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ON GLOBAL QUASI-ANALYTIC SOLUTIONS OF THE DEGENERATE KIRCHHOFF EQUATION

By

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§ 1. Introduction

The global solvability on the Cauchy problem of the degenerate Kirchhoff equation with real analytic data has been well investigated. Then a natural question arises: Isn't it possible to weaken the regularity of the initial data to any other ultradifferentiable functions involving certain Gevery class or quasi-analyticity? It is the pourpose of this paper to show some non-small quasi-analytic initial data provides an affirmative answer for this question in the Cauchy problem

$$(1.1)\qquad \begin{cases} \partial_t^2 u + M((Au, u)_{L^2})Au = f(t, x), & (t, x) \in (0, T) \times R^n \\ u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), & x \in R^n. \end{cases}$$

Here, $Au(t,x) = \sum_{i,j=1}^n D_j(a_{ij}(x)D_i u(t,x))$, $D_j = (1/\sqrt{-1})(\partial/\partial x_j)$ and $(Au(t,\cdot), u(t,\cdot))_{L^2}$ denotes an inner product of $Au(t,x)$ and $u(t,x)$ in $L^2(R_x^n)$. The nonlinear part $M(\eta)$ is an arbitrary positive function in $C^1([0,\infty))$.

Historically, the treatment by S. N. Bernstein [2] for this problem in 1940 is the first case in search of mathematical concern. He used Fourier series and proved the existence of one dimensional time global real analytic solution of the simplest form of Kirchhoff equation

$$u_{tt} - (1 + a\int_{-\pi}^{\pi} |u_x|^2\, dx)u_{xx} = 0$$

with analytic and periodic initial data in $\Omega = [-\pi, \pi]$.

The next bench mark study obtained by S. I. Pohozaev [13] included the initial-boundary value problem in a bounded damain $\Omega \subset R_x^n$ with Dirichlet conditions and real analytic data, whose proof was due to Galerkin method.

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Different approach toward (1.1) was developed by A. Arosio and S. Spagnolo [1] whose challenge was to consolidate the solvability of (1.1) even though the nonlinear part $M(\cdot)$ degenerates, i.e. $M(\eta) \geq 0$ $(\eta \geq 0)$. This weakly hyherbolicity was retained in the study of P. D'Ancona and S. Spagnolo [3], who proved the time global existence of periodic and real analytic solutions. And their research prompted recent attempt of K. Kajitani and K. Yamaguti [7], which proved the existence and uniqness of space-time global solution of (1.1) with real analytic data and degenerate conditions for both A and $M(\eta)$.

Apart from these trends, the first breakthrough to weaken the regulality of initial data in (1.1) was brought by K. Nishihara [12], which outstands among a lot of endeavors searching relaxed regularity than real analyticity for initial data. He assumed the initial data *quasi-analytic*, and his method deeply affects the attempt of this paper. The main difference lying between his study and this paper is in the assumptions; he employed $A = -\Delta$ while we assumed A was degenerate elliptic.

Let us state assumptions.

First, let A be *degenrate elliptic*; i.e. $[a_{ij}(x); i,j = 1, \dots, n]$ is a real symmetric matrix

$$a(x,\xi) = \sigma(A)(x,\xi) = \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq 0 \tag{1.2}$$

for $x \in R^n$ and $\xi = (\xi_1, \dots, \xi_n) \in R^n$.

Each conponent of $[a_{ij}(x); i,j = 1, \dots, n]$ should be real analytic in the sense that there are constants $c_0 > 0$ and $\rho_0 > 0$ such that

$$|D_x^\alpha a_{ij}(x)| \leq c_0 \rho_0^{-|\alpha|} |\alpha|! \tag{1.3}$$

for $x \in R_x^n$, $\alpha = (\alpha_1, \dots, \alpha_n) \in N^n$ and $i,j = 1, \dots, n$.

The nonlinear part $M(\eta) \in C^1([0, \infty))$ satisfies

$$M(\eta) \geq m_0 > 0 \tag{1.4}$$

for $\eta \in [0, \infty)$.

Let us introduce several functional spaces.

For $s \in R$ and $\rho > 0$, $H_\rho^s = \{u(x) \in L^2(R_x^n);\ \langle\xi\rangle^s e^{\rho q(\xi)} \hat{u}(\xi) \in L^2(R_\xi^n)\}$ defines a Hilbert space, where $\hat{u}(\xi)$ stands for Fourier transform of u, $\langle\xi\rangle = (1 + \xi_1^2 + \dots \xi_n^2)^{1/2}$ and $q(\xi) = (\langle\xi\rangle / \log(1 + \langle\xi\rangle))$. For $\rho < 0$, $H_{-\rho}^{-s}$ defines the dual space of H_ρ^s. For $\rho = 0$, $H^s = H_0^s$ denotes the usual Sobolev space. Note that the dual space of H_ρ^s equals to $H_{-\rho}^{-s}$ for any $s, \rho \in R$.

For $\rho \in R$, let us define an operator $e^{\rho q(D)}$ from H^s_ρ to H^s as follows

$$e^{\rho q(D)}u(x) = \int_{R^n_\xi} e^{ix\cdot\xi+\rho q(\xi)}\hat{u}(\xi)\tilde{d}\xi$$

for $u \in H^s_\rho$, where $\tilde{d}\xi = (2\pi)^{-n}d\xi$. Note that $(e^{\rho q(D)})^{-1} = e^{-\rho q(D)}$ maps H^s to H^s_ρ.
Then, the result puts it;

Theorem 1.1. *Assume that* (1.2) $\sim$ (1.4) *are valid. Let* $\varepsilon > 0$ *and* $T > 0$ *be arbitraly given real numbers and* $0 < \rho_1 < \rho_0/\sqrt{n}$. *Put* $\rho(t) = \rho_1 e^{-\gamma t}$ *for* $\gamma > 0$. *Then there exists* $\gamma > 0$ *such that for any* $u_0 \in H^{4+\varepsilon}_{\rho_1}$, $u_1 \in H^{3+\varepsilon}_{\rho_1}$ *and for any* $f(t,x)$ *satisfying* $e^{\rho(t)q(D)}f \in C^0([0,T];H^3)$, *the Cauchy Problem* (1.1) *has the unique solution* $u(t,x)$ *satisfying* $e^{\rho(t)q(D)}u \in \bigcap_{j=0}^2 C^{2-j}([0,T];H^j)$.

§2. Preliminaries

If $\lambda(\xi) \in C^\infty(R^n_\xi)$ satisfies

$$1 \leq \lambda(\xi) \leq A_0\langle\xi\rangle, \quad |\partial^\alpha_\xi \lambda(\xi)| \leq A_\alpha \lambda(\xi)^{1-|\alpha|}$$

$\lambda(\xi)$ is said to be a basic weight function. A_0 and A_α are constants depending only on index. The class of pseudo-differential operators of order m, denoting S^m_λ, is the collection of $a(x,\xi) \in C^\infty(R^n_x \times R^n_\xi)$ whose derivatives satisfy

$$|a^{(\alpha)}_{(\beta)}(x,\xi)| \leq C_{\alpha\beta}\lambda(\xi)^{m-|\alpha|}$$

for $x, \xi \in R^n$ and for multi-indices α, $\beta \in N^n$, where $a^{(\alpha)}_{(\beta)}(x,\xi) = \left(\frac{\partial}{\partial\xi}\right)^\alpha \cdot \left(\frac{1}{\sqrt{-1}}\frac{\partial}{\partial x}\right)^\beta a(x,\xi)$. In the case $\lambda(\xi) = \langle\xi\rangle$, we rather write S^m in stead of $S^m_{\langle\xi\rangle}$, the usual class of pseudo-differential operators. S^m_λ defines a *Fréchét* space equipped with semi-norms $|a|^{(m)}_l = \max_{|\alpha|+|\beta| \leq l} \sup_{R^n_x \times R^n_\xi}\{a^{(\alpha)}_{(\beta)}(x,\xi)\lambda(\xi)^{-m+|\alpha|}\}$ $(l = 0,1,2,\dots)$.

$$a(x,D)u(x) = \int_{R^n_\xi} e^{ix\cdot\xi}a(x,\xi)\hat{u}(\xi)\tilde{d}\xi$$

for $u \in \mathscr{S}$, defines a pseudo-differential operator $a(x,D)$ where $\mathscr{S}$ denotes the *Schwartz space of rapidly decreasing functions* in R^n.

For function $\zeta_\nu(\xi) = (\zeta_{\nu,1}(\xi),\dots,\zeta_{\nu,n}(\xi))$, $\zeta_{\nu,k}(\xi) = \nu\sin\left(\frac{\xi_k}{\nu}\right)$ $(k = 1,\dots,n;$ $\nu > 0)$, let $\lambda_\nu(\xi) = \langle\zeta_\nu(\xi)\rangle$, then $\lambda_\nu(\xi)$ defines a weight function. $\zeta_\nu(\xi)$ and $\lambda_\nu(\xi)$

have properties

$$\begin{cases} \text{(i)} \quad |\zeta_\nu(\xi)| \le min(|\xi|, \sqrt{n}\nu) \\ \text{(ii)} \quad |\partial_\xi^\alpha \zeta_\nu(\xi)| \le Z_\alpha \lambda_\nu(\xi)^{1-|\alpha|} \\ \text{(iii)} \quad \zeta_\nu(\xi) \to \xi \quad (\nu \to \infty, \text{compact convergence}) \end{cases}$$

and

$$\begin{cases} \text{(i)} \quad |\lambda_\nu(\xi)| \le min(\langle\xi\rangle, \sqrt{1+n\nu^2}) \\ \text{(ii)} \quad |\partial_\xi^\alpha \lambda_\nu(\xi)| \le L_\alpha \lambda_\nu(\xi)^{1-|\alpha|} \\ \text{(iii)} \quad \lambda_\nu(\xi) \to \langle\xi\rangle \quad (\nu \to \infty, \text{compact convergence}) \end{cases}$$

respectively.

