}$$

$$= \sum_{i,k} x_k (\det U) \delta_{ki} \boldsymbol{e}_i - (\det U)\boldsymbol{x} = 0.$$

Then (1) has been proved. Now let us put $\boldsymbol{u} = \boldsymbol{u}_1 \wedge \boldsymbol{u}_2 \wedge \boldsymbol{u}_3$ and $\boldsymbol{v} = \boldsymbol{v}_1 \wedge \boldsymbol{v}_2 \wedge \boldsymbol{v}_3$. Using (3), for any $\boldsymbol{a} \in \bigwedge^3(\boldsymbol{C}^9)$ we have

$$((\boldsymbol{u} \times \boldsymbol{w})\boldsymbol{v} - (\boldsymbol{v} \times \boldsymbol{w})\boldsymbol{u}, \boldsymbol{a})$$

$$= (((\boldsymbol{u} \times \boldsymbol{w})\boldsymbol{v}_1) \wedge \boldsymbol{v}_2 \wedge \boldsymbol{v}_3, \boldsymbol{a}) - (((\boldsymbol{u} \times \boldsymbol{w})\boldsymbol{v}_2) \wedge \boldsymbol{v}_1 \wedge \boldsymbol{v}_3, \boldsymbol{a})$$

$$+ (((\boldsymbol{u} \times \boldsymbol{w})\boldsymbol{v}_3) \wedge \boldsymbol{v}_1 \wedge \boldsymbol{v}_2, \boldsymbol{a} - (((\boldsymbol{v} \times \boldsymbol{w})\boldsymbol{u}_1) \wedge \boldsymbol{u}_2 \wedge \boldsymbol{u}_3, \boldsymbol{a})$$

$$+ (((\boldsymbol{v} \times \boldsymbol{w})\boldsymbol{u}_2) \wedge \boldsymbol{u}_1 \wedge \boldsymbol{u}_3, \boldsymbol{a}) - (((\boldsymbol{v} \times \boldsymbol{w})\boldsymbol{u}_3) \wedge \boldsymbol{u}_1 \wedge \boldsymbol{u}_2, \boldsymbol{a})$$

$$= -(\boldsymbol{u}, \boldsymbol{w})(\boldsymbol{v}, \boldsymbol{a}) + \sum_{i=1}^{3} \sum_{j=1}^{3} (\boldsymbol{u}_i \wedge \boldsymbol{u}_{i+1} \wedge \boldsymbol{v}_j, \boldsymbol{w})(\boldsymbol{u}_{i+2} \wedge \boldsymbol{v}_{i+1} \wedge \boldsymbol{v}_{j+2}, \boldsymbol{a})$$

$$+ (\boldsymbol{v}, \boldsymbol{w})(\boldsymbol{u}, \boldsymbol{a}) - \sum_{i=1}^{3} \sum_{j=1}^{3} (\boldsymbol{u}_i \wedge \boldsymbol{v}_j \wedge \boldsymbol{v}_{j+1}, \boldsymbol{w})(\boldsymbol{u}_{i+1} \wedge \boldsymbol{u}_{i+2} \wedge \boldsymbol{v}_{j+2}, \boldsymbol{a})$$

$$= -(\boldsymbol{u} \wedge \boldsymbol{v}, \boldsymbol{w} \wedge \boldsymbol{a}) = -(*(*(\boldsymbol{u} \wedge \boldsymbol{v}) \wedge \boldsymbol{w}), \boldsymbol{a}).$$

Then (2) has been proved. □

From Lemmas 1.1, 1.2 and 2.1, we can prove that $\mathfrak{g}$ becomes a Lie algebra. Furthermore we have the following:

THEOREM 2.2. *The Lie algebra* $\mathfrak{g}$ *is a complex simple Lie algebra of type* E_8.

PROOF. For a subset $I = \{i,j,k\}$ $(i<j<k)$ of $\{1,2,\dots,9\}$, we put

$$\boldsymbol{e}_I = \boldsymbol{e}_i \wedge \boldsymbol{e}_j \wedge \boldsymbol{e}_k \in \bigwedge^3(\boldsymbol{C}^9).$$

Let $\mathfrak{a}$ be a non-zero ideal of $\mathfrak{g}$ and let us put

$$\mathfrak{q} = \bigwedge^3(\boldsymbol{C}^9) \oplus \bigwedge^3(\boldsymbol{C}^9).$$

There are three cases to be considered: (a) $\mathfrak{sl}(9,\boldsymbol{C}) \cap \mathfrak{a} = \{0\}$ and $\mathfrak{q} \cap \mathfrak{a} = \{0\}$, (b) $\mathfrak{sl}(9,\boldsymbol{C}) \cap \mathfrak{a} \neq \{0\}$, (c) $\mathfrak{q} \cap \mathfrak{a} \neq \{0\}$.

Case (a): Let $p : \mathfrak{g} \to \mathfrak{sl}(9,\boldsymbol{C})$ denote the projection. If $p(\mathfrak{a}) = 0$, then $\mathfrak{a}$ is contained in $\mathfrak{q}$, which contradicts to $\mathfrak{q} \cap \mathfrak{a} = \{0\}$. For this reason $p(\mathfrak{a})$ is a non-zero ideal of $\mathfrak{sl}(9,\boldsymbol{C})$, hence we have $p(\mathfrak{a}) = \mathfrak{sl}(9,\boldsymbol{C})$. For an element $X = \sum_{i=1}^{8} E_{ii} - 8E_{99} \in \mathfrak{sl}(9,\boldsymbol{C})$, there exists an element $(\boldsymbol{u},\boldsymbol{v}) = (\sum_I u_I \boldsymbol{e}_I, \sum_J v_J \boldsymbol{e}_J) \in \mathfrak{q}$ such that $(X,\boldsymbol{u},\boldsymbol{v}) \in \mathfrak{a}$. Since $[(X,0,0),(X,\boldsymbol{u},\boldsymbol{v})] = (0, X\boldsymbol{u}, -{}^tX\boldsymbol{v}) \in \mathfrak{q} \cap \mathfrak{a} = \{0\}$, we have

$$0 = X\boldsymbol{u} = \sum_I u_I X \boldsymbol{e}_I = 3\sum_{I \not\ni 9} u_I \boldsymbol{e}_I - 6\sum_{I \ni 9} u_I \boldsymbol{e}_I,$$
$$0 = -{}^tX\boldsymbol{v} = -3\sum_{J \not\ni 9} v_J \boldsymbol{e}_J + 6\sum_{J \ni 9} v_J \boldsymbol{e}_J,$$

i.e., $u_I = 0$ and $v_J = 0$. Then $0 \neq (X,\boldsymbol{u},\boldsymbol{v}) = (X,0,0) \in \mathfrak{sl}(9,\boldsymbol{C}) \cap \mathfrak{a} = \{0\}$. This is a contradiction.

Case (b): Since $\mathfrak{sl}(9,\boldsymbol{C}) \cap \mathfrak{a}$ is a non-zero ideal of $\mathfrak{sl}(9,\boldsymbol{C})$, we have $\mathfrak{sl}(9,\boldsymbol{C}) \subset \mathfrak{a}$. For any $\boldsymbol{e}_i \wedge \boldsymbol{e}_j \wedge \boldsymbol{e}_k \in \bigwedge^3(\boldsymbol{C}^9)$, put

$$X = \frac{1}{3}(E_{ii} + E_{jj} + E_{kk}) - E_{ll}.$$

Since $(X,0,0) \in \mathfrak{sl}(9,\boldsymbol{C}) \subset \mathfrak{a}$, we see that

$$(0, \boldsymbol{e}_i \wedge \boldsymbol{e}_j \wedge \boldsymbol{e}_k, 0) = [(X,0,0),(0,\boldsymbol{e}_i \wedge \boldsymbol{e}_j \wedge \boldsymbol{e}_k, 0)] \in \mathfrak{a},$$
$$(0,0,\boldsymbol{e}_i \wedge \boldsymbol{e}_j \wedge \boldsymbol{e}_k) = [(X,0,0),(0,0,-\boldsymbol{e}_i \wedge \boldsymbol{e}_j \wedge \boldsymbol{e}_k)] \in \mathfrak{a}.$$

It follows that $\mathfrak{q} \subset \mathfrak{a}$. Hence we have $\mathfrak{a} = \mathfrak{g}$.

Case (c): Let $R = (0,\boldsymbol{u},\boldsymbol{v})$ be a non-zero element of $\mathfrak{q} \cap \mathfrak{a}$. In case $\boldsymbol{u} \neq 0$, we put $\boldsymbol{u} = \sum_I u_I \boldsymbol{e}_I$. Without loss of generality, we may assume $u_{\{123\}} = 1$. Putting $S_{ij} = (E_{ii} - E_{jj}, 0, 0) \in \mathfrak{g}$ and $T = (0,0,\boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \wedge \boldsymbol{e}_4) \in \mathfrak{g}$, we have

$$\begin{aligned}0 &\neq \mathrm{ad}(T)\mathrm{ad}(S_{37})\mathrm{ad}(S_{27})\mathrm{ad}(S_{17})\mathrm{ad}(S_{36})\mathrm{ad}(S_{25})\mathrm{ad}(S_{14})R \\ &= (-E_{34},0,0) \in \mathfrak{sl}(9,\boldsymbol{C}) \cap \mathfrak{a}.\end{aligned}$$

Then we can reduce this case to case (b). In case $v \neq 0$, we can similarly reduce to case (b).

Thus the simplicity of $\mathfrak{g}$ has been proved. On the other hand, since the dimension of $\mathfrak{g}$ is clearly 248, we see that $\mathfrak{g}$ is a Lie algebra of type E_8. □

Let us define a conjugate linear transformation γ and an inner product $\langle R_1, R_2\rangle$ on $\mathfrak{g}$ as follows:

$$\gamma(X, \boldsymbol{u}, \boldsymbol{v}) = (-{}^t\overline{X}, -\overline{\boldsymbol{v}}, -\overline{\boldsymbol{u}}),$$
$$\langle R_1, R_2\rangle = -B_{\mathfrak{g}}(R_1, \gamma R_2),$$

where $B_{\mathfrak{g}}$ is the Killing form of $\mathfrak{g}$. We shall show that this inner product is positive definite Hermitian. Now, let us consider another symmetric bilinear form defined by

$$B_1((X_1, \boldsymbol{u}_1, \boldsymbol{v}_1), (X_2, \boldsymbol{u}_2, \boldsymbol{v}_2)) = \mathrm{tr} X_1 X_2 + (\boldsymbol{u}_1, \boldsymbol{v}_2) + (\boldsymbol{u}_2, \boldsymbol{v}_1).$$

Using (1.1), Lemmas 1.1 and 1.2, we see that B_1 is $\mathfrak{g}$-invariant. Since $\mathfrak{g}$ is simple, there exists some $\alpha \in \boldsymbol{C}$ such that $B_{\mathfrak{g}} = \alpha B_1$ ([5]). For $R = (E_{11} - E_{22}, 0, 0) \in \mathfrak{g}$, we have $B_1(R, R) = 2$. On the other hand, we have $B_{\mathfrak{g}}(R, R) = 120$ by straightforward calculation. It follows that $B_{\mathfrak{g}} = 60B_1$. Hence we have

$$\langle R_1, R_2\rangle = 60\mathrm{tr} X_1^t\overline{X}_2 + 60(\boldsymbol{u}_1, \overline{\boldsymbol{u}}_2) + 60(\boldsymbol{v}_1, \overline{\boldsymbol{v}}_2),$$

for $R_i = (X_i, \boldsymbol{u}_i, \boldsymbol{v}_i) \in \mathfrak{g}$. It follows that $\langle R_1, R_2\rangle$ is a positive definite Hermitian inner product on $\mathfrak{g}$. Using Lemma 1.2 (3) and (4), we see that γ holds the Lie bracket. Then it is clear that

$$\mathfrak{g}^\gamma = \{R \in \mathfrak{g} \mid \gamma(R) = R\}$$
$$= \{(X, \boldsymbol{u}, -\overline{\boldsymbol{u}}) \in \mathfrak{g} \mid X \in \mathfrak{su}(9), \boldsymbol{u} \in \textstyle\bigwedge^3(\boldsymbol{C}^9)\}$$

is the compact real form of $\mathfrak{g}$. Furthermore, from Proposition 1.4, we see that the group

$$E_8 = \{\alpha \in \mathrm{Aut}(\mathfrak{g}) \mid \langle \alpha R_1, \alpha R_2\rangle = \langle R_1, R_2\rangle\}$$

is a simply connected compact simple Lie group of type E_8.

Put $\omega = \exp(2\pi i/3) \in \boldsymbol{C}$ and define a transformation $w : \mathfrak{g} \to \mathfrak{g}$ as follows:

$$w(X, \boldsymbol{u}, \boldsymbol{v}) = (X, \omega \boldsymbol{u}, \omega^2 \boldsymbol{v}).$$

It is clear that $w \in E_8$ and $w^3 = 1$.

Let σ denote the inner automorphism of group G induced by $s \in G$, i.e., $\sigma(x) = sxs^{-1}$. We put G^s instead of G^σ.

THEOREM 2.3. *The subgroup* $(E_8)^w$ *of* E_8 *is isomorphic to the group* $SU(9)/\boldsymbol{Z}_3$.

PROOF. We define a map $\varphi : SU(9) \to (E_8)^w$ by

$$\varphi(A)(X, \boldsymbol{u}, \boldsymbol{v}) = (\mathrm{Ad}(A)X, A\boldsymbol{u}, {}^t A^{-1}\boldsymbol{v}).$$

For $Y \in \mathfrak{su}(9)$, we have

$$\begin{aligned}\exp(\mathrm{ad}(Y,0,0))(X,\boldsymbol{u},\boldsymbol{v}) &= (\exp(\mathrm{ad}(Y))X, (\exp Y)\boldsymbol{u}, (\exp(-{}^tY))\boldsymbol{v}) \\ &= (\mathrm{Ad}(\exp Y)X, (\exp Y)\boldsymbol{u}, {}^t(\exp Y)^{-1}\boldsymbol{v}) \\ &= \varphi(\exp Y)(X,\boldsymbol{u},\boldsymbol{v}).\end{aligned}$$

Hence $\varphi(A)$ is an automorphism of $\mathfrak{g}$. Furthermore, using Lemma 1.1, we have

$$\langle \varphi(A)R_1, \varphi(A)R_2 \rangle = \langle R_1, R_2 \rangle.$$

Hence we see $\varphi(A) \in E_8$. It is clear that $w\varphi(A) = \varphi(A)w$. Thus the map φ is well-defined. Obviously φ is a homomorphism. We shall show that φ is surjective. The Lie algebra of $(E_8)^w$ is isomorphic to

$$\{R \in \mathfrak{g} \mid \gamma R = R, wR = R\} = \{(X,0,0) \in \mathfrak{g} \mid X \in \mathfrak{su}(9)\} \cong \mathfrak{su}(9).$$

Thus the differential of φ is surjective. Since $(E_8)^w$ is connected ([6]), φ is surjective.

At last, we shall show that $\mathrm{Ker}\,\varphi = \{I, \omega I, \omega^2 I\}$. Let A be an element of $\mathrm{Ker}\,\varphi$. Since $\mathrm{Ad}(A)X = X$, we have $A = \zeta^m I$ where $m \in \boldsymbol{Z}$ and $\zeta = \exp(2\pi i/9) \in \boldsymbol{C}(\zeta^3 = \omega)$. Since

$$(X,\boldsymbol{u},\boldsymbol{v}) = \varphi(\zeta^m I)(X,\boldsymbol{u},\boldsymbol{v}) = (X, \zeta^{3m}\boldsymbol{u}, \zeta^{6m}\boldsymbol{v}) = (X, \omega^m\boldsymbol{u}, \omega^{2m}\boldsymbol{v}),$$

we have $m \equiv 0 \bmod 3$. Then we see that $\mathrm{Ker}\,\varphi = \{I, \omega I, \omega^2 I\} \cong \boldsymbol{Z}_3$.

Therefore $SU(9)/\boldsymbol{Z}_3 \cong (E_8)^w$ has been proved. □

This theorem means that $(E_8)^w$ is a subgroup of type A_8.

§3. The subgroup of type $A_2 \times E_6$

In this section, we denote the complexification of any real vector space S by $S^{\boldsymbol{C}}$. And the complex conjugation of $S^{\boldsymbol{C}}$ with respect to the real form S is denoted by τ instead of $^-$ in §1, §2 and §4, because we have to distinguish it from the following canonical involution $^-$ of Cayley algebra. The complex conjugation in $\boldsymbol{C} = \boldsymbol{R}^C$ is also denoted by τ.

Let $\mathfrak{C}$ be the division Cayley algebra over $\boldsymbol{R}$. We denote the canonical involution of $\mathfrak{C}$ by $\bar{x}$ $(x \in C)$. Let

$$\mathfrak{J} = \{U \in M(3, \mathfrak{C}) \mid U = {}^t\bar{U}\}$$
$$= \left\{ \begin{pmatrix} a_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & a_2 & x_1 \\ x_2 & \bar{x}_1 & a_3 \end{pmatrix} \in M(3, \mathfrak{C}) \,\middle|\, \begin{matrix} a_i \subset \boldsymbol{R} \\ x_i \in \mathfrak{C} \end{matrix} \right\}$$

be the Jordan algebra over $\boldsymbol{R}$ with respect to the Jordan multiplication

$$U \circ V = \frac{1}{2}(UV + VU).$$

In $\mathfrak{J}^C$, a symmetric inner product (U, V), a positive definite Hermitian inner product $\langle U, V \rangle$, a cross product $U \times V$, a cubic form (U, V, W) and the determinant $\det U$ are defined respectively by

$$(U, V) = \mathrm{tr}(U \circ V), \quad \langle U, V \rangle = (\tau U, V),$$
$$U \times V = U \circ V - \frac{1}{2}(\mathrm{tr}(U)V + \mathrm{tr}(V)U) + \frac{1}{2}\{\mathrm{tr}(U)\mathrm{tr}(V) - (U, V)\}I,$$
$$(U, V, W) = (U, V \times W) = (U \times V, W), \quad \det U = (U, U, U),$$

where I means the 3×3 unit matrix.