It might be significant to emphasize that $\zeta_{\nu,k}(\xi)$ provides approximating difference quotient to D_{x_k}. In fact, the identity $e^{ix\xi} = \cos x\xi + i \sin x\xi$ presents

$$\int e^{ix\xi} \nu \sin \frac{\xi_k}{\nu} \hat{u}(\xi) \tilde{d}\xi = \frac{\nu}{2i}\left(u\left(x_1, \dots, x_k + \frac{1}{\nu}, \dots, x_n\right) - u\left(x_1, \dots, x_k - \frac{1}{\nu}, \dots, x_n\right)\right).$$

Replacing D_{x_k} to $\zeta_{\nu,k}(\xi)$, we obtain the Cauchy problem for the *difference equation*

$$\text{(2.1)} \qquad \begin{cases} \partial_t^2 u_\nu + M(\eta_\nu(t)) A_\nu u_\nu = f(t,x), \quad (t,x) \in (0,T) \times R^n, \\ u_\nu(0,x) = u_0(x), \quad \partial_t u_\nu(0,x) = u_1(x), \quad x \in R^n, \end{cases}$$

where

$$A_\nu u_\nu(t,x) = \sum_{i,j=1}^n \zeta_{\nu,j}(D)(a_{aj}(x)\zeta_{\nu,i}(D) u_\nu(t,x))$$

and

$$\eta_\nu(t) = (A_\nu u_\nu(t,\cdot), u_\nu(t,\cdot))_{L_x^2}.$$

We must propose an energy estimate for (2.1), which will establish Lemma 2.2.

LEMMA 2.1. *Let $F(\eta) = \int_0^\eta M(s)ds$. Define $e_\nu(t)$ as*

$$\text{(2.2)} \qquad e_\nu(t)^2 = \tfrac{1}{2}\{\|\partial_t u_\nu(t,\cdot)\|_{L_x^2}^2 + F(\eta_\nu(t))\}, \quad 0 \le t \le T,$$

for $u_\nu \in C^2([0,T];L^2)$ *which is supposed to be the solution of* (2.1). *Then,*

$$e_\nu(t) \leq (\|u_1\|^2 + F(\eta_\nu(0)))^{1/2} + \int_0^T \|f(s,\cdot)\| ds \tag{2.3}$$

for $t \in [0,T]$.

PROOF. Taking time derivatives of both sides of (2.2), we have

$$\begin{aligned} 2e_\nu(t)e'_\nu(t) &= \frac{1}{2}\{(\partial_t^2 u_\nu, \partial_t u_\nu)_{L_x^2} + (\partial_t u_\nu, \partial_t^2 u_\nu)_{L_x^2} + M(\eta_\nu(t))\eta'_\nu(t)\} \\ &= Re\{-M(\eta_\nu(t))(A_\nu u_\nu, \partial_t u_\nu)_{L_x^2} + (f, \partial_t u_\nu)_{L_x^2} + M(\eta_\nu(t))(A_\nu u_\nu, \partial_t u_\nu)_{L_x^2}\} \\ &\leq \|f\|_{L_x^2}\|\partial_t u_\nu\|_{L_x^2} \leq 2\|f\|_{L_x^2} e_\nu(t)^2 \end{aligned}$$

after taking (2.1) and Schwarz inequality into account. Integration with respect to t of the inequality above completes the proof. q.e.d.

The next lemma is a direct conclusion of the previous one.

LEMMA 2.2. *Let* $u_0 \in H_0^1$, $u_1 \in L^2$ *and* $f \in C^0([0,T];L^2)$. *Then the solution of* (2.1) *satisfies*

$$\|\partial_t u_\nu(t,\cdot)\|_{L_x^2} \leq C_T \tag{2.4}$$

$$\|u_\nu(t,\cdot)\|_{L_x^2} \leq C_T \tag{2.5}$$

$$\eta_\nu(t) \leq C_T \tag{2.6}$$

for $t \in [0,T]$. *The constants may depend on* T *but not on* ν.

PROOF. (1.3) leads to

$$\eta_\nu(0) = \sum_{i,j=1}^n (a_{ij}\zeta_{\nu,j}(D)u_0, \zeta_{\nu,i}(D)u_0)_{L_x^2} \leq nc_0^2 \sum_{j=1}^n \|\zeta_{\nu,j}(D)u_0\|_{L_x^2}^2 \leq n^2 c_0^2 \|u_0\|_{H^1}^2,$$

which implies that (2.3) and (2.6) $e_\nu(t)$ has positive upper bound independent of ν. Thus (2.4) is proved. (2.4) derives (2.5). (1.4) implies

$$F(\eta_\nu(t)) = \int_0^{\eta_\nu(t)} M(s)\, ds \geq m_0 \eta_\nu(t),$$

which implies

$$\eta_\nu(t) \leq \frac{1}{m_0} F(\eta_\nu(t)) \leq \frac{2}{m_0} e_\nu(t)^2$$

for $t \in [0, T]$. Since $e_\nu(t)$ is uniformly bounded in ν, (2.6) is proved. q.e.d.

§3. Some Properties on PsDOp

To begin with, let us state some well known facts on pseudo-differential operators. Here $S^m_{\lambda_\nu}$ is the class of symbols of pseudo-differential operators introduced in the previous section.

LEMMA 3.1. (i) *Let $a_\nu(x, \xi) \in S^m_{\lambda_\nu}$ and $s \in R$. There exists a constant $C_s > 0$ independent of ν such that*

$$\|\langle D\rangle^s a_\nu(x, D)u\|_{L^2} \leq C_s|a_\nu|_{l_0}^{(m)} \|\langle D\rangle^{s+m} u\|_{L^2} \tag{3.1}$$

for $u \in H^{s+m}$.

(ii) *Let $a_\nu(x, \xi) \in S^2_{\lambda_\nu}$ be non negative. Then some positive constants C_1 and C_2 independent of ν exist and satisfy*

$$Re(a_\nu(x, D)u, u)_{L^2} \geq -C_1\|u\|_{L^2} \tag{3.2}$$

and

$$\begin{aligned}\sum_{|\alpha|=1} \{\|\langle D\rangle^s \lambda_\nu(D)^{-1} a_{\nu(\alpha)}(x, D)u\|^2_{L^2} + \|\langle D\rangle^s a^{(\alpha)}_\nu(x, D)u\|^2_{L^2}\} \\ \leq C_2\{2C_1\|\langle D\rangle^s u\|^2_{L^2} + Re(a_\nu(x, D)\langle D\rangle^s u, \langle D\rangle^s u)_{L^2}\}\end{aligned} \tag{3.3}$$

for $u \in H^{s+2}$.

PROOF. For (i), refer to [8] for example. For (ii), consult [4] and [10]. q.e.d.

Now, we are able to come up with the pseudo-differential operators characterizing quasi-analyticity. By $q_\nu(\xi)$, we define

$$q_\nu(\xi) = \frac{\lambda_\nu(\xi)}{\log(1 + \lambda_\nu(\xi))}, \tag{3.4}$$

where $\lambda_\nu(\xi)$ is the symbol prescribed in the previous section. It is easy to observe

$q_\nu(\xi)$ defines a basic weight function, so inequalities

$$1 \le q_\nu(\xi) \le Q_0\langle\xi\rangle, \quad |q_\nu^{(\alpha)}(\xi)| \le Q_\alpha q_\nu(\xi)\lambda_\nu(\xi)^{-|\alpha|} \tag{3.5}$$

are satisfied with some positive constants Q_0 and Q_α depending only on index α. Let us define another pseudo-differential operator for $u \in L_x^2$ and by

$$a(\rho, x, D)u = e^{\rho q_\nu(D)} a(x) e^{-\rho q_\nu(D)} u,$$

where $a(x)$ is a real analytic function in terms of

$$|D_x^\alpha a(x)| \le c_0 \rho_0^{-|\alpha|} |\alpha|! \quad (x \in R^n, \alpha \in N^n). \tag{3.6}$$

The next lemma 3.2 provides an asymptotic expansion of $a(\rho, x, \xi)$.

Proposition 3.2. *Suppose that $a(x)$ satisfies* (3.6). *Then, $a(\rho; x, D)$ is a pseudo-differential operator of order* 0 *whose symbol has the expansion*

$$a(\rho; x, \xi) = a(x) + \rho a_{1\nu}(x, \xi) + \rho^2 a_{2\nu}(\rho; x, \xi) + r_\nu(\rho; x, \xi), \tag{3.7}$$

where

$$a_{1\nu}(x, \xi) = -\sum_{|\alpha|=1} a_{(\alpha)}(x) q_\nu^{(\alpha)}(\xi) \in S_{\lambda_\nu}^0, \tag{3.8}$$

and $a_{2\nu}$ and r_ν respectively satisfy

$$\frac{a_{2\nu}}{q_\nu^2} \in S_{\lambda_\nu}^{-2}, \tag{3.9}$$

$$r_\nu \in S_{\lambda_\nu}^{-1}. \tag{3.10}$$

Proof. Let $u \in \mathscr{S}$. Then, we can write

$$\begin{aligned} &e^{\rho q_\nu(D)}(a \cdot e^{-\rho q_\nu(D)} u)(x) \\ &= \int e^{ix\cdot\eta + \rho q_\nu(\eta)} \tilde{d}\eta \int e^{-iy\cdot\eta} (a \cdot e^{-\rho q_\nu(D)} u)(y)\, dy \\ &= \lim_{\delta\to+0} \int e^{ix\cdot\eta + \rho q_\nu(\eta) - \delta|\eta|^2} \tilde{d}\eta \int e^{-iy\cdot\eta - \delta|x-y|^2} (a \cdot e^{-\rho q_\nu(D)} u)(y)\, dy \\ &= \lim_{\delta\to+0} \iiint e^{i(x-y)\cdot\eta + \rho q_\nu(\eta) - \delta|x-y|^2 - \delta|\eta|^2} a(y) e^{iy\cdot\xi - \rho q_\nu(\xi)} \hat{u}(\xi)\, \tilde{d}\eta\, dy\, \tilde{d}\xi \\ &= \lim_{\delta\to+0} \int e^{ix\cdot\delta} a_\delta(x, \xi) \hat{u}(\xi)\, \tilde{d}\xi, \end{aligned}$$

where $a_\delta(x,\xi)$ is given by

$$a_\delta(x,\xi) = \iint e^{-iy\cdot\eta-\delta|y|^2-\delta|\xi+\eta|^2+\rho(q_\nu(\xi+\eta)-q_\nu(\xi))}a(x+y)\,dy\,\tilde{d}\eta.$$

Let us define $w_\nu(\xi)$ by

$$\begin{aligned} q_\nu(\xi+\eta)-q_\nu(\xi) &= \sum_{j=1}^{n}\eta_j\int_0^1(\partial_{\xi_j}q_\nu)(\xi+\theta\eta)\,d\theta \\ &= \eta\cdot w_\nu(\xi,\eta), \end{aligned}$$

and we can rewrite $a_\delta(x,\xi)$ by using the Stokes formula

$$\begin{aligned} a_\delta(x,\eta) &= \int_{R^n}\int_{R^n} e^{-i(y-i\rho w_\nu(\xi,\eta))\cdot\eta-\delta|y|^2-\delta|\xi+\eta|^2}a(x+y)\,dy\,\tilde{d}\eta \\ &= \int_{R^n}\tilde{d}\eta\int_{R^n-iw_\nu(\xi,\eta)} e^{-iz\cdot\eta-\delta(z+i\rho w(\xi,\eta))^2-\delta|\xi+\eta|^2}a(x+z+i\rho w_\nu(\xi,\eta))\,dz \\ &= \int_{R^n}\tilde{d}\eta\int_{R^n} e^{-iy\cdot\eta-\delta(y+i\rho w_\nu(\xi,\eta))^2-\delta|\xi+\eta|^2}a(x+y+i\rho w_\nu(\xi,\eta))\,dy \end{aligned}$$

for $\rho<\rho_0/n$, where we write $z^2=\sum_{j=1}^n|z_j|^2$ for $z\in C^n$. Thus, by Taylor's expansion, we obtain

$$\begin{aligned} \lim_{\delta\to+0}a_\delta(x,\xi) &= Os-\iint e^{-iy\cdot\eta}a(x+y+i\rho w_\nu(\xi,\eta))\,dy\,\tilde{d}\eta \\ &= a(x+i\rho w_\nu(\xi,0))+r(\rho;x,\xi), \end{aligned}$$

where

$$\begin{aligned} r(\rho;x,\xi) &= \lim_{\delta\to 0}\iint e^{-iy\cdot\eta-\delta(y+i\rho w_\nu(\xi,\eta)^2-\delta|\xi+\eta|^2} \\ &\times\sum_{|\alpha|+|\beta|=1}\int_0^1\partial_\eta^\alpha\{D_y^\beta a(x+\theta y+i\rho w_\nu(\xi,\theta))\}\,d\theta\,dy\,\tilde{d} \end{aligned}$$

satisfies (3.10) (See, for instance, Lemma 2.4 in [8]). Taylor's expansion again to $a(x+i\rho w_\nu(\xi,0))$ yields

$$\begin{aligned} & a(x+i\rho w_\nu(\xi,0)) \\ &\quad = a(x+i\rho\partial_\xi q_\nu(\xi)) \\ &\quad = a(x)+i\rho a_{1\nu}(x,\xi)+\rho^2 a_{2\nu}(\rho;x,\xi), \end{aligned}$$

where $a_{1\nu}(x,\xi)$ and $a_{2\nu}(\rho;x,\xi)$ satisfy (3.8) and (3.9) respectively. q.e.d.

§4. A Priori Estimates of Solutions for the Transformed Problem

Let $0 < T < \infty$ and $\rho(t)$ be a positive valued function $\rho(t) = \rho_0 e^{-\gamma t}(t \in [0, \infty))$ with positive parameter γ. We shall transform unknown function u_ν in (2.1) into v_ν by means of pseudo-differential operator $e^{\rho(t)q_\nu(D)}$, where $q_\nu(D)$ is introduced in section 2.

Let $v_\nu(t,x) = e^{\rho(t)q_\nu(D)}u_\nu(t,x)$, and we observe this transforms (2.1) to

$$(4.1)\qquad \begin{cases} (\partial_t - Q_{\nu t})^2 v_\nu(t) + M(\eta_\nu(t))A_{Q_\nu}v_\nu(t) = g_\nu(t), & t \in (0,T) \\ v_\nu(0) = v_0, \\ \partial_t v_\nu(0) = v_1, \end{cases}$$

where $Q_\nu(t) = \rho(t)q_\nu(D)$, $Q_{\nu t}(t) = \rho_t(t)q_\nu(D)$ and $A_{Q_\nu} = e^{Q_\nu(t)}A_\nu e^{-Q_\nu(t)}$. Initial data and g_ν are set by

$$\begin{aligned} g_\nu(t,x) &= e^{Q_\nu(t)}f(t,x), \\ v_0(x) &= e^{Q_\nu(0)}u_0(x), \\ v_1(x) &= Q_{\nu t}(0)e^{Q_\nu(0)}u_0(x) + e^{Q_\nu(0)}u_1(x). \end{aligned}$$

It is an immediate consequence of Proposition 3.2 that A_{Q_ν} has the expansion

$$(4.2)\qquad A_{Q_\nu} = A_\nu + \rho(t)a_{1\nu}(x,D) + \rho(t)^2 a_{2\nu}(\rho(t);x,D) + r_\nu(\rho(t);x,D),$$

where

$$\begin{aligned} a_\nu(x,\xi) &= \sum_{i,j=1}^{n} a_{ij}(x)\zeta_{\nu,i}(\xi)\zeta_{\nu,j}(\xi), \\ a_{1\nu}(x,\xi) &= -\sum_{|\alpha|=1} a_{\nu(\alpha)}(x,\xi)q_\nu^{(\alpha)}(\xi), \quad \frac{a_{1\nu}}{q_\nu} \in S^1_{\lambda_\nu}, \end{aligned}$$

and

$$\frac{a_{2\nu}}{q_\nu^2} \in C^0([0,T];S^0_{\lambda_\nu}), \quad \frac{r_\nu}{q_\nu} \in C^0([0,T];S_{\lambda_\nu^0}).$$

We shall adopt an energy $E_{\nu,s}(t)$ for unknown function v_ν prescribed in (4.1). We put

$$(4.3)\qquad \begin{aligned} E_{\nu,s}(t)^2 = \frac{1}{2}\{&\|(\partial_t - Q_{\nu t})v_\nu(t)\|^2_{H^s} + M(\eta_\nu(t))(A_\nu\langle D\rangle^s v_\nu(t), \langle D\rangle^s v_\nu(t))_{L^2} \\ &+ \|\,|Q_{\nu t}|^{1/2}v_\nu(t)\|^2_{H^s} + \|\,|Q_{\nu t}|v_\nu(t)\|^2_{H^s}\} \end{aligned}$$

and

$$\eta_\nu(t) = (A_\nu u_\nu(t), u_\nu(t))_{L^2_x}.$$

Differentiating (4.3), we gain

$$\begin{aligned}
&(4.4) && 2E'_{\nu,s}(t)E_{\nu,s}(t) = \frac{1}{2}M'(\eta_\nu(t))\eta'_\nu(t)(A_\nu\langle D\rangle^s v_\nu(t), \langle D\rangle^s v_\nu(t))_{L^2}\\
&(4.5) && \quad + Re((\partial_t - Q_{\nu t})v_\nu(t), -M(\eta_\nu(t))A_{Q_\nu}v_\nu(t) + g_\nu)_{H^s}\\
&(4.6) && \quad + Re((\partial_t - Q_{\nu t})v_\nu, Q_{\nu t}(\partial_t - Q_{\nu t})v_\nu(t))_{H^s}\\
&(4.7) && \quad + M(\eta_\nu(t))Re(A_\nu\langle D\rangle^s v_\nu, (\partial_t - Q_{\nu t})\langle D\rangle^s v_\nu(t))_{L^2}\\
&(4.8) && \quad + M(\eta_\nu(t))Re(A_\nu\langle D\rangle^s v_\nu(t), Q_{\nu t}\langle D\rangle^s v_\nu(t))_{L^2}\\
&(4.9) && \quad + Re(|Q_{\nu t}|^{1/2}v_\nu(t), (\partial_t - Q_{\nu t})|Q_{\nu t}|^{1/2}v_\nu(t))_{H^s}\\
&(4.10) && \quad - Re(|Q_{\nu t}|^{1/2}v_\nu(t), |Q_{\nu t}|^{3/2}v_\nu(t))_{H^s}\\
&(4.11) && \quad + Re(|Q_{\nu t}|v_\nu(t), (\partial_t - Q_{\nu t})|Q_{\nu t}|v_\nu(t))_{H^s}\\
&(4.12) && \quad - Re(|Q_{\nu t}|^{3/2}v_\nu(t), |Q_{\nu t}|^{3/2}v_\nu(t))_{H^s},
\end{aligned}$$

after taking (4.1) into account. Obviously the terms (4.6), (4.10) and (4.12) are negative,

$$\begin{aligned}
(4.13)\quad & Re((\partial_t - Q_{\nu t})v_\nu(t), Q_{\nu t}(\partial_t - Q_{\nu t})v_\nu(t))_{H^s} - Re(|Q_{\nu t}|^{1/2}v_\nu(t), |Q_{\nu t}|^{3/2}v_\nu(t))_{H^s}\\
&\quad - Re(|Q_{\nu t}|^{3/2}v_\nu(t), |Q_{\nu t}|^{3/2}v_\nu(t))_{H^s}\\
&= -\| |Q_{\nu t}|^{1/2}(\partial_t - Q_{\nu t})v_\nu(t)\|^2_{H^s} - \| |Q_{\nu t}|v_\nu(t)\|^2_{H^s} - \| |Q_{\nu t}|^{3/2}v_\nu(t)\|^2_{H^s}
\end{aligned}$$

and (4.5) and (4.7) provides

$$\begin{aligned}
& Re((\partial_t - Q_{\nu t})v_\nu(t), -M(\eta_\nu(t))A_{Q_\nu}v_\nu(t) + g_\nu)_{H^s}\\
&\quad + Re(M(\eta_\nu(t))\langle D\rangle^{-s}A_\nu\langle D\rangle^s v_\nu, (\partial_t - Q_{\nu t})v_\nu(t))_{H^s}\\
&\le \|g_\nu\|_{H^s}\|(\partial_t - Q_{\nu t})v_\nu(t)\|_{H^s}\\
&\quad + M(\eta_\nu(t))Re((\partial_t - Q_{\nu t})v_\nu(t), (\langle D\rangle^{-s}A_\nu\langle D\rangle^s - A_{Q_\nu})v_\nu(t))_{H^s}\\
(4.14)\quad &\le 2^{1/2}\|g_\nu\|_{H^s}E_{\nu,s}(t) + \frac{1}{4}\| |Q_{\nu t}|^{1/2}(\partial_t - Q_{\nu t})v_\nu(t)\|^2_{H^s}\\
&\quad + M(\eta_\nu(t))^2\| |Q_{\nu t}|^{-1/2}((\langle D\rangle^s A_\nu\langle D\rangle^{-s} - A_{Q_\nu})v_\nu(t)\|^2_{H^s}.
\end{aligned}$$