In [10], Yokota realized a complex simple Lie algebra $\mathfrak{e}_6^C$ of type E_6 as

$$\mathfrak{e}_6^C = \{\phi \in \mathrm{Hom}_C(\mathfrak{J}^C, \mathfrak{J}^C) \mid (\phi U, U, U) = 0\}$$

and he showed that the group

$$E_6 = \left\{ \alpha \in \mathrm{Iso}_C(\mathfrak{J}^C) \,\middle|\, \begin{matrix} \det \alpha U = \det U \\ \langle \alpha U, \alpha V \rangle = \langle U, V \rangle \end{matrix} \right\}$$

is a simply connected compact simple Lie group of type E_6, whose Lie algebra is

$$\mathfrak{e}_6 = \{\phi \in \mathfrak{e}_6^C \mid \langle \phi U, V \rangle + \langle U, \phi V \rangle = 0\}$$
$$= \{\phi \in \mathfrak{e}_6^C \mid -\tau{}^t\phi\tau = \phi\},$$

where ${}^t\phi$ means the transpose of ϕ with respect to (U, V). For $U, V \in \mathfrak{J}^C$, define

$U \vee V \in \mathfrak{e}_6^C$ by

$$(U \vee V)X = \frac{1}{2}(V, X)U + \frac{1}{6}(U, V)X - 2V \times (U \times X), \quad (X \in \mathfrak{J}^C).$$

Now, we consider a complex 81-dimensional vector space $\boldsymbol{C}^3 \otimes \mathfrak{J}^C$. We denote each element of $\boldsymbol{C}^3 \otimes \mathfrak{J}^C$ in matrix form as

$$\boldsymbol{U} = (U_i) = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} \quad (U_i \in \mathfrak{J}^C).$$

For $\phi \in \mathrm{Hom}(\mathfrak{J}^C, \mathfrak{J}^C)$, $X = (x_{ij}) \in M(3, \boldsymbol{C})$ and $\boldsymbol{U} = (U_i) \in \boldsymbol{C}^3 \otimes \mathfrak{J}^C$, define $\phi\boldsymbol{U}$, $X\boldsymbol{U} \in \boldsymbol{C}^3 \otimes \mathfrak{J}^C$ as follows:

$$\phi\boldsymbol{U} = \begin{bmatrix} \phi U_1 \\ \phi U_2 \\ \phi U_3 \end{bmatrix}, \quad X\boldsymbol{U} = \begin{bmatrix} x_{11}U_1 + x_{12}U_2 + x_{13}U_3 \\ x_{21}U_1 + x_{22}U_2 + x_{23}U_3 \\ x_{31}U_1 + x_{32}U_2 + x_{33}U_3 \end{bmatrix}.$$

For $\boldsymbol{U} = (U_i)$, $\boldsymbol{V} = (V_i) \in \boldsymbol{C}^3 \otimes \mathfrak{J}^C$, let us define a symmetric inner product $(\boldsymbol{U}, \boldsymbol{V})$, a positive definite Hermitian inner product $\langle \boldsymbol{U}, \boldsymbol{V} \rangle$, a cross product $\boldsymbol{U} \times \boldsymbol{V}$, an element $\boldsymbol{U} \circ \boldsymbol{V}$ of $\mathfrak{sl}(3, \boldsymbol{C})$ and an element $\boldsymbol{U} \vee \boldsymbol{V}$ of $\mathfrak{e}_6^C$ by

$$(\boldsymbol{U}, \boldsymbol{V}) = (U_1, V_1) + (U_2, V_2) + (U_3, V_3),$$

$$\langle \boldsymbol{U}, \boldsymbol{V} \rangle = \langle U_1, V_1 \rangle + \langle U_2, V_2 \rangle + \langle U_3, V_3 \rangle,$$

$$\boldsymbol{U} \times \boldsymbol{V} = \begin{bmatrix} U_2 \times V_3 - V_2 \times U_3 \\ U_3 \times V_1 - V_3 \times U_1 \\ U_1 \times V_2 - V_1 \times U_2 \end{bmatrix},$$

$$\boldsymbol{U} \circ \boldsymbol{V} = \begin{bmatrix} (U_1, V_1) & (U_1, V_2) & (U_1, V_3) \\ (U_2, V_1) & (U_2, V_2) & (U_2, V_3) \\ (U_3, V_1) & (U_3, V_2) & (U_3, V_3) \end{bmatrix} - \frac{1}{3}(\boldsymbol{U}, \boldsymbol{V})I,$$

$$\boldsymbol{U} \vee \boldsymbol{V} = U_1 \vee V_1 + U_2 \vee V_2 + U_3 \vee V_3,$$

respectively.

Next, let us consider a complex vector space direct sum

$$\mathfrak{m} = \mathfrak{sl}(3, \boldsymbol{C}) \oplus \mathfrak{e}_6^C \oplus \boldsymbol{C}^3 \otimes \mathfrak{J}^C \oplus \boldsymbol{C}^3 \otimes \mathfrak{J}^C.$$

Furthermore, let us define an anti-symmetric bilinear product on $\mathfrak{m}$ as follows:

$$(X, \phi, \boldsymbol{U}, \boldsymbol{V}) = [(X_1, \phi_1, \boldsymbol{U}_1, \boldsymbol{V}_1), (X_2, \phi_2, \boldsymbol{U}_2, \boldsymbol{V}_2)]$$

where

$$\begin{cases} X = [X_1, X_2] + \frac{1}{4} U_1 \circ V_2 - \frac{1}{4} U_2 \circ V_1, \\ \phi = [\phi_1, \phi_2] + \frac{1}{2} U_1 \vee V_2 - \frac{1}{2} U_2 \vee V_1, \\ U = \phi_1 U_2 - \phi_2 U_1 + X_1 U_2 - X_2 U_1 - V_1 \times V_2, \\ V = -{}^t\phi_1 V_2 + {}^t\phi_2 V_1 - {}^tX_1 V_2 + {}^tX_2 V_1 + U_1 \times U_2. \end{cases}$$

Let $\mathfrak{e}_8^C$ be a complex simple Lie algebra of type E_8 which was realized in [4];

$$\mathfrak{e}_8^C = \mathfrak{e}_7^C \oplus \mathfrak{P}^C \oplus \mathfrak{P}^C \oplus C \oplus C \oplus C.$$

Define a map $\mu : \mathfrak{e}_8^C \to \mathfrak{m}$ as follows:

$$\mu(\Phi(\phi, S, T, \nu), (U, V, \xi, \eta), (W, Y, \zeta, \omega), r, s, t)$$
$$= \left(\begin{bmatrix} \frac{2}{3}\nu & -\frac{1}{2}\xi & \frac{1}{2}\zeta \\ \frac{1}{2}\omega & -\frac{1}{3}\nu - r & t \\ \frac{1}{2}\eta & s & -\frac{1}{3}\nu + r \end{bmatrix}, \phi, \begin{bmatrix} -2S \\ W \\ U \end{bmatrix}, \begin{bmatrix} -2T \\ V \\ -Y \end{bmatrix} \right).$$

We can prove that μ is isomorphism by straightforward calculation. Thus we have

THEOREM 3.1. *The Lie algebra* $\mathfrak{m}$ *is a complex simple Lie algebra of type* E_8.

Using the Killing form of $\mathfrak{e}_8^C$ which was obtained in [4], we see that the Killing form $B_{\mathfrak{m}}$ is

$$B_{\mathfrak{m}}(R_1, R_2) = 60\,\mathrm{tr}(X_1 X_2) + \frac{5}{2} B_{\mathfrak{e}_6^C}(\phi_1, \phi_2) + 15(U_1, V_2) + 15(U_2, V_1),$$

for $R_i = (X_i, \phi_i, U_i, V_i) \in \mathfrak{m}$. We define a conjugate linear transformation γ and an inner product $\langle R_1, R_2 \rangle$ on $\mathfrak{m}$ as follows:

$$\gamma(X, \phi, U, V) = (-\tau\,{}^tX, -\tau\,{}^t\phi\tau, -\tau V, -\tau U)$$
$$\langle R_1, R_2 \rangle = -B_{\mathfrak{m}}(R_1, \gamma R_2).$$

Then we have

$$\langle R_1, R_2 \rangle = 60\,\mathrm{tr}\, X_1(\tau\,{}^tX_2) + \frac{5}{2} B_{\mathfrak{e}_6^C}(\phi_1, \tau\,{}^t\phi_2\tau) + 15\langle U_1, U_2 \rangle + 15\langle V_1, V_2 \rangle.$$

This implies that $\langle R_1, R_2\rangle$ is a positive definite Hermitian inner product on $\mathfrak{m}$. Using the following equations:

$$\tau^t(\boldsymbol{U}\circ\boldsymbol{V}) = (\tau\boldsymbol{V})\circ(\tau\boldsymbol{U}) \quad \text{and} \quad \tau^t(\boldsymbol{U}\vee\boldsymbol{V}) = (\tau\boldsymbol{V})\vee(\tau\boldsymbol{U}),$$

we see that γ holds the Lie bracket. Then it is clear that $\mathfrak{m}^\gamma = \{R\in\mathfrak{m}\,|\,\gamma(R) = R\}$ is the compact real form of $\mathfrak{m}$. Furthermore, from Proposition 1.4, we see that the group

$$E_8 = \{\alpha\in\mathrm{Aut}(\mathfrak{m})\,|\,\langle\alpha R_1, \alpha R_2\rangle = \langle R_1, R_2\rangle\}$$

is a simply connected compact simple Lie group of type E_8.

Let us define a transformation $\delta:\mathfrak{m}\to\mathfrak{m}$ as follows:

$$\delta(X,\phi,\boldsymbol{U},\boldsymbol{V}) = (X,\phi,\omega\boldsymbol{U},\omega^2\boldsymbol{V}).$$

It is clear that $\delta\in E_8$ and $\delta^3 = 1$.

THEOREM 3.2. *The subgroup* $(E_8)^\delta$ *is isomorphic to the group* $(SU(3)\times E_6)/\boldsymbol{Z}_3$.

PROOF. For any $A\in SU(3)$, we define a linear transformation $\psi(A)$ on $\mathfrak{m}$ by

$$\psi(A)(X,\phi,\boldsymbol{U},\boldsymbol{V}) = (\mathrm{Ad}(A)X,\phi,A\boldsymbol{U},{}^tA^{-1}\boldsymbol{V}).$$

Obviously the map $\psi: SU(3)\to GL(\mathfrak{m})$ is a homomorphism. Furthermore, since

$$\exp(\mathrm{ad}(Y,0,0,0)) = \psi(\exp Y), \quad (Y\in\mathfrak{su}(3)),$$

we see that $\psi(A)$ is an automorphism of $\mathfrak{m}$. For any $\boldsymbol{U},\boldsymbol{V}\in\boldsymbol{C}^3\otimes\mathfrak{J}^{\boldsymbol{C}}$, we have

$$\langle A\boldsymbol{U}, A\boldsymbol{V}\rangle = \langle\boldsymbol{U},\boldsymbol{V}\rangle.$$

It follows that

$$\langle\psi(A)R_1, \psi(A)R_2\rangle = \langle R_1, R_2\rangle.$$

Hence $\psi(A)\in E_8$. Similarly, for any $\alpha\in E_6$, we define a linear transformation $\mu(\alpha)$ on $\mathfrak{m}$ by

$$\mu(\alpha)(X,\phi,\boldsymbol{U},\boldsymbol{V}) = (X,\mathrm{Ad}(\alpha)\phi,\alpha\boldsymbol{U},{}^t\alpha^{-1}\boldsymbol{V}),$$

where ${}^t\alpha$ means the transpose of α with respect to (U,V). It is clear that the map

$\mu : E_6 \to GL(\mathfrak{m})$ is homomorphism. Since

$$\alpha U_1 \times \alpha U_2 = {}^t\alpha^{-1}(U_1 \times U_2), \quad (\alpha U_1, {}^t\alpha^{-1}u_2) = (U_1, U_2), \quad (U_i \in \mathfrak{J}^C),$$

$\mu(\alpha)$ is an automorphism of $\mathfrak{m}$. Since

$$B_{\mathfrak{e}_6^C}(\mathrm{Ad}(\alpha)\phi_1, \tau^t(\mathrm{Ad}(\alpha)\phi_2)\tau) = B_{\mathfrak{e}_6^C}(\phi_1, \tau^t\phi_2\tau),$$

we have

$$\langle \mu(\alpha)R_1, \mu(\alpha)R_2 \rangle = \langle R_1, R_2 \rangle.$$

Hence $\mu(\alpha) \in E_8$. It is clear that $\psi(A)\mu(\alpha) = \mu(\alpha)\psi(A)$.

Furthermore we define a map $\varphi : SU(3) \times E_6 \to (E_8)^\delta$ by

$$\varphi(A, \alpha) = \psi(A)\mu(\alpha).$$

Since $\delta = \psi(\omega I)$, we have $\delta\varphi(A, \alpha) = \varphi(A, \alpha)\delta$. Thus the map φ is well-defined. Obviously φ is a homomorphism. Also we can prove that φ is surjective as in proof of Theorem 2.3.

Next, we shall show that $\mathrm{Ker}\,\varphi = \{(I, 1), \omega I, \omega^2 1), (\omega^2 I, \omega 1)\}$. For $(A, \alpha) \in \mathrm{Ker}\,\varphi$, we have $A = \omega^m I$ and $\alpha = \omega^n 1$ where $n, m \in \boldsymbol{Z}$. Since

$$(X, \phi, U, V) = \varphi(\omega^m I, \omega^n 1)(X, \phi, U, V) = (X, \phi, \omega^{m+n} U, \omega^{2(m+n)} V),$$

we have that $m + n \equiv 0 \bmod 3$. Thus we have

$$\mathrm{Ker}\,\varphi = \{(I, 1), (\omega I, \omega^2 1), (\omega^2 I, \omega 1)\} \cong \boldsymbol{Z}_3.$$

Therefore $(SU(3) \times E_6)/\boldsymbol{Z}_3 \cong (E_8)^\delta$ has been proved. □

This theorem means that $(E_8)^\delta$ is a subgroup of type $A_2 \times E_6$.

§4. The subgroup of type $A_4 \times A_4$

In this section, let us consider a complex vector space direct sum

$$\mathfrak{l} = \mathfrak{l}_0 \oplus \mathfrak{l}_1 \oplus \mathfrak{l}_2 \oplus \mathfrak{l}_3 \oplus \mathfrak{l}_4$$

where

$$\mathfrak{l}_0 = \mathfrak{sl}(5, \boldsymbol{C}) \oplus \mathfrak{sl}(5, \boldsymbol{C}),$$
$$\mathfrak{l}_1 = \textstyle\bigwedge^1(\boldsymbol{C}^5) \otimes \bigwedge^2(\boldsymbol{C}^5), \quad \mathfrak{l}_2 = \bigwedge^2(\boldsymbol{C}^5) \otimes \bigwedge^1(\boldsymbol{C}^5),$$
$$\mathfrak{l}_3 = \textstyle\bigwedge^2(\boldsymbol{C}^5) \otimes \bigwedge^1(\boldsymbol{C}^5), \quad \mathfrak{l}_4 = \bigwedge^1(\boldsymbol{C}^5) \otimes \bigwedge^2(\boldsymbol{C}^5).$$

We define an anti-symmetric bilinear product on $\mathfrak{l}$ as follows:

$$[\mathfrak{l}_0, \mathfrak{l}_0] \subset \mathfrak{l}_0, \qquad [(X_1, Y_1), (X_2, Y_2)] = ([X_1, X_2], [Y_1, Y_2]),$$

$$[\mathfrak{l}_0, \mathfrak{l}_1] \subset \mathfrak{l}_1, \qquad [(X, Y), \boldsymbol{x} \otimes \boldsymbol{a}] = (X\boldsymbol{x}) \otimes \boldsymbol{a} + \boldsymbol{x} \otimes (Y\boldsymbol{a}),$$

$$[\mathfrak{l}_0, \mathfrak{l}_2] \subset \mathfrak{l}_2, \qquad [(X, Y), \boldsymbol{b} \otimes \boldsymbol{y}] = (X\boldsymbol{b}) \otimes \boldsymbol{y} + \boldsymbol{b} \otimes (-{}^tY\boldsymbol{y}),$$

$$[\mathfrak{l}_0, \mathfrak{l}_3] \subset \mathfrak{l}_3, \qquad [(X, Y), \boldsymbol{c} \otimes \boldsymbol{z}] = (-{}^tX\boldsymbol{c}) \otimes \boldsymbol{z} + \boldsymbol{c} \otimes (Y\boldsymbol{z}),$$

$$[\mathfrak{l}_0, \mathfrak{l}_4] \subset \mathfrak{l}_4, \qquad [(X, Y), \boldsymbol{w} \otimes \boldsymbol{d}] = (-{}^tX\boldsymbol{w}) \otimes \boldsymbol{d} + \boldsymbol{w} \otimes (-{}^tY\boldsymbol{d}),$$

$$[\mathfrak{l}_1, \mathfrak{l}_4] \subset \mathfrak{l}_0, \qquad [\boldsymbol{x} \otimes \boldsymbol{a}, \boldsymbol{w} \otimes \boldsymbol{d}] = (-(\boldsymbol{a}, \boldsymbol{d})\boldsymbol{x} \times \boldsymbol{w}, (\boldsymbol{x}, \boldsymbol{w})\boldsymbol{a} \times \boldsymbol{d}),$$

$$[\mathfrak{l}_2, \mathfrak{l}_3] \subset \mathfrak{l}_0, \qquad [\boldsymbol{b} \otimes \boldsymbol{y}, \boldsymbol{c} \otimes \boldsymbol{z}] = ((\boldsymbol{y}, \boldsymbol{z})\boldsymbol{b} \times \boldsymbol{c}, (\boldsymbol{b}, \boldsymbol{c})\boldsymbol{z} \times \boldsymbol{y}),$$

$$[\mathfrak{l}_1, \mathfrak{l}_1] \subset \mathfrak{l}_2 \quad \text{and} \quad [\mathfrak{l}_4, \mathfrak{l}_4] \subset \mathfrak{l}_3,$$
$$[\boldsymbol{x}_1 \otimes \boldsymbol{a}_1, \boldsymbol{x}_2 \otimes \boldsymbol{a}_2] = (\boldsymbol{x}_1 \wedge \boldsymbol{x}_2) \otimes *(\boldsymbol{a}_1 \wedge \boldsymbol{a}_2),$$