Since

$$\langle D\rangle^{-s}A_\nu\langle D\rangle^s = A_\nu + \tilde{r}_\nu(x, D), \quad \tilde{r}_\nu(x,\xi) \in S^1_{\lambda_\nu}$$

and using several symbol calculations together with Lemma 3.1 and (4.2), we will see the last term of (4.14) has estimate

$$\begin{aligned}
&M(\eta_\nu(t))^2\| \, |Q_{\nu t}|^{-1/2}(\langle D\rangle^{-s}A_\nu\langle D\rangle^s - A_{Q_\nu})v_\nu(t)\|_{H^s}^2 \\
&\quad \le M(\eta_\nu(t))^2\{4\rho(t)^2\| \, |Q_{\nu t}|^{-1/2}a_{1\nu}(x,D)v_\nu(t)\|_{H^s}^2 \\
&\qquad + 4\rho(t)^4\| \, |Q_{\nu t}|^{-1/2}a_{2\nu}(\rho(t);x,D)v_\nu(t)\|_{H^s}^2 \\
&\qquad + 4\| \, |Q_{\nu t}|^{-1/2}r_\nu(\rho(t);x,D)v_\nu(t)\|_{H^s}^2 \\
&\qquad + 4\| \, |Q_{\nu t}|^{-1/2}\tilde{r}_\nu(x,D)v_\nu\|_{H^s}^2\}
\end{aligned}$$

$$\begin{aligned}
(4.15) \qquad &\le M(\eta_\nu(t))^2\left\{c\frac{1}{\gamma^2}(A_\nu|Q_{\nu t}|^{1/2}v_\nu(t), |Q_{\nu t}|^{1/2}v_\nu(t))_{H^s} + c\frac{1}{\gamma^2}\| \, |Q_{\nu t}|^{1/2}v_\nu\|_{H^s}^2\right. \\
&\qquad \left. + c\frac{1}{\gamma^4}\| \, |Q_{\nu t}|^{3/2}v_\nu(t)\|_{H^s}^2 + c\left(\frac{e^{2\gamma T}}{\gamma^2} + \frac{e^{4\gamma T}}{\gamma^4}\right)E_{\nu,s}(t)_{H^s}^2\right\}.
\end{aligned}$$

In fact, repeated applications of Lemma 3.1 bring about

$$\begin{aligned}
&4\rho(t)^2\| \, |Q_{\nu t}|^{-1/2}a_{1\nu}(x,D)v_\nu(t)\|_{H^s}^2 \\
&\quad \le 4(n+1)\rho(t)^2\sum_{|\alpha|=1}\left\|\frac{1}{|\gamma\rho(t)|}\, q_\nu(D)^{-1/2}q_\nu^{(\alpha)}(D)A_{\nu(\alpha)}q_\nu(D)^{-1/2}\cdot|Q_{\nu t}|^{1/2}v_\nu(t)\right\|_{H^s}^2 \\
&\quad \le \frac{c_n}{\gamma^2}\{Re(A_\nu\langle D\rangle^s|Q_{\nu t}|^{1/2}v_\nu(t), \langle D\rangle^s|Q_{\nu t}|^{1/2}v_\nu(t))_{L^2} + c\| \, |Q_{\nu t}|^{1/2}v_\nu(t)\|_{H^s}^2\}.
\end{aligned}$$

Likewise we get

$$\begin{aligned}
&4\rho(t)^4\| \, |Q_{\nu t}|^{-1/2}a_{2\nu}(x,D)v_\nu(t)\|_{H^s}^2 \\
&\quad = 4\rho(t)^4\left\|\frac{1}{\gamma\rho(t)|^2}\, q_\nu(D)^{-1/2}a_{2\nu}q_\nu(D)^{3/2}\cdot|Q_{\nu t}|^{3/2}v_\nu\right\|_{H^s}^2 \\
&\quad \le \frac{c}{\gamma^4}\| \, |Q_{\nu t}|^{3/2}v_\nu\|_{H^s}^2, \\
&4\| \, |Q_{\nu t}|^{-1/2}r_\nu(\rho(t);x,D)v_\nu(t)\|_{H^s}^2 \\
&\quad = 4\left\|\frac{1}{|\gamma\rho(t)|}\, q_\nu(D)^{-1/2}r_\nu q_\nu(D)^{-1/2}\cdot|Q_{\nu t}|^{1/2}v_\nu\right\|_{H^s}^2 \\
&\quad \le \frac{c}{\gamma^2\rho(t)^2}\| \, |Q_{\nu t}|^{1/2}v_\nu\|_{H^s}^2 \le \frac{ce^{2\gamma T}}{\gamma^2}\, E_{\nu,s}(t)^2,
\end{aligned}$$

and

$$
\begin{aligned}
&4\| \, |Q_{vt}|^{-1/2}\tilde{r}_v(x, D)v_v(t)\|_{H^s}^2 \\
&= 4\,\frac{1}{|\gamma\rho(t)|}\|q_v(D)^{-1/2}\tilde{r}_v v_v\|_{H^s}^2 \\
&\leq \frac{c}{\gamma^2\rho(t)^2}\|\lambda_v(D)q_v(D)^{1/2}v_v\|_{H^s}^2 \leq \frac{c}{\gamma^2\rho(t)^2}\|q_v(D)v_v\|_{H^s}^2 \\
&= \frac{c}{\gamma^3\rho(t)^3}\| \, |Q_{vt}|v_v\|_{H^s}^2 \leq \frac{ce^{4\gamma T}}{\gamma^4}\,E_{v,s}(t)^2.
\end{aligned}
$$

Thus we have checked (4.15).

Since the $S^1_{\lambda_v}$-term of $\sigma(|Q_{vt}|^{-1/2}A_v|Q_{vt}|^{1/2})$ is purely imaginary number, that is essentially positive valued $S^2_{\lambda_v}$-symbol with $S^0_{\lambda_v}$-remainder. So an application of lemma 3.1 to (3.3) derives,

(4.16)

$$
\begin{aligned}
(4.8) &= -M(\eta_v(t))Re(|Q_{vt}|^{1/2}A_v|Q_{vt}|^{-1/2}\cdot\langle D\rangle^s Q_{vt}|^{1/2}v_v, \langle D\rangle^S|Q_{vt}|^{1/2}v_v)_{L^2} \\
&\leq -M(\eta_v(t))Re(A_v\langle D\rangle^s|Q_{vt}|^{1/2}v_v, \langle D\rangle^s|Q_{vt}|^{1/2}v_v)_{L^2} + cM(\eta_v(t))\| \, |Q_{vt}|^{1/2}v_v\|_{H^s}^2 \\
&\leq -M(\eta_v(t))(A_v|Q_{vt}|^{1/2}v_v, |Q_{vt}|^{1/2}v_v)_{H^s} + cM(\eta_v(t))E_{v,s}(t)^2.
\end{aligned}
$$

Meanwhile we can compute

$$
\begin{aligned}
(4.9) + (4.11) &= Re(|Q_{vt}|^{1/2}v_v, \partial_t(|Q_{vt}|^{1/2}v_v) - Q_{vt}|Q_{vt}|^{1/2}v_v)_{H^s} \\
&\quad + Re(|Q_{vt}|w_v, \partial_t(|Q_{vt}|v_v) - Q_{vt}|Q_{vt}|v_v)_{H^s} \\
&= -\frac{\gamma}{2}\| \, |Q_{vt}|^{1/2}v_v\|_{H^s}^2 + Re(|Q_{vt}|^{1/2}v_v, |Q_{vt}|^{1/2}(\partial_t - Q_{vt})v_v)_{H^s} \\
&\quad + Re(|Q_{vt}|v_v, |Q_{vt}|(\partial_t - Q_{vt})v_v)_{H^s} - \gamma\| \, |Q_{vt}|v_v\|_{H^s}^2 \\
&\leq \frac{1}{3}\| \, |Q_{vt}|^{3/2}v_v\|_{H^s}^2 + \tfrac{3}{4}\| \, |Q_{vt}|^{1/2}(\partial_t - Q_{vt})v_v\|_{H^s}^2 \\
&\quad - \gamma\left\| \, |Q_{vt}|v_v\right\|_{H^s}^2 - \left(\frac{\gamma}{2} - 1\right)\left\| \, |Q_{vt}|^{1/2}v_v\right\|_{H^s}^2
\end{aligned}
\tag{4.17}
$$

by using Schwarz inequality and relation $\partial_t|Q_{vt}| = -\gamma|Q_{vt}|$.

Summing up from (3.13) to (3.16), we come to

$$
\begin{aligned}
&2E_{v,s}(t)E'_{v,s}(t) \\
&\le \frac{1}{2}M'(\eta_v(t))\eta'_v(t)(A_v\langle D\rangle^s v_v, \langle D\rangle^s v_v)_{L^2} + \|(\partial_t - Q_{vt})v_v\|_{H^s}\|g_v\|_{H^s} \\
&\quad - \left(\frac{1}{3} - \frac{cm_0^4}{\gamma^4}\right)\| \, |Q_{v4}|^{3/2}v_v\|_{H^s}^2 - \left(1 - \frac{cm_0^3}{\gamma^2}\right)(A_v|Q_{vt}|^{1/2}v_v, |Q_{vt}|^{1/2}v_v)_{H^s} \\
&\quad - \left(\frac{\gamma}{2} - 1 - M_0 - \frac{M_0^2 e^{2T\gamma}}{\gamma^2} - \frac{M_0^2 e^{4T_\gamma}}{\gamma^4}\right)E_{v,s}(t)^2 \\
&\le \frac{1}{2}M'(\eta_v(t))\eta'_v(t)(A_v v_v, v_v)_{L^2_x} \\
&\quad + 2^{1/2}\|g_v\|_{H^s}E_{v,s}(t) + M_0^2\left(\frac{e^{2T\gamma}}{\gamma^2} + \frac{e^{4T_\gamma}}{\gamma^4}\right)E_{v,s}(t)^2, \qquad (4.18)
\end{aligned}
$$

if we take $\gamma > 0$ so large that

$$\gamma > max\{(cm_0^3)^{1/2}, (3cm_0^4)^{1/4}, 2(M_0+1)\},$$

where

$$m_0 \le M(\eta_v(t)) \le \sup_{0\le\eta\le C_T} M(\eta) = M_0. \qquad (4.19)$$

The constant C_T appeared in (4.19) and in (2.6) is same.

Only (4.4) remains unsolved.

LEMMA 4.1. *Let T be a linear, symmetric and positive operator in L^2, then*

$$|(Tu, v)_{H^s}| \le (Tu, u)_{L^2}^{1/2}(Tv, v)_{L^2}^{1/2}.$$

The statement is rather elementary and acceptable without proof.