$$[\mathfrak{l}_2, \mathfrak{l}_2] \subset \mathfrak{l}_4 \quad \text{and} \quad [\mathfrak{l}_3, \mathfrak{l}_3] \subset \mathfrak{l}_1,$$
$$[\boldsymbol{b}_1 \otimes \boldsymbol{y}_1, \boldsymbol{b}_2 \otimes \boldsymbol{y}_2] = *(\boldsymbol{b}_1 \wedge \boldsymbol{b}_2) \otimes (\boldsymbol{y}_1 \wedge \boldsymbol{y}_2),$$

$$[\mathfrak{l}_1, \mathfrak{l}_2] \subset \mathfrak{l}_3 \quad \text{and} \quad [\mathfrak{l}_4, \mathfrak{l}_3] \subset \mathfrak{l}_2,$$
$$[\boldsymbol{x} \otimes \boldsymbol{a}, \boldsymbol{b} \otimes \boldsymbol{y}] = *(\boldsymbol{b} \wedge \boldsymbol{x}) \otimes *(*(\boldsymbol{a}) \wedge \boldsymbol{y}),$$

$$[\mathfrak{l}_2, \mathfrak{l}_4] \subset \mathfrak{l}_1 \quad \text{and} \quad [\mathfrak{l}_3, \mathfrak{l}_1] \subset \mathfrak{l}_4,$$
$$[\boldsymbol{b} \otimes \boldsymbol{y}, \boldsymbol{w} \otimes \boldsymbol{d}] = *(*(\boldsymbol{b}) \wedge \boldsymbol{w}) \otimes *(\boldsymbol{d} \wedge \boldsymbol{y}).$$

In order to prove the Jacobi identity, we show the following:

LEMMA 4.1. *For* $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \bigwedge^1(\boldsymbol{C}^5)(= \boldsymbol{C}^5)$ *and* $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \bigwedge^2(\boldsymbol{C}^5)$, *we have*

$$*(\boldsymbol{a}) \wedge *(\boldsymbol{b} \wedge \boldsymbol{c}) + *(\boldsymbol{b}) \wedge *(\boldsymbol{c} \wedge \boldsymbol{a}) + *(\boldsymbol{c}) \wedge *(\boldsymbol{a} \wedge \boldsymbol{b}) = 0, \tag{1}$$

$$*(\boldsymbol{a} \wedge *(*(\boldsymbol{b}) \wedge \boldsymbol{x})) + *(\boldsymbol{b} \wedge *(*(\boldsymbol{a}) \wedge \boldsymbol{x})) + \boldsymbol{x} \wedge *(\boldsymbol{a} \wedge \boldsymbol{b}) = 0, \tag{2}$$

$$*(*(\boldsymbol{x} \wedge \boldsymbol{y}) \wedge \boldsymbol{z}) = (\boldsymbol{x}, \boldsymbol{z})\boldsymbol{y} - (\boldsymbol{y}, \boldsymbol{z})\boldsymbol{x}, \tag{3}$$

$$\boldsymbol{x} \wedge *(*(\boldsymbol{a}) \wedge \boldsymbol{y}) + *(\boldsymbol{y} \wedge *(\boldsymbol{a} \wedge \boldsymbol{x})) - (\boldsymbol{x}, \boldsymbol{y})\boldsymbol{a} = 0, \tag{4}$$

$$*(\boldsymbol{a} \wedge *(\boldsymbol{b} \wedge \boldsymbol{x})) - *(*(\boldsymbol{b}) \wedge *(*(\boldsymbol{a}) \wedge \boldsymbol{x})) - (\boldsymbol{a}, \boldsymbol{b})\boldsymbol{x} = 0, \tag{5}$$

$$\boldsymbol{a} \times *(\boldsymbol{b} \wedge \boldsymbol{x}) + \boldsymbol{b} \times *(\boldsymbol{a} \wedge \boldsymbol{x}) - \boldsymbol{x} \times *(\boldsymbol{a} \wedge \boldsymbol{b}) = 0, \tag{6}$$

$$*(*(\boldsymbol{a}) \wedge \boldsymbol{x}) \times \boldsymbol{y} - *(*(\boldsymbol{a}) \wedge \boldsymbol{y}) \times \boldsymbol{x} + \boldsymbol{a} \times (\boldsymbol{x} \wedge \boldsymbol{y}) = 0, \tag{7}$$

$$(\boldsymbol{a} \times \boldsymbol{b})\boldsymbol{c} = *(*(\boldsymbol{a} \wedge \boldsymbol{c}) \wedge \boldsymbol{b}) - 1/5(\boldsymbol{a}, \boldsymbol{b})\boldsymbol{c} - (\boldsymbol{b}, \boldsymbol{c})\boldsymbol{a}, \tag{8}$$

$$(\boldsymbol{x} \times \boldsymbol{y})\boldsymbol{a} = - *(\boldsymbol{y} \wedge *(\boldsymbol{x} \wedge \boldsymbol{a})) + 3/5(\boldsymbol{x}, \boldsymbol{y})\boldsymbol{a}. \tag{9}$$

PROOF. (1): Let us put $\boldsymbol{a} = \boldsymbol{a}_1 \wedge \boldsymbol{a}_2$, $\boldsymbol{b} = \boldsymbol{a}_3 \wedge \boldsymbol{a}_4$, $\boldsymbol{c} = \boldsymbol{a}_5 \wedge \boldsymbol{a}_6$ and $\boldsymbol{a}_i = \sum_{j=1}^5 a_{ij}\boldsymbol{e}_j$. Since

$$\begin{aligned}(*(*(\boldsymbol{a}) \wedge *(\boldsymbol{b} \wedge \boldsymbol{c})), \boldsymbol{x}) &= (\boldsymbol{a}, *(\boldsymbol{b} \wedge \boldsymbol{c}) \wedge \boldsymbol{x}) \\ &= (\boldsymbol{a}_1, *(\boldsymbol{b} \wedge \boldsymbol{c}))(\boldsymbol{a}_2, \boldsymbol{x}) - (\boldsymbol{a}_2, *(\boldsymbol{b} \wedge \boldsymbol{c}))(\boldsymbol{a}_1, \boldsymbol{x}) \\ &= (\boldsymbol{a}_1 \wedge \boldsymbol{b} \wedge \boldsymbol{c}, \boldsymbol{e}_1 \wedge \cdots \wedge \boldsymbol{e}_5)(\boldsymbol{a}_2, \boldsymbol{x}) \\ &\quad - (\boldsymbol{a}_2 \wedge \boldsymbol{b} \wedge \boldsymbol{c}, \boldsymbol{e}_1 \wedge \cdots \wedge \boldsymbol{e}_5)(\boldsymbol{a}_1, \boldsymbol{x}),\end{aligned}$$

we have

$$\begin{aligned}&*(*(\boldsymbol{a}) \wedge *(\boldsymbol{b} \wedge \boldsymbol{c}) + *(\boldsymbol{b}) \wedge *(\boldsymbol{c} \wedge \boldsymbol{a}) + *(\boldsymbol{c}) \wedge *(\boldsymbol{a} \wedge \boldsymbol{b})) \\ &= \sum_{j=1}^{5} \sum_{i=1}^{6} (-1)^i (\boldsymbol{a}_1 \wedge \cdots \wedge \boldsymbol{a}_{i-1} \wedge \boldsymbol{a}_{i+1} \wedge \cdots \wedge \boldsymbol{a}_6, \boldsymbol{e}_1 \wedge \cdots \wedge \boldsymbol{e}_5) a_{ij}\boldsymbol{e}_j \\ &= -\sum_{j=1}^{5} \det \begin{bmatrix} a_{1j} & a_{11} & \cdots & a_{15} \\ a_{2j} & a_{21} & \cdots & a_{25} \\ \vdots & \vdots & \ddots & \vdots \\ a_{6j} & a_{61} & \cdots & a_{65} \end{bmatrix} \boldsymbol{e}_j = 0.\end{aligned}$$

(2): Using (1), we have

$$\begin{aligned}&(*(\boldsymbol{a} \wedge *(*(\boldsymbol{b}) \wedge \boldsymbol{x})) + *(\boldsymbol{b} \wedge *(*(\boldsymbol{a}) \wedge \boldsymbol{x})) + \boldsymbol{x} \wedge *(\boldsymbol{a} \wedge \boldsymbol{b}), \boldsymbol{c}) \\ &= (*(\boldsymbol{b}) \wedge \boldsymbol{x}, \boldsymbol{c} \wedge \boldsymbol{a}) + (*(\boldsymbol{a}) \wedge \boldsymbol{x}, \boldsymbol{b} \wedge \boldsymbol{c}) + (*(\boldsymbol{c}) \wedge \boldsymbol{x}, \boldsymbol{a} \wedge \boldsymbol{b}) \\ &= (*(\boldsymbol{x}), *(\boldsymbol{b}) \wedge *(\boldsymbol{c} \wedge \boldsymbol{a}) + *(\boldsymbol{a}) \wedge *(\boldsymbol{b} \wedge \boldsymbol{c}) + *(\boldsymbol{c}) \wedge *(\boldsymbol{a} \wedge \boldsymbol{b})) = 0.\end{aligned}$$

(3): For any $\boldsymbol{v} \in \bigwedge^1(\boldsymbol{C}^5) = \boldsymbol{C}^5$, we have

$$(*(*(\boldsymbol{x} \wedge \boldsymbol{y}) \wedge \boldsymbol{z}), \boldsymbol{v}) = (\boldsymbol{x} \wedge \boldsymbol{y}, \boldsymbol{z} \wedge \boldsymbol{v}) = (\boldsymbol{x}, \boldsymbol{z})(\boldsymbol{y}, \boldsymbol{v}) - (\boldsymbol{y}, \boldsymbol{z})(\boldsymbol{x}, \boldsymbol{v}).$$

Then (3) has proved. (4) and (5): Let us put $\boldsymbol{a} = \boldsymbol{a}_1 \wedge \boldsymbol{a}_2$. Since

$$(\boldsymbol{x} \wedge \boldsymbol{a}, \boldsymbol{y} \wedge \boldsymbol{b}) = (\boldsymbol{x}, \boldsymbol{y})(\boldsymbol{a}, \boldsymbol{b}) - (\boldsymbol{a}_1, \boldsymbol{y})(\boldsymbol{x} \wedge \boldsymbol{a}_2, \boldsymbol{b}) + (\boldsymbol{a}_2, \boldsymbol{y})(\boldsymbol{x} \wedge \boldsymbol{a}_1, \boldsymbol{b}),$$

$$\begin{aligned}(*(\boldsymbol{b}) \wedge \boldsymbol{x}, *(\boldsymbol{a}) \wedge \boldsymbol{y}) &= (\boldsymbol{a}, \boldsymbol{y} \wedge *(*(\boldsymbol{b}) \wedge \boldsymbol{x})) \\ &= (\boldsymbol{a}_1, \boldsymbol{y})(\boldsymbol{x} \wedge \boldsymbol{a}_2, \boldsymbol{b}) - (\boldsymbol{a}_2, \boldsymbol{y})(\boldsymbol{x} \wedge \boldsymbol{a}_1, \boldsymbol{b}),\end{aligned}$$

we have

$$(\boldsymbol{x} \wedge \boldsymbol{a}, \boldsymbol{y} \wedge \boldsymbol{b}) + (*(\boldsymbol{b}) \wedge \boldsymbol{x}, *(\boldsymbol{a}) \wedge \boldsymbol{y}) = (\boldsymbol{x}, \boldsymbol{y})(\boldsymbol{a}, \boldsymbol{b}).$$

Using this identity, we have

$$(x \wedge *(*(a) \wedge y) + *(y \wedge *(a \wedge x)) - (x,y)a, b)$$
$$= (*(b) \wedge x, *(a) \wedge y) + (x \wedge a, y \wedge b) - (x,y)(a,b) = 0$$
$$(*(a \wedge *(b \wedge x)) - *(*(b) \wedge *(*(a) \wedge x)) - (a,b)x, y)$$
$$= (x \wedge b, y \wedge a) + (*(a) \wedge x, *(b) \wedge y) - (x,y)(a,b) = 0.$$

(6): Since

$$((x \times y)z, v) = -(x \wedge z, y \wedge v) + 4/5(x,y)(z,v)$$
$$= (y,z)(x,v) - 1/5(x,y)(z,v),$$

we have

$$(x \times y)z = (y,z)x - 1/5(x,y)z. \tag{10}$$

For $v, w \in \bigwedge^1(C^5) = C^5$, we have

$$((a \times *(b \wedge x))v, w)$$
$$= (v \wedge a, w \wedge *(b \wedge x)) - 3/5(*(a \wedge b), x)(v,w)$$
$$= (v,w)(a, *(b \wedge x)) - (a_1, w)(w \wedge a_2, *(b \wedge x))$$
$$+ (a_2, w)(w \wedge a_1, *(b \wedge x)) - 3/5(*(a \wedge b), x)(v,w)$$
$$= 2/5(*(a \wedge b), x)(v,w) - (a_1, w)(b \wedge x \wedge w \wedge a_2, e_1 \wedge \cdots \wedge e_5)$$
$$+ (a_2, w)(b \wedge x \wedge w \wedge a_1, e_1 \wedge \cdots \wedge e_5)$$
$$((b \times *(a \wedge x))v, w)$$
$$= (x \wedge a, w \wedge *(b \wedge v)) - 3/5(*(a \wedge b), x)(v,w)$$
$$= (x,w)(*(a \wedge b), v) + (a_1, w)(b \wedge x \wedge w \wedge a_2, e_1 \wedge \cdots \wedge e_5)$$
$$- (a_2, w)(b \wedge x \wedge w \wedge a_1, e_1 \wedge \cdots \wedge e_5) - 3/5(*(a \wedge b), x)(v,w).$$

Using (10), we have

$$(a \times *(b \wedge x))v + (b \times *(a \wedge x))v = (x,w) * (a \wedge b) - 1/5(*(a \wedge b), x)v$$
$$= (x \times *(a \wedge b))v.$$

(7): We have

$$((\boldsymbol{a}\times(\boldsymbol{x}\wedge\boldsymbol{y}))\boldsymbol{v},\boldsymbol{w}) = (\boldsymbol{x}\wedge\boldsymbol{y}\wedge\boldsymbol{w},\boldsymbol{v}\wedge\boldsymbol{a}) - 3/5(\boldsymbol{a},\boldsymbol{x}\wedge\boldsymbol{y})(\boldsymbol{v},\boldsymbol{w})$$
$$= (\boldsymbol{x},\boldsymbol{v})(\boldsymbol{y}\wedge\boldsymbol{w},\boldsymbol{a}) - (\boldsymbol{y},\boldsymbol{v})(\boldsymbol{x}\wedge\boldsymbol{w},\boldsymbol{a}) + 2/5(\boldsymbol{a},\boldsymbol{x}\wedge\boldsymbol{y})(\boldsymbol{v},\boldsymbol{w})$$
$$= (\boldsymbol{x},\boldsymbol{v})(*(*(\boldsymbol{a})\wedge\boldsymbol{y}),\boldsymbol{w}) - (\boldsymbol{y},\boldsymbol{v})(*(*(\boldsymbol{a})\wedge\boldsymbol{x}),\boldsymbol{w}) + 2/5(\boldsymbol{a},\boldsymbol{x}\wedge\boldsymbol{y})(\boldsymbol{v},\boldsymbol{w}).$$

On the other hand, using (10), we have

$$(*(*(\boldsymbol{a})\wedge\boldsymbol{x})\times\boldsymbol{y})\boldsymbol{v} = (\boldsymbol{y},\boldsymbol{v})*(*(\boldsymbol{a})\wedge\boldsymbol{x}) - 1/5(*(*(\boldsymbol{a})\wedge\boldsymbol{x}),\boldsymbol{v})\boldsymbol{y}$$
$$= (\boldsymbol{y},\boldsymbol{v})*(*(\boldsymbol{a})\wedge\boldsymbol{x}) - 1/5(\boldsymbol{a},\boldsymbol{x}\wedge\boldsymbol{v})\boldsymbol{y},$$
$$-(*(*(\boldsymbol{a})\wedge\boldsymbol{x})\times\boldsymbol{y})\boldsymbol{v} = -(\boldsymbol{x},\boldsymbol{v})*(*(\boldsymbol{a})\wedge\boldsymbol{y}) - 1/5(\boldsymbol{a},\boldsymbol{x}\wedge\boldsymbol{v})\boldsymbol{y}.$$

Hence (7) has been proved. (8): Let us put $\boldsymbol{a}=\boldsymbol{a}_1\wedge\boldsymbol{a}_2$ and $\boldsymbol{c}=\boldsymbol{c}_1\wedge\boldsymbol{c}_2$. Since

$$((\boldsymbol{a}\times\boldsymbol{b})\boldsymbol{v},\boldsymbol{w}) = (\boldsymbol{a}\wedge\boldsymbol{v},\boldsymbol{b}\wedge\boldsymbol{w}) - 3/5(\boldsymbol{a},\boldsymbol{b})(\boldsymbol{v},\boldsymbol{w})$$
$$= -(\boldsymbol{a}_1,\boldsymbol{w})(\boldsymbol{x}\wedge\boldsymbol{a}_2,\boldsymbol{b}) + (\boldsymbol{a}_2,\boldsymbol{w})(\boldsymbol{x}\wedge\boldsymbol{a}_1,\boldsymbol{b}) + 2/5(\boldsymbol{a},\boldsymbol{b})(\boldsymbol{v},\boldsymbol{w}),$$

we have

$$(\boldsymbol{a}\times\boldsymbol{b})\boldsymbol{c} = -(\boldsymbol{a}_1\wedge\boldsymbol{c}_1,\boldsymbol{b})\boldsymbol{a}_2\wedge\boldsymbol{c}_2 + (\boldsymbol{a}_1\wedge\boldsymbol{c}_2,\boldsymbol{b})\boldsymbol{a}_2\wedge\boldsymbol{c}_1$$
$$+ (\boldsymbol{a}_2\wedge\boldsymbol{c}_1,\boldsymbol{b})\boldsymbol{a}_1\wedge\boldsymbol{c}_2 - (\boldsymbol{a}_2\wedge\boldsymbol{c}_2,\boldsymbol{b})\boldsymbol{a}_1\wedge\boldsymbol{c}_1 + 4/5(\boldsymbol{a},\boldsymbol{b})\boldsymbol{c}.$$