It is a quick result of Lemma 4.1 that

$$
\begin{aligned}
\eta'_v(t) &= 2Re(A_v u_v(t), \partial_t u_v(t))_{L^2_x} \\
&\le 2(A_v u_v, u_v)_{L^2_x}^{1/2}(A_v\partial_t u_v, \partial_t u_v)_{L^2_x}^{1/2} \\
&\le c(A_v\partial_t u_v, \partial_t u_v)_{L^2_x}^{1/2},
\end{aligned}
$$

where we used (2.6) again. With this inequality and $M'(\eta_v(t)) \le \max_{0\le\eta\le C}|M'(\eta)|$, we find

$$\eta'_v(t) \le cna_0\|\lambda_v(D)\partial_t u_v\|_{L^2_x}$$

hence

$$\frac{1}{2}M'({}_{\nu}(t))\eta_{\nu}'(t)(A_{\nu}\langle D\rangle^{s}u_{\nu},\langle D\rangle^{s}u_{\nu})_{L_x^2} \leq \frac{cna_0}{m_0}\|\lambda_{\nu}(D)\partial_t u_{\nu}\|_{L^2}E_{\nu,s}(t)^2. \tag{4.20}$$

LEMMA 4.2. *Let* $P_0(s) = \exp(\rho_1 e^{-\gamma T s}/\log(1+s))$. *Then,* $N_0(s^{1/2})$ *is continuous, increasing, and convex function if we define*

$$N_0(s) = \begin{cases} cs^{\sigma} & (\sigma > 2, 0 \leq s \leq s_0) \\ P_0(s) + (cs_0^{\sigma} + P_0(s_0)) & (s_0 \leq s). \end{cases}$$

LEMMA 4.3 [5]. *Let* ϕ *and* ψ *be continuous and strictly increasing. We define* $\mathcal{M}_{\phi}(f)$ *by*

$$\mathcal{M}_{\phi}(f) = \phi^{-1}\left(\int \phi(f(x))q(x)\,dx\right)$$

where f *and* q *are the nonnegative function such that* $\int q(x)dx = 1$ *and* $\int \phi(f(x))q(x)dx$ *exists. Then in order that* $\mathcal{M}_{\phi}(f) \leq \mathcal{M}_{\psi}(f)$ *for all* f, *it is necessary and sufficient that* $\psi \circ \phi^{-1}$ *should be convex.*

Lemma 4.3 is a direct quotation from famous [5], we accept it here without *proof.*

Now let us try analogous estimates for $\|\lambda_{\nu}(D)\partial_t u_{\nu}(t)\|_{L_x^2}$ like Nisihara did. When $\int_{R^n}|\partial_t u_{\nu}(t,x)|^2\,dx \geq 1$, we see $(1/\|\partial_t u_{\nu}\|_{L_x^2}) \leq 1$, so

$$\begin{aligned} \|\lambda_{\nu}(D)\partial_t u_{\nu}(t)\|_{L^2} &= (\|\partial_t u_{\nu}\|_{L_x^2})\left(\int_{R_\xi^n}\frac{|\partial_t\hat{u}_{\nu}(t,\xi)|^2}{\|\partial_t u_{\nu}\|_{L_x^2}}\lambda_{\nu}(\xi)^2\,d\xi\right)^{1/2} \\ &\leq CN_0^{-1}\left(\int_{R_\xi^n}\frac{|\partial_t\hat{u}_{\nu}(t,\xi)|^2}{\|\partial_t\hat{u}_{\nu}\|_{L_x^2}}N_0(\lambda_{\nu}(\xi))\,d\xi\right) \\ &\leq CN_0^{-1}\left(\int_{R_\xi^n}N_0(\lambda_{\nu}(\xi))|\partial_t\hat{u}_{\nu}(t,\xi)|^2\,d\xi\right) \end{aligned} \tag{4.21}$$

is assured if we recall Lemma 4.2 and Lemma 4.3.

When $\int_{R^n}|\partial_t u_{\nu}(t,x)|^2\,dx < 1$, let us adopt

$$p_{\theta,\nu}(\xi,t) = (1 - \|\partial_t u_{\nu}\|_{L_x^2}^2)\varphi_{\theta}(\xi),$$

where $\varphi_{\theta}(\xi) = \theta^{-n}\varphi(\theta^{-1}\xi)$ with $\int \varphi(\xi)d\xi = 1$ and $0 < \theta < 1$ is a Friedrichs' mollifier. It is easily checked that $p_{\theta,\nu}(\xi,t)$ satisfies

$$0<\int_{R^n_\xi} p_{\theta,\nu}(\xi,t)\,d\xi,\quad \int_{R^n_\xi}(|\partial_t\hat{u}_\nu(t,\xi)|^2+p_{\theta,\nu}(\xi,t))\,d\xi=1.$$

Applying $p_{\theta,\nu}$ to Lemma 4.3, we get

$$\begin{aligned}\|\lambda_\nu(\xi)\partial_t u_\nu(t)\|_{L^2_x} &\le \left(\int_{R^n_\xi}\lambda_\nu(xi)^2(|\partial_t\hat{u}_\nu(t,xi)|^2+p_{\theta,\nu}(\xi,t))\,d\xi\right)^{1/2}\\ &\le N_0^{-1}\left(\int_{R^n_\xi}N_0(\lambda_\nu(\xi))(|\partial_t\hat{u}_\nu(t,\xi)|^2+p_{\theta,\nu}(\xi,t))\,d\xi\right)\\ &\le N_0^{-1}\left(\int_{R^n_\xi}N_0(\lambda_\nu(\xi))(|\partial_t\hat{u}_\nu(t,\xi)|^2\,d\xi+\sup_{|\xi|\le\theta}N_0(\langle\xi\rangle)\right),\end{aligned}\tag{4.22}$$

which implies

$$\|\lambda_\nu(D)\partial_t u_\nu(t)\|_{L^2_x}\le N_0^{-1}\left(\int_{R^n_\xi}N_0(\lambda_\nu(\xi))|\partial_t\hat{u}_\nu(t,\xi)|^2\,d\xi+N_0(1)\right)$$

by letting $\theta\to 0$. Since $N_0(\lambda_\nu(\xi))\le Ce^{2p(t)q_\nu(\xi)}$, we have reached for $s\ge 0$,

$$\begin{aligned}\|\lambda_\nu(D)\partial_t u_\nu(t)\|_{L^2_x}&\le CN_0^{-1}\left(c\int_{R^n_\xi}|e^{p(t)q_\nu(\xi)}\langle\xi\rangle^s\hat{u}_\nu(t,\xi)|^2\,d\xi+N_0(1)\right)\\ &\le CN_0^{-1}(c(\|(\partial_t-Q_{\nu t})v_\nu(t)\|^2_{L^2_x}+1))\\ &\le CN_0^{-1}(c(E_{\nu,s}(t)^2+1)).\end{aligned}\tag{4.23}$$

Assembling (4.18), (4.20) and (4.23), we have found the differential ineqality that $E_{\nu,s}(t)$ must obey:

$$E'_{\nu,s}(t)\le 2^{1/2}\|g(t)\|_{H^s}+\frac{c(T)}{m_0}E_{\nu,s}(t)N_0^{-1}(c(E_{\nu,s}(t)^2+1)),\quad 0\le t\le T,\tag{4.24}$$

where $g(t)=e^{p(t)q(D)}f(t)$.

Now we can state our final conclusion.

PROPOSITION 4.4. *Let $T>0$ and $s\ge 0$. $E_\nu(t)$ defined by* (4.3) *satisfies* (4.24) *in $0\le t\le T$. Moreover, if $v_\nu(t)\in C^0([0,T];H^s)$ and $E_{\nu,s}(0)$ takes independent value of ν, $E_{\nu,s}(t)$ is uiformly bounded in ν and t, namely*

$$E_{\nu,s}(t)\le B^{-1}\left(\int_0^T\|g(\tau)\|_{H^s}\,d\tau\right)\quad(0\le t\le T),$$

where B^{-1} is a positive function in $C([0,\infty))$.

To prove Proposotion 4.4, we have to quote a lemma.

LEMMA 4.5 [12]. *Let* $\alpha \in C([0,\infty))$ *and* $\beta \in C((0,\infty))$ *be nondecreasing functions whose ranges are* $[0,\infty)$ *and* $(0,\infty)$ *respectively. Let* $\gamma \in C([0,\infty)$ *be a nonnegative function. If they satisfy*

$$\alpha(t) \leq c + \int_0^t (\gamma(s) + \beta(\alpha(s)))\, ds \quad (0 \leq t < \infty),$$

where c *is a positive constant, then*

$$\alpha(t) \leq B^{-1}(B_0) < \infty \quad \left(0 \leq t < \int_0^t \left(\frac{\gamma(\tau)}{\beta(c)} + 1\right) d\tau \leq B_0\right)$$

for any fixed number B_0 *less than* $B(\infty)$, *where*

$$B(t) = \int_c^t \frac{ds}{\beta(s)} \quad (t \geq 0).$$

Moreover, if $B(\infty) = \infty$, *then*

$$\alpha(t) \leq B^{-1}(t) \tag{4.25}$$

for all $t \geq 0$.

PROOF. Let $h(t) = c + \int_0^t (\gamma(s) + \beta(\alpha(s)))\, ds$. Then the definition of $B(t)$ derives

$$\begin{aligned} \frac{d}{dt} B(h(t)) &= \frac{\gamma(t) + \beta(\alpha(t))}{\beta(h(t))} \\ &\leq \frac{\gamma(t)}{\beta(c)} + 1. \end{aligned}$$

Hence

$$B(h(t)) \leq B(h(0)) + \int_0^t \left(\frac{\gamma(\tau)}{\beta(c)} + 1\right) d\tau = \int_0^t \left(\frac{\gamma(\tau)}{\beta(c)} + 1\right) d\tau$$

and

$$\alpha(t) \leq h(t) \leq B^{-1}\left(\int_0^t \left(\frac{\gamma(\tau)}{\beta(c)} + 1\right) d\tau\right).$$

If $B : [c,\infty) \to [0, B(\infty))$ then $B^{-1} : [0, B(\infty)) \to [c,\infty)$, and if there exists some upper bound $B_0 < B(\infty)$ and we get

$$B(h(t)) \le \int_0^t \left(\frac{\gamma(\tau)}{\beta(c)} + 1\right) d\tau \le B_0 < B(\infty),$$

therefore

$$\alpha(t) \le h(t) \le B^{-1}\left(\int_0^t \left(\frac{\gamma(\tau)}{\beta(c)} + 1\right) d\tau\right) < \infty. \qquad \text{q.e.d.}$$

If we accept Lemma 4.5, and if we take $c = E_{v,s}(0) + 1$, $\alpha(t) = E_{v,s}(t)$, $\beta(t) = tN_0^{-1}(t^2 + 1)$ and $\gamma(t) = \|g(t)\|_{H^1}$, the inequality in Proposition 4.5 immediately follows since

$$B(\infty) = \int_c^\infty \frac{d\tau}{\tau N_0^{-1}(\tau^2 + 1)} = \infty,$$

which characterizes *quasi-analyticity.*

§5. Local Solution

Our task here is presenting a proposition which gurantees the existence of local solution of the Cauchy problem (4.1) for every fixed v. Throughout this section we employ the abbreviation v for v_v to avoid complexity.