On the other hand, for $\boldsymbol{d}\in\bigwedge^2(\boldsymbol{C}^5)$, we have

$$(*(*(\boldsymbol{a}\wedge\boldsymbol{c})\wedge\boldsymbol{b}),\boldsymbol{d})$$
$$= (\boldsymbol{a}\wedge\boldsymbol{c},\boldsymbol{b}\wedge\boldsymbol{d})$$
$$= (\boldsymbol{a},\boldsymbol{b})(\boldsymbol{c},\boldsymbol{d}) - (\boldsymbol{a}_1\wedge\boldsymbol{c}_1,\boldsymbol{b})(\boldsymbol{a}_2\wedge\boldsymbol{c}_2,\boldsymbol{d}) + (\boldsymbol{a}_1\wedge\boldsymbol{c}_2,\boldsymbol{b})(\boldsymbol{a}_2\wedge\boldsymbol{c}_1,\boldsymbol{d})$$
$$+ (\boldsymbol{a}_2\wedge\boldsymbol{c}_1,\boldsymbol{b})(\boldsymbol{a}_1\wedge\boldsymbol{c}_2,\boldsymbol{d}) - (\boldsymbol{a}_2\wedge\boldsymbol{c}_2,\boldsymbol{b})(\boldsymbol{a}_1\wedge\boldsymbol{c}_1,\boldsymbol{d}) + (\boldsymbol{c},\boldsymbol{b})(\boldsymbol{a},\boldsymbol{d}).$$

Hence (8) has been proved. (9): Using (10), we have

$$(\boldsymbol{x}\times\boldsymbol{y})\boldsymbol{a} = (\boldsymbol{y},\boldsymbol{a}_1)\boldsymbol{x}\wedge\boldsymbol{a}_2 - (\boldsymbol{y},\boldsymbol{a}_2)\boldsymbol{x}\wedge\boldsymbol{a}_1 - 2/5(\boldsymbol{x},\boldsymbol{y})\boldsymbol{a}.$$

On the other hand, we have

$$(-*(\boldsymbol{y}\wedge*(\boldsymbol{x}\wedge\boldsymbol{a})),\boldsymbol{b}) = -(\boldsymbol{x}\wedge\boldsymbol{a},\boldsymbol{y}\wedge\boldsymbol{b})$$
$$= (\boldsymbol{y},\boldsymbol{a}_1)(\boldsymbol{x}\wedge\boldsymbol{a}_2,\boldsymbol{b}) - (\boldsymbol{y},\boldsymbol{a}_2)(\boldsymbol{x}\wedge\boldsymbol{a}_1,\boldsymbol{b}) - (\boldsymbol{x},\boldsymbol{y})(\boldsymbol{a},\boldsymbol{b})$$

Hence, (9) has proved. □

From Lemmas 1.1, 1.2 and 4.1, we can prove that $\mathfrak{l}$ becomes a graded (i.e., $[\mathfrak{l}_k, \mathfrak{l}_l] \subset \mathfrak{l}_m$ where $m \equiv k + l \mod 5$) Lie algebra. Furthermore we have the following:

THEOREM 4.2. *The Lie algebra* $\mathfrak{l}$ *is a complex simple Lie algebra of type* E_8.

PROOF. Let $\mathfrak{a}$ be a non-zero ideal of $\mathfrak{l}$ and let us put

$$\mathfrak{l}_{01} = \{(X, 0) \in \mathfrak{l}_0 | X \in \mathfrak{sl}(5, \boldsymbol{C})\} \cong \mathfrak{sl}(5, \boldsymbol{C}),$$
$$\mathfrak{l}_{02} = \{(0, Y) \in \mathfrak{l}_0 | Y \in \mathfrak{sl}(5, \boldsymbol{C})\} \cong \mathfrak{sl}(5, \boldsymbol{C}),$$
$$\mathfrak{q} = \mathfrak{l}_1 \oplus \mathfrak{l}_2 \oplus \mathfrak{l}_3 \oplus \mathfrak{l}_4.$$

There are three cases to be considered: (a) $\mathfrak{l}_{01} \cap \mathfrak{a} = \{0\}$, $\mathfrak{l}_{02} \cap \mathfrak{a} = \{0\}$ and $\mathfrak{q} \cap \mathfrak{a} = \{0\}$, (b) $\mathfrak{l}_{01} \cap \mathfrak{a} \neq \{0\}$ or $\mathfrak{l}_{02} \cap \mathfrak{a} \neq \{0\}$, (c) $\mathfrak{q} \cap \mathfrak{a} \neq \{0\}$.

Case (a): Let $p_i : \mathfrak{l} \to \mathfrak{l}_{0i}$ $(i = i, 2)$ denote the projection. If $p_1(\mathfrak{a}) = \{0\}$ and $p_2(\mathfrak{a}) = \{0\}$, then $\mathfrak{a}$ is contained in $\mathfrak{q}$, which contradicts to $\mathfrak{q} \cap \mathfrak{a} = \{0\}$. Hence, without loss of generality, we many assume $p_1(\mathfrak{a}) = \mathfrak{l}_{01}$, because $\mathfrak{l}_{01}$ is a simple Lie algebra. For $X = \sum_{i=1}^{4} E_{ii} - 4E_{55} \in \mathfrak{sl}(5, \boldsymbol{C})$, there exists $(Y, \alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathfrak{l}_{02} \oplus \mathfrak{q}$ such that $(X, Y, \alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathfrak{a}$. Since

$$\begin{aligned}&[(X, 0), (X, Y, \alpha_1, \alpha_2, \alpha_3, \alpha_4)]\\ &\quad = (0, 0, [X, \alpha_1], [X, \alpha_2], [X, \alpha_3], [X, \alpha_4]) \in \mathfrak{q} \cap \mathfrak{a} = \{0\},\end{aligned}$$

we have $[X, \alpha_i] = 0$ $(i = 1, 2, 3, 4)$. Since any eigenvalue of $\operatorname{ad} X$ is not 0, we have $\alpha_i = 0$. Then we have $(X, Y) \in \mathfrak{l}_0 \cap \mathfrak{a}$. Since

$$[(X, Y), (E_{45}, 0)] = (5E_{45}, 0) \in \mathfrak{l}_{01} \cap \mathfrak{a},$$

we have $\mathfrak{l}_{01} \cap \mathfrak{a} \neq \{0\}$. This is contradiction.

Case (b): We may assume $\mathfrak{l}_{01} \cap \mathfrak{a} \neq \{0\}$. Since $\mathfrak{l}_{01}$ is simple, we have $\mathfrak{l}_{01} \subset \mathfrak{a}$. Since $[\mathfrak{l}_{01}, \mathfrak{l}_i] = \mathfrak{l}_i$ $(i \geq 1)$, we have $\mathfrak{q} \subset \mathfrak{a}$. Since

$$\begin{aligned}\mathfrak{a} \supset [\mathfrak{l}_1, \mathfrak{l}_4] \ni &[e_1 \otimes (e_1 \wedge e_2), e_1 \otimes (e_1 \wedge e_3)]\\ &= (0, -E_{23}),\end{aligned}$$

we have $\mathfrak{l}_{02} \cap \mathfrak{a} \neq \{0\}$. It follows that $\mathfrak{l}_{02} \subset \mathfrak{a}$. Hence we have $\mathfrak{a} = \mathfrak{l}$.

Case (c): Let $R = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ $(\alpha_i \in \mathfrak{l}_i)$ be a non-zero element of $\mathfrak{q} \cap \mathfrak{a}$. In case $\alpha_1 \neq 0$, we put $\alpha_1 = \sum_{i,j<k} \alpha_{ijk} e_i \otimes (e_j \wedge e_k)$. Without loss of generality, we may assume $\alpha_{112} = 1$. Putting $S_{ijkl} = (E_{ii} - E_{jj}, E_{kk} - E_{ll}) \in \mathfrak{l}_0$ and $T =$

$e_2 \otimes e_1 \wedge e_2 \in \mathfrak{l}_4$, we have

$$\mathrm{ad}(T)\mathrm{ad}(S_{1523})\mathrm{ad}(S_{1415})\mathrm{ad}(S_{1314})\mathrm{ad}(S_{1213})R = (-E_{12}, 0) \in \mathfrak{l}_{01} \cap \mathfrak{a}.$$

Then we can reduce this case to case (b). In case $\alpha_i \neq 0$ $(i = 2, 3, 4)$, we can similarly reduce to case (b).

Thus the simplicity of $\mathfrak{l}$ has been proved. On the other hand, since the dimension of $\mathfrak{l}$ is clearly 248, we see that $\mathfrak{l}$ is a Lie algebra of type E_8. □

Let us define a conjugate linear transformation γ and an inner product $\langle R_1, R_2 \rangle$ on $\mathfrak{l}$ as follows:

$$\gamma(X, Y, \boldsymbol{x} \otimes \boldsymbol{a}, \boldsymbol{b} \otimes \boldsymbol{y}, \boldsymbol{c} \otimes \boldsymbol{z}, \boldsymbol{w} \otimes \boldsymbol{d}) = (-{}^t\overline{X}, -{}^t\overline{Y}, \overline{\boldsymbol{w}} \otimes \overline{\boldsymbol{d}}, \overline{\boldsymbol{c}} \otimes \overline{\boldsymbol{z}}, \overline{\boldsymbol{b}} \otimes \overline{\boldsymbol{y}}, \overline{\boldsymbol{x}} \otimes \overline{\boldsymbol{a}}),$$

$$\langle R_1, R_2 \rangle = -B_{\mathfrak{l}}(R_1, \gamma R_2).$$

As in §2, we obtain

$$\begin{aligned} B_{\mathfrak{l}}(R_1, R_2) = {} & 60\mathrm{tr}\, X_1 X_2 + 60\mathrm{tr}\, Y_1 Y_2 - 60(\boldsymbol{x}_1, \boldsymbol{w}_2)(\boldsymbol{a}_1, \boldsymbol{d}_2) \\ & - 60(\boldsymbol{x}_2, \boldsymbol{w}_1)(\boldsymbol{a}_2, \boldsymbol{d}_1) - 60(\boldsymbol{y}_1, \boldsymbol{z}_2)(\boldsymbol{b}_1, \boldsymbol{c}_2) - 60(\boldsymbol{y}_2, \boldsymbol{z}_1)(\boldsymbol{b}_2, \boldsymbol{c}_1), \\ \langle R_1, R_2 \rangle = {} & 60\mathrm{tr}\, X_1^t \overline{X}_2 + 60\mathrm{tr}\, Y_1^t \overline{Y}_2 + 60(\boldsymbol{x}_1, \overline{\boldsymbol{x}}_2)(\boldsymbol{a}_1, \overline{\boldsymbol{a}}_2) + 60(\boldsymbol{y}_1, \overline{\boldsymbol{y}}_2)(\boldsymbol{b}_1, \overline{\boldsymbol{b}}_2) \\ & + 60(\boldsymbol{z}_1, \overline{\boldsymbol{z}}_2)(\boldsymbol{c}_1, \overline{\boldsymbol{c}}_2) + 60(\boldsymbol{w}_1, \overline{\boldsymbol{w}}_2)(\boldsymbol{d}_1, \overline{\boldsymbol{d}}_2). \end{aligned}$$

Thus $\langle R_1, R_2 \rangle$ is a positive definite Hermitian inner product on $\mathfrak{l}$. Using Lemma 1.2 (3) and (4), we see that γ holds the Lie bracket. Then it is clear that $\mathfrak{l}^\gamma = \{R \in \mathfrak{l} \mid \gamma(R) = R\}$ is the compact real form of $\mathfrak{l}$. Furthermore, from Proposition 1.4, we see that the group

$$E_8 = \{\alpha \in \mathrm{Aut}(\mathfrak{l}) \mid \langle \alpha R_1, \alpha R_2 \rangle = \langle R_1, R_2 \rangle\}$$

is a simply connected compact simple Lie group of type E_8.

Put $\eta = \exp(2\pi i/5) \in \boldsymbol{C}$ and define a transformation $\iota : \mathfrak{l} \to \mathfrak{l}$ as follows:

$$\iota(X, Y, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = (X, Y, \eta\alpha_1, \eta^2\alpha_2, \eta^3\alpha_3, \eta^4\alpha_4).$$

It is clear that $\iota \in E_8$ and $\iota^5 = 1$.

THEOREM 4.3. *The subgroup* $(E_8)^\iota$ *of* E_8 *is isomorphic to the group* $(SU(5) \times SU(5))/\boldsymbol{Z}_5$.

PROOF. For any $A \in SU(5)$, we define a linear transformation $\psi_1(A)$ of $\mathfrak{l}$ by

$$\begin{aligned} & \psi_1(A)(X, Y, \boldsymbol{x} \otimes \boldsymbol{a}, \boldsymbol{b} \otimes \boldsymbol{y}, \boldsymbol{c} \otimes \boldsymbol{z}, \boldsymbol{w} \otimes \boldsymbol{d}) \\ & = (\mathrm{Ad}(A)X, Y, (A\boldsymbol{x}) \otimes \boldsymbol{a}, (A\boldsymbol{b}) \otimes \boldsymbol{y}, ({}^tA^{-1}\boldsymbol{c}) \otimes \boldsymbol{z}, ({}^tA^{-1}\boldsymbol{w}) \otimes \boldsymbol{d}). \end{aligned}$$

Obviously the map $\psi_1 : SU(5) \to GL(\mathfrak{l})$ is a homomorphism. For any $Z \in \mathfrak{su}(5)$, we have $(Z, 0) \in \mathfrak{l}_0$ and

$$\begin{aligned}
&\exp(\mathrm{ad}(Z,0))(X, Y, \boldsymbol{x} \otimes \boldsymbol{a}, \boldsymbol{b} \otimes \boldsymbol{y}, \boldsymbol{c} \otimes \boldsymbol{z}, \boldsymbol{w} \otimes \boldsymbol{d}) \\
&= (\exp(\mathrm{ad}(Z))X, Y, ((\exp Z)\boldsymbol{x}) \otimes \boldsymbol{a}, \\
&\qquad ((\exp Z)\boldsymbol{b}) \otimes \boldsymbol{y}, ((\exp(-{}^tZ))\boldsymbol{c}) \otimes \boldsymbol{z}, ((\exp(-{}^tZ))\boldsymbol{w}) \otimes \boldsymbol{d}) \\
&= (\mathrm{Ad}(\exp Z)X, Y, ((\exp Z)\boldsymbol{x}) \otimes \boldsymbol{a}, \\
&\qquad ((\exp Z)\boldsymbol{b}) \otimes \boldsymbol{y}, ({}^t(\exp Z)^{-1}\boldsymbol{c}) \otimes \boldsymbol{z}, ({}^t(\exp Z)^{-1}\boldsymbol{w}) \otimes \boldsymbol{d}) \\
&= \psi_1(\exp Z)(X, Y, \boldsymbol{x} \otimes \boldsymbol{a}, \boldsymbol{b} \otimes \boldsymbol{y}, \boldsymbol{c} \otimes \boldsymbol{z}, \boldsymbol{w} \otimes \boldsymbol{d}).
\end{aligned}$$

It follows that $\psi_1(A)$ is an automorphism of $\mathfrak{l}$. Using Lemma 1.1, we have

$$\langle \psi_1(A)R_1, \psi_1(A)R_2 \rangle = \langle R_1, R_2 \rangle.$$

Hence $\psi_1(A) \in E_8$. Similarly for any $B \in SU(5)$, we define a linear transformation $\psi_2(B)$ of $\mathfrak{l}$ by

$$\begin{aligned}
&\psi_2(B)(X, Y, \boldsymbol{x} \otimes \boldsymbol{a}, \boldsymbol{b} \otimes \boldsymbol{y}, \boldsymbol{c} \otimes \boldsymbol{z}, \boldsymbol{w} \otimes \boldsymbol{d}) \\
&= (X, \mathrm{Ad}(B)Y, \boldsymbol{x} \otimes (B\boldsymbol{a}), \boldsymbol{b} \otimes ({}^tB^{-1}\boldsymbol{y}), \boldsymbol{c} \otimes (B\boldsymbol{z}), \boldsymbol{w} \otimes ({}^tB^{-1}\boldsymbol{d})).
\end{aligned}$$

It is clear that the map $\psi_2 : SU(5) \to GL(\mathfrak{l})$ is a homomorphism, $\psi_2(B) \in E_8$ and $\psi_1(A)\psi_2(B) = \psi_2(B)\psi_1(A)$.

Furthermore we define a map $\varphi : SU(5) \times SU(5) \to (E_8)^t$ by

$$\varphi(A, B) = \psi_1(A)\psi_2(B).$$

Since $\iota = \psi_1(\eta I)$, we have $\iota\varphi(A, B) = \varphi(A, B)\iota$. Thus the map φ is well-defined. Obviously φ is a homomorphism. Also we can prove that φ is surjective as in proof of Theorem 2.3.

At last, we shall show that $\mathrm{Ker}\,\varphi = \{(\eta^m I, \eta^n I) | m + 2n \equiv 0 \bmod 5\}$. For $(A, B) \in \mathrm{Ker}\,\varphi$, we have $A = \eta^m I$ and $B = \eta^n I$ $(m, n \in \boldsymbol{Z})$. Since

$$\begin{aligned}
(X, Y, \alpha_1, \alpha_2, \alpha_3, \alpha_4) &= \varphi(\eta^m I, \eta^n I)(X, Y, \alpha_1, \alpha_2, \alpha_3, \alpha_4) \\
&= (X, Y, \eta^{m+2n}\alpha_1, \eta^{2m-n}\alpha_2, \eta^{-2m-n}\alpha_3, \eta^{-m-2n}\alpha_4) \\
&= (X, Y, \eta^{m+2n}\alpha_1, \eta^{2(m+2n)}\alpha_2, \eta^{-2(m+2n)}\alpha_3, \eta^{-(m+2n)}\alpha_4),
\end{aligned}$$

we have that $m+2n \equiv 0 \bmod 5$. Then we have

$$\mathrm{Ker}\varphi = \{(I,I), (\eta I, \eta^2 I), (\eta^2 I, \eta^4 I), (\eta^3 I, \eta I), (\eta^4 I, \eta^3 I)\} \cong \boldsymbol{Z}_5.$$

Therefore $(SU(5) \times SU(5))/\boldsymbol{Z}_5 \cong (E_8)'$ has been proved. □

This theorem means that $(E_8)'$ is a subgroup of type $A_4 \times A_4$.

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