PROPOSITION 5.1. *Let* $\rho(t) = \rho_1 e^{-\gamma t}$ *and* $Q_v = \rho(t) q_v(D)$ *and* $s \ge 0$. *Suppose* $v_0, v_1 \in H^s$. *For each fixed* v, *the Cauchy problem*

$$\text{(5.1)} \qquad \begin{cases} (\partial_t - Q_{vt})^2 v(t) + M(\eta_v(t)) A_{Q_v} v = g_v(t), & t_0 \le t \le T \\ v(t_0) = v_0, \partial_t v(t_0) = v_1 \end{cases}$$

has a unique solution $v(t) \in C^2([t_0, t_0 + T_v]; H^s)$.

At first, let us assure the solution of the ordinary differential equation

$$\begin{cases} (\partial_t - Q_{vt}) w(t,x) = h(t,x), & 0 \le t \le T \\ w(t_0) = w_0 \end{cases}$$

can be written by

$$w(t,x) = K[h](t,x) + e^{Q_v(t) - Q_v(t_0)} w_0(x)$$

if we define an operator K by

$$K[h] = \int_{t_0}^t e^{Q_v(t) - Q_v(s)} h(s,x)\, ds.$$

This operator rewrites (5.1)

$$v(t,x) = K \circ K[F[v]](t,x) + K[e^{Q_\nu(t)-Q_\nu(t_0)}(v_1 - Q_{\nu t}(t_0))v_0](t,x) + e^{Q_\nu(t)-Q_\nu(t_0)}v_0(x), \tag{5.2}$$

where $F[v] = g_\nu - M(\eta(t))A_{Q_\nu}v$, and then we are able to define a sequence $\{v_{(k)}\}_{k=0,1,2,\dots}$ as

$$\begin{cases} v_{(0)}(t,x) = K[e^{Q_\nu(t)-Q_\nu(t_0)}(v_1 - Q_{\nu t}(t_0)v_0)](t,x) + e^{Q_\nu(t)-Q_\nu(t_0)}v_0(x) \\ v_{(k)}(t,x) = K \circ K[F[v_{(k-1)}]](t,x) + v_{(0)}(x), k = 1,2,\dots, \end{cases} \tag{5.3}$$

which would be convergent in $C^2([t_0, t_0 + T_\nu]; H^s)$. Hence we get

$$v_{(k+1)} - v_{(k)} = K \circ K[F[v_{(k)}] - F[v_{(k-1)}]].$$

All we have to do is to show that F is Lipschtz continuous in metric $\|\cdot\|_{L^2}$ and $LK \circ K$ defines a contraction for sufficiently small life span T_ν, where L is the Lipschtz constant. Since $\rho(t)$ is decreasing, we have

$$\begin{aligned} \|K[h](t,\cdot)\|_{H^s} &\le \int_{t_0}^{t} \|e^{(\rho(t)-\rho(s))q_\nu(D)}h(s,x)\|_{H^s}\,ds \\ &\le \int_{t_0}^{t} \|e^{(\rho(t)-\rho(s))q_\nu(\xi)}\hat{h}(s)\|_{H^s}\,ds \\ &\le \int_{t_0}^{t_0+T_\nu} \|\hat{h}(s)\|_{H^s}\,ds \le T_\nu \sup_{t_0 \le s \le t_0+T_0} \|h(s)\|_{H^s} \end{aligned}$$

and also get

$$\sup_{0\le t\le T_\nu} \|K \circ K[h](t,\cdot)\|_{H^s} \le T_\nu^2 \sup_{t_0 \le t \le t_0+T_\nu} \|h\|_{H^s}. \tag{5.4}$$

Recalling $|\lambda_\nu(\xi)| \le min(\langle\xi\rangle, \sqrt{1+n\nu^2})$, and we get

$$\begin{aligned} |\eta_\nu(t,u_{(k)}) - \eta_\nu(t,u_{(k-1)})| &= |(A_\nu(u_{(k)} - u_{(k-1)}), u_{(k)})_{L^2} + (A_\nu u_{(k-1)}, u_{(k)} - u_{(k-1)})_{L^2}| \\ &\le C_{\nu,n,a_0}(\|u_{(k)}\|_{L^2} + \|u_{(k-1)}\|_{L^2})\|u_{(k)} - u_{(k-1)}\|_{H^s} \le 2C_{\nu,n,a_0}\|v_{(k)} - v_{(k-1)}\|_{H^s}, \end{aligned}$$

where C_{ν,n,a_0} is independent of k and $u_{(k)} = e^{-\rho(t)q_\nu(D)}v_{(k)}$.

Now let us make sure $\|v_{(k)}\|_{H^s} \le C_\nu$ holds uniformly in k. We will check it inductively. First, we can assume $\|v_{(0)}\|_{H^s} \le C_{0\nu}$ and $\|g_\nu\|_{H^s} \le C_{0\nu}$. We may take $C_\nu \ge C_{0\nu}$. Then this assumption and (5.4) yield $\|v_{(k)}\|_{L^2} \le T_\nu^2\|F[v_{(k-1)}]\|_{H^s} + C_{0\nu}$

and

$$\|F[v_{(k-1)}]\|_{L_2} = \|g_\nu - M(\eta_\nu(e^{-Q_\nu}v_{(k-1)})A_{Q_\nu}v_{(k-1)}\|_{L^2}$$
$$\leq C_{0\nu} + \left(\sup_{0\leq\eta\leq C_{0\nu}C_\nu} M(\eta)\right)C_{0\nu}C_\nu.$$

The last inequality is true because A_{Q_ν} is a H^s-bounded operator for each fixed ν and $\|A_{Q_\nu}v_{(k-1)}\|_{H^s} \leq C'\|v_{(k-1)}\|_{H^s} \leq C'C_\nu$ if we assume $\|v_{(k-1)}\|_{H^s} \leq C_\nu$ and take $C' \leq C_{0\nu}$. Hence we get

$$\|v_{(k)}\|_{L^2} \leq T_\nu^2\left(C_{0\nu} + C_{0\nu}C_\nu \sup_{0\leq\eta\leq C_{0\nu}C_\nu} M(\eta)\right) + C_{0\nu} \leq C_\nu$$

if we choose T_ν so small that $T_\nu^2 \leq \left(\dfrac{C_\nu - C_{0\nu}}{C_{0\nu} + C_{0\nu}C_\nu \sup_{0\leq\eta\leq C_{0\nu}C_\nu} M(\eta)}\right)$. Thus, our assertion is verified.

With the last result we get

$$\begin{aligned}
&\|F[v_{(k)}] - F[v_{(k-1)}]\|_{H^s}\\
&\quad\leq M(\eta_\nu(t,u_{(k)}))\|A_{Q_\nu}(v_{(k)} - v_{(k-1)})\|_{H^s} + |M(\eta_\nu(t,u_{(k)}))\\
&\qquad - M(\eta_\nu(t,u_{(k-1)}))|\,\|A_{Q_\nu}v_{(k-1)}\|_{H^s}\\
&\quad\leq c_\nu \max_{0\leq\eta\leq c} M(\eta)\|v_{(k)} - v_{(k-1)}\|_{H^s}\\
&\qquad + c_\nu \max_{0\leq\eta c}|M'(\eta)|\,\|v_{(k-1)}\|_{H^s}|\eta_\nu(t,u_{(k)}) - \eta_\nu(t,u_{(k-1)})|\\
&\quad\leq L_\nu\|v_{(k)} - v_{(k-1)}\|_{H^s},
\end{aligned}$$

hence

$$\sup_{t_0\leq t\leq t_0+T_\nu}\|v_{(k+1)} - v_{(k)}\|_{H^s} \leq L_\nu T_\nu^2 \sup_{t_0\leq t\leq t_0+T_\nu}\|v_{(k)} - v_{(k-1)}\|_{H^s}.$$

If we take T_ν so small that $L_\nu T_\nu^2 < 1$, we can conclude the sequence we defined above converges to $v_\nu \in C^2([t_0, t_0 + T_\nu; H^s)$.

Note each initial surface $t = t_0$ affects neither the Lipschitz constant nor T_ν. So we are able to prolong the gained solution $v_\nu(t) \in C^2([t_0, t_0 + T_\nu]; H^s)$ to $v_\nu(t) \in C^2([t_0 + T_\nu, t_0 + 2T_\nu]; H^s)$. Iteration of these process up to T, an arbitrary given edge, makes our solution turn out to be global one. It is clear that $v_\nu(t)$ also belongs to $C^1([0,T]; H^s)$ and $C^0([0,T]; L^2)$ by (5.2). Thus,

PROPOSOTION 5.2. *If v_0, $v_1 \in H^s$ and $g_\nu(t) \in C^0([0,T]; H^s)$, the Cauchy problem* (3.1) *has a unique solution $v_\nu(t) \in C^2([0,T]; H^s)$.*

§6. Proof of Theorem 1.1

In section 4, we found $E_{\nu,s}(t)$ is *uniformly bounded*; in this section, we will prove $\{v_\nu(t)\}_{\nu>0}$ is *equi-continuous* in $C^0([0,T];H^s)$. Then, Ascoli-Arzela's theorem gurantees the existence of subsequence $\{v_{\nu p}(t)\}_{p=1,2,\dots}$ converging in $C^0([0,T];H^s)$. The way of picking up subsequence is the same as the proof of original version of Ascoli-Arzela's theorem (c.f. Kumano-Go [8]).

We have already proved in Proposition 4.4 that for $s \geq 0$

$$E_{\nu,s}(t) \leq C, \quad 0 \leq t \leq T$$

and replacement of $w_\nu(t)$ with $v_\nu(t)$ in (4.3) and (5.1) yields

$$\|(\partial_t - Q_{\nu t})v_{\nu(t)}\|_{H^s} \leq C, \quad \| \, |Q_{\nu t}| v_\nu(t)\|_{H^s} \leq C.$$

These two lead to $\|\partial_t v_\nu(t)\|_{H^s} - \| \, |Q_{\nu t}| v_\nu(t)\|_{H^s} \leq \|(\partial_t - Q_{\nu t})v_\nu(t)\|_{H^s} \leq C$ hence $\|\partial_t v_\nu(t)\|_{H^s} \leq 2C$. Thus

$$\|v_\nu(t)\|_{H^s} \leq \int_0^T \|\partial_t v_\nu(\tau)\|_{H^s} d\tau + \|v_0\|_{H^s} \leq 2CT + \|v_0\|_{H^s}, \tag{6.1}$$

which implies $v_\nu(t)$ is uniformly bounded in $C^0([0,T];H^s)$. Integration of both sides of $\|\partial_t v_\nu(t)\|_{H^s} \leq 2C$ derives

$$\|v_\nu(t) - v_\nu(t')\|_{H^s} \leq \int_{t'}^t \|\partial_t v_\nu(\tau)\|_{H^s} d\tau \leq 2C|t - t'|, \tag{6.2}$$

which means $v_\nu(t)$ equi-continuous. Therefore there exists a subsequence $\{v_{\nu_p}(t)\}_{p=1,2,\dots}$ *weakly* converging to $v(t) \in C^0([0,T];H^s)$, where $v(t) = \lim_{p\to\infty} v_{\nu_p}(t)$. If we set $s > 2$, $u(t) = e^{-Q(t)}v(t)$ would be the solution to (1.1). However, it is uncertain yet that $\eta_{\nu_p}(t) \to (Au(t,\cdot), u(t,\cdot))_{L^2}$ as $p \to \infty$ and $v(t)$ satisfies (1.1).

Back to the previous section, (3.1) has a unique solution $v_\nu(t)$ for each fixed ν and let us define $u_\nu(t) = e^{-Q_\nu(t)}v_\nu(t)$ to satisfy (2.1). If we can prove

$$u_\nu(t) \to u(t) \quad \text{strongly in } C^0([0,T];H^1) \quad \textit{as } \nu \to \infty,$$

we can complete the proof of main theorem. We have to prepare several lemmas to accomplish it.

LEMMA 6.1. *Let $p(x,\xi) \in C^\infty(R^n_x \times R^n_\xi)$ be a symbol in S^m and let $p_\nu(x,\xi) = p(x,\zeta_\nu(\xi))$ in $S^m_{\lambda_\nu}$. Then, for any compact subset K of R^n_ξ,*

$$p\nu^{(\alpha)}_{(\beta)}(x,\xi) \to p^{(\alpha)}_{(\beta)}(x,\xi) \quad \text{uniformly on } R^n_x \times K \quad (\nu \to \infty).$$

LEMMA 6.2. *Let* $p(x,\xi)\in C^\infty(R^n_x\times R^n_\xi)$ *be a symbol in* S^m *and let* $p_\nu(x,\xi)=p(x,\zeta_\nu(\xi))$ *in* $S^m_{\lambda_\nu}$. *Then, for* $m,m'\in R$ *and* $\ell-0\in N\cup\{0\}$

$$\lim_{\nu\to\infty}|\sigma(\langle D\rangle^{-m+m'}(p_\nu(x,D)-p(x,D))\langle D\rangle^{-m'-\varepsilon})(x,\xi)|_{l_0}^{(0)}=0$$

for any positive ε.

PROOF. The proof of Lemma 6.1 is seen in Kumano-go [6] (page 237, Lemma 3.3).

Lemma 6.2 follows Lemma 6.1.

Let $\tilde p_\nu(x,\xi)=p_\nu(x,\xi)-p(x,\xi)$. We are able to describe the symbol above as

$$\begin{aligned}&\sigma(\langle D\rangle^{-m+m'}(p_\nu(x,D)-p(x,D))\langle D\rangle^{-m'-\varepsilon}\\&=Os-\iint_{R^n_y\times R^n_\eta}e^{-iy\eta}\langle\xi+\eta\rangle^{-m+m'}\langle\xi\rangle^{-m'-\varepsilon}\tilde p_\nu(x+y,\xi)\,dy\,d\eta\\&=\iint_{R^n_y\times R^n_\eta}e^{-iy\eta}h_\nu(x,\xi;y,\eta)\,dy\,d\eta,\end{aligned}$$

where $h_\nu(x,\xi;y,\eta)=\langle\eta\rangle^{-2l}\langle D_y\rangle^{2l}(\langle y\rangle^{-2l}\langle D_\eta\rangle^{2l}\langle\xi+\eta\rangle^{-m+m'}\langle\xi\rangle^{-m'-\varepsilon}\tilde p_\nu(x+y,\xi))$.

Let us decompose pre-integrated function into several segments. For arbitrary given positive radius R, we set three segments $\{|\xi|\le R\}$, $\{|\xi|>R,\ |\xi|<2|\eta|\}$ and $\{|\xi|>R,|\xi|\ge 2|\eta|\}$. In the first segment, we replace any ξ-related quantities with their suprema. We also use facts that if $|\xi|\ge 2|\eta|$, then $\langle\xi+\eta\rangle\ge\langle\xi\rangle-|\eta|\ge\frac12\langle\xi\rangle$ and if $|\xi|<2|\eta|$, then $\lambda_\nu(\xi)\le\langle\xi\rangle\le 2\langle\eta\rangle$.

Thus, repeated applications of Leibniz formula with some inequality like $\langle\xi+\eta\rangle\le 2\langle\xi\rangle\langle\eta\rangle$ and $|\partial_\xi^\gamma\langle\xi\rangle|\le C_\gamma\langle\xi\rangle^{1-|\gamma|}$ yield

$$\begin{aligned}&|h^{(\alpha)}_{\nu(\beta)}(x,\xi;y,\eta)|\\&=|\langle\eta\rangle^{-2l}\langle D_y\rangle^{2l}(\langle y\rangle^{-2l}\langle D_\eta\rangle^{2l}\partial_\xi^\alpha(\langle\xi+\eta\rangle^{-m+m'}\langle\xi\rangle^{-m'-\varepsilon}\tilde p_{\nu(\beta)}(x+y,\xi))|\\&\le\langle\eta\rangle^{-2l}\sum_{\alpha'\le\alpha}\binom{\alpha}{\alpha'}\sum_{\alpha''\le\alpha'}\binom{\alpha'}{\alpha''}\sum_{j=0}^{2l}\binom{2l}{j}|\langle D_y\rangle^{2l-j}\langle y\rangle^{-2l}|\,|\partial_\xi^{\alpha-\alpha''}\langle D_\eta\rangle^{2l}\langle\xi+\eta\rangle^{-m+m'}|\\&\quad\times|\partial_\xi^{\alpha''}\langle\xi\rangle^{-m-\varepsilon}|\,|\langle D_y\rangle^j\tilde p^{(\alpha-\alpha'-\alpha'')}_{\nu(\beta)}(x+y,\xi)|\\&\le C_{l\alpha}\sum_{\alpha'\le\alpha}\binom{\alpha}{\alpha'}\sum_{\alpha''\le\alpha'}\binom{\alpha'}{\alpha''}\langle\eta\rangle^{-2l}\langle y\rangle^{-2l}\langle\xi\rangle^{-\varepsilon-m-|\alpha''|}\langle\xi+\eta\rangle^{-(m-m')-|\alpha-\alpha'|-2l}\\&\quad\times\max_{\substack{\alpha'\le\alpha\\|\beta'|\le|\beta|+2l}}\sup_{x\in R^n}|\tilde p^{(\alpha')}_{\nu(\beta')}(x,\xi)|\end{aligned}$$

$$\le \begin{cases} C_{\alpha\beta}(R)\langle\eta\rangle^{-2l}\langle y\rangle^{-2l} \max\limits_{\substack{\alpha'\le\alpha \\ |\beta'|\le|\beta|+2l}} \sup\limits_{\substack{x\in R^n \\ |\xi|\le R}} |\tilde{p}_{\nu(\beta')}^{(\alpha')}(x,\xi)| \quad (|\xi|\le R) \\ C'_{\alpha\beta}\langle\eta\rangle^{-2l}\langle y\rangle^{-2l}\langle\xi\rangle^{-\varepsilon}\langle\xi\rangle^{-2l} \quad (|\xi|\ge R, |\xi|\ge 2|\eta|) \\ C''_{\alpha\beta}\langle\eta\rangle^{-2l+|m|+|\alpha|}\langle y\rangle^{-2l}\langle\xi\rangle^{-\varepsilon-|m|-|\alpha|} \quad (|\xi|\ge R, |\xi|<2|\eta|). \end{cases}$$

Hence if we take $l > ((|m|+n+1+\ell_0)/2)(|\alpha|+|\beta|\le\ell_0)$, the integral exists and

$$\lambda_\nu(\xi)^{|\alpha|}\left|\iint_{R_y^n\times R_\eta^n} e^{-iy\eta}h_\nu(x,\xi;y,\eta)\,dy\,d\eta\right|$$
$$\le \begin{cases} C_{\alpha\beta}(R)(1+R^2)^{\ell_0/2} \max\limits_{\substack{\alpha'\le\alpha \\ |\beta'|\le|\beta|+2l}} \sup\limits_{\substack{x\in R^n \\ |\xi|\le R}} |\tilde{p}_{\nu(\beta')}^{(\alpha')}(x,\xi)| \quad (|\xi|\le R) \\ C'_{\alpha\beta}R^{-\varepsilon} \quad (|\xi|\ge R). \end{cases}$$

Since Lemma 6.1 gurantees $\lim_{\nu\to\infty}\sup_{\substack{x\in R^n \\ |\xi|\le R}}\max_{\substack{\alpha'\le\alpha \\ \beta'\le|\beta|+2l}}|\tilde{p}_{\nu(\beta')}^{(\alpha')}(x,\xi)|=0$, we can say

$$\limsup_{\nu\to\infty}|\sigma(\langle D\rangle^{-m+m'}(p_\nu(x,D)-p(x,D))\langle D\rangle^{-m'-\varepsilon}|_{\ell_0}^{(0)} \le C_{l,\ell_0}R^{-\varepsilon} \text{ for any } R>0,$$

which confirms Lemma 6.2. q.e.d.

LEMMA 6.3. *Let $p(x,\xi)\in C^\infty(R_x^n\times R_\xi^n)$ be a symbol in S^0 and let $p_\nu(x,\xi)=p(x,\zeta_\nu(\xi))$ in $S_{\lambda_\nu}^0$. Suppose $u\in H^\varepsilon$ for a given positive real number ε. Then,*

$$\lim_{\nu\to 0}\|p_\nu(x,D)u-p(x,D)u\|_{L^2}=0.$$

PROOF. Taking account of L^2-boundedness of $(p_\nu(x,D)-p(x,D))\langle D\rangle^{-\varepsilon}$ and previous lemma, we get

$$\|p_\nu(x,D)u-p(x,D))u\|_{L^2}=\|(p_\nu(x,D)-p(x,D))\langle D\rangle^{-\varepsilon}\cdot\langle D\rangle^{\varepsilon}u\|_{L^2}$$
$$\le C|(p_\nu(x,\xi)-p(x,\xi))\langle\xi\rangle^{-\varepsilon}|_\ell^{(0)}\|u\|_{H^\varepsilon}\to 0. \quad \text{q.e.d.}$$

Lemma 5.2 showed us the Caucy problem (2.1) has a unique solution $u_\nu=e^{-Q_\nu(t)}v_\nu(t)$ for each fixed $\nu>0$. Let us write the counterpart v'_ν for ν' and set $w_{\nu\nu'}=u_\nu-u_{\nu'}$. Then $w_{\nu\nu'}$ satisfies

$$\text{(6.4)} \qquad \begin{cases} \partial_t^2 w_{\nu\nu'}+M(\eta_\nu(t))A_\nu w_{\nu\nu'}=G_{\nu\nu'}(t,x) \\ w_{\nu\nu'}(0,x)=0, \quad \partial_t w_{\nu\nu'}(0,x)=0, \end{cases}$$

where

$$G_{\nu\nu'}(t,x) = -(M(\eta_\nu(t))A_\nu - M(\eta_{\nu'}(t))A_{\nu'})u_{\nu'}(t).$$

In order to show

$$\|w_{\nu\nu'}(t)\|_{H^1} \le c \int_0^t \|G_{\nu\nu'}(s)\|_{H^1}\, ds, \tag{6.5}$$

$$\lim_{\nu\nu' \to \infty} \sup_{t\in[0,T]} \|G_{\nu\nu'}(t)\|_{H^1} = 0. \tag{6.6}$$

It is useful to investigate the energy

$$e_{\nu\nu'}(t)^2 = \frac{1}{2}\{\|\partial_t w_{\nu\nu'}(t)\|_{H^1}^2 + M(\eta_\nu(t))(A_\nu\langle D\rangle w_{\nu\nu'}(t), \langle D_{\nu\nu'}(t))_{L^2}\}. \tag{6.7}$$

The both the derivative of $e_{\nu\nu'}(t)$ and the fact $|\partial_t M(\eta_\nu(t))| \le CM(\eta_\nu(t))$ lead us to

$$\begin{aligned}
&2e_{\nu\nu'}(t)e'_{\nu\nu'}(t) \\
&\quad = Re(\partial_t^2 w_{\nu\nu'}(t), \partial_t w_{\nu\nu'}(t)\partial_t)_{H^1} + (\partial_t M(\eta_\nu(t)))(A_\nu\langle D\rangle w_{\nu\nu'}(t), \langle D\rangle w_{\nu\nu'}(t))_{L^2} \\
&\qquad + M(\eta_\nu(t))Re(A_\nu\langle D\rangle\partial_t w_{\nu\nu'}(t), \langle D\rangle w_{\nu\nu'}(t))_{L^2} \\
&\quad \le -M(\eta_\nu(t))Re((\langle D\rangle A_\nu - A_\nu\langle D\rangle)w_{\nu\nu'}(t), \langle D\rangle\partial_t w_{\nu\nu'}(t))_{L^2} \\
&\qquad + Ce_{\nu\nu'}(t)^2 + e_{\nu\nu'}(t)\|G_{\nu\nu'}(t)\|_{H^1}.
\end{aligned} \tag{6.8}$$

Let us find the estimate of the first term of (6.8). Putting $\sigma(A_\nu)(x,\xi) = a_\nu(x,\xi) \in S^2_{\lambda_\nu}$, we can represent

$$\sigma(\langle D\rangle A_\nu - A_\nu\langle D\rangle)(x,\xi) = \sum_{|\alpha|=1} a_{\nu(\alpha)}(x,\xi)\omega_\alpha(\xi) + r_\nu(x,\xi),$$

where $\omega_\alpha(xi) = \partial_\xi^\alpha\langle\xi\rangle$ and the remainder $r_\nu(x,\xi) \in S^1_{\lambda_\nu}$. Hence

$$\begin{aligned}
&|Re((\langle D\rangle A_\nu - A_\nu\langle D\rangle)w_{\nu\nu'}(t), \langle D\rangle\partial_t w_{\nu\nu'}(t))_{L^2}| \\
&\quad \le C\sum_{|\alpha|=1} \|a_{\nu(\alpha)}(x,\xi)w_{\nu\nu'}(t)\|_{L^2}\|\partial_t w_{\nu\nu'}(t)\|_{H^1} + C\|w_{\nu\nu'}(t)\|_{H^1}\|\partial_t w_{\nu\nu'}(t)\|_{H^1},
\end{aligned}$$

and it is a quick result of (3.3) of Lemma 3.1 that

$$\begin{aligned}
\|a_{\nu(\alpha)}(x,\xi)w_{\nu\nu'}(t)\|_{L^2}^2 &\le C((A_\nu\langle D\rangle w_{\nu\nu'}(t), \langle D\rangle w_{\nu\nu'}(t))_{L^2} + C\|w_{\nu\nu'}(t)\|_{H^1}^2) \\
&\le Ce_{\nu\nu'}(t)^2 + C\left(\int_0^t e_{\nu\nu'}(\tau)\, d\tau\right)^2.
\end{aligned}$$

So we get

$$e'_{\nu\nu'}(t) \leq Ce_{\nu\nu'}(t) + C\int_0^t e_{\nu\nu'}(\tau)\,d\tau + C\|G_{\nu\nu'}(t)\|_{H^1}. \tag{6.9}$$

The calculation over $\|G_{\nu\nu'}(t)\|_{H^1}$ is remained. By its definition

$$\begin{aligned} &\|G_{\nu\nu'}(t)\|_{H^1} \\ &\leq \|(M(\eta_\nu(t)) - M(\eta_{\nu'}(t)))A_{\nu'}v_{\nu'}\|_{H^1} + \|M(\eta_{\nu'}(t))(A_{\nu'} - A_\nu)v_{\nu'})\|_{H^1} \\ &\leq C|M(\eta_\nu(t)) - M(\eta_{\nu'}(t))| \qquad (6.10) \\ &\quad + C\|(A_{\nu'} - A_\nu)v_{\nu'})\|_{H^1}, \qquad (6.11) \end{aligned}$$

where we took $s = 3$ and used $\|A_{\nu'}v_{\nu'}\|_{H^1} \leq C$, $\|v_{\nu'}\|_{H^3} \leq C$ and $M(\eta_\nu(t)) \leq C$, results of section 4 and Lemma 2.2. The last two terms have the following estimates.

$$\begin{aligned} (6.11) &= \|\langle D\rangle(A_{\nu'} - A_\nu)\langle D\rangle^{-3-\varepsilon}\langle D\rangle^{3+\varepsilon}v_{\nu'})\|_{L^2} \\ &\leq C|\sigma(\langle D\rangle(A_{\nu'} - A_\nu)\langle D\rangle^{-3-\varepsilon}|_{\ell'}^{(0)}\|\langle D\rangle^{3+\varepsilon}v_{\nu'})\|_{L^2} \\ &\leq C'|\sigma(\langle D\rangle(A_{\nu'} - A_\nu)\langle D\rangle^{-3-\varepsilon}|_{\ell'}^{(0)}, \end{aligned}$$

if we choose $s \geq 3 + \varepsilon$ in $E_{\nu,s}(t)$. Writing

$$\begin{aligned} (6.10) &= C|M'(\eta_{\nu'}(t) + \theta(\eta_\nu(t) - \eta_{\nu'}(t)))|\,|\eta_\nu(t) - \eta_{\nu'}(t)| \\ &\leq C'M'_0|\eta_\nu(t) - \eta_{\nu'}(t)| \\ &\leq C'M'_0(|(A_\nu - A_{\nu'})u_\nu, u_\nu)_{L^2}| + |(A_{\nu'}w_{\nu\nu'}, u_\nu)_{L^2}| + |(A_{\nu'}u_{\nu'}, w_{\nu\nu'})_{L^2}|), \end{aligned}$$

and applying Lemma 6.3 to the first term of the last inequality, we get

$$\begin{aligned} (6.10) &\leq C|\sigma((A_\nu - A)\langle D\rangle^{-2-\varepsilon})|_{\ell''}^{(0)}\|u_\nu\|_{H^{2+\varepsilon}}^2 + C\|w_{\nu\nu'}\|_{H^1}(\|u_\nu\|_{H^1} + \|u_{\nu'}\|_{H^1}) \\ &\leq C'|((A_\nu - A)\langle D\rangle^{-2-\varepsilon})|_{\ell''}^{(0)} + C'\int_0^t e_{\nu\nu'}(\tau)\,d\tau. \qquad (6.12) \end{aligned}$$

Combining all together from (6.9) to (6.12), we come to

$$\begin{aligned} e_{\nu\nu'}(t) &\leq c_1e_{\nu\nu'}(t) + c_2\int_0^t e_{\nu\nu'}(\tau)\,d\tau \\ &\quad + c_3|\sigma((A_\nu - A)\langle D\rangle^{-2-\varepsilon})|_{\ell''}^{(0)} + c_4|\sigma(\langle D\rangle(A_{\nu'} - A_\nu)\langle D\rangle^{-3-\varepsilon}\langle D\rangle^{3+\varepsilon}v_{\nu'}))|_{\ell''}^{(0)} \end{aligned}$$

hence

$$\begin{aligned} e'_{\nu\nu'}(t) &\leq c_1 \int_0^t e_{\nu\nu'}(\tau)\,d\tau \\ &\leq C(T)\{|\sigma((A_\nu - A)\langle D\rangle^{-2-\varepsilon})|_{\ell''}^{(0)} + |\sigma(\langle D\rangle(A_{\nu'} - A_\nu)\langle D\rangle^{-3-\varepsilon})|_{\ell'}^{(0)}\} \\ &\to 0(\nu, \nu' \to \infty), \end{aligned}$$

which implies

$$\sup_{t\in[0,T]} \|u_\nu(t) - u_{\nu'}(t)\|_{H^1} \to 0(\nu, \nu' \to \infty).$$

Therefore we can conclude $\eta_{\nu_p}(t) \to (Au(t,\cdot), u(t,\cdot))_{L^2}$ as $p \to \infty$. Thus we have proved Theorem 1.1.

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