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ON THE CAUCHY PROBLEM WITH SMALL ANALYTIC DATA FOR NONLINEAR WEAKLY HYPERBOLIC SYSTEMS

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Abstract. In this paper we investigate the life span of the Cauchy Problem for nonlinear systems of the form

$$(*)\qquad \begin{cases} \partial_t u = f(t, x, u, \partial_1 u, \dots, \partial_n u) \\ u(0, x) = \varepsilon\phi(x). \end{cases}$$

Assuming that $(*)$ is weakly hyperbolic and has the solution $u \equiv 0$ with $\phi \equiv 0$, we prove that

i) lifespan $T_\varepsilon \to \in \infty$ as $\varepsilon \to 0$.
T_ε admits the asymptotic estimate
$T_\varepsilon \geq \psi^{-1}(\mu \log \log(1/\varepsilon)$, where $\psi(t) = \int_0^t |f(\tau)| d\tau, \mu > 0$.
ii) $u = 0$ is a stable solution.

In order to get this fact, we first consider the case of linear systems and then apply to nonlinear systems.

§ 1. Introduction

The Cauchy-Kovalevski theorem assures that the Cauchy problem for the first order systems with analytic coefficients and analytic data are locally solvable in the class of analytic functions. Here we must pay attention to the fact that this theorem can be applied to any type of systems, but gives only the local solvability. For this reason, Bony and Schapira restricted the type of weakly hyperbolic systems, and showed the global solvability for linear systems (see [BS]). And then, their results were extended by D'Ancona and Spagnolo to

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the following nonlinear weakly hyperbolic systems with small analytic data (see [DS]).

(A) $$\partial_t u = f(u, \partial_1 u, \dots, \partial_n u).$$

Noting that the system (A) doesn't contain variables t, x, we shall consider the following more general systems.

(B) $$\partial_t u = f(t, x, u, \partial_1 u, \dots, \partial_n u).$$

In order to investigate the system (B), we first consider the linear system

(1) $$\begin{cases} \partial_t u = \sum_{h=1}^{n} A_h(t,x)\partial_h u + B(t,x)u + g(t,x) \\ u(0,x) = u_0(x), \end{cases}$$

where

$A_h(t,x), B(t,x)$ are $N \times N$ matrices which are analytic in $\boldsymbol{R}_x^n$ and satisfy,

(2) $$|\partial_x^\alpha A_h(t,x)| \le M_a(t)\rho_a^{-|\alpha|}\alpha!, |\partial_x^\alpha B(t,x)| \le M_b(t)\rho_b^{-|\alpha|}\alpha! \quad \text{a.e. on } (0, \infty)$$

for $^\forall \alpha \in \boldsymbol{N}$, $^\forall x \in \boldsymbol{R}^n$ with $^\exists M_a(t)$, $^\exists M_b(t) \in L^1_{\text{loc}}(0, \infty)$, $^\exists \rho_a$, $^\exists \rho_b > 0$.

Now we assume that (1) is weakly hyperbolic, i.e.

(3) $$\text{“}\sum_{h=1}^{n} \xi_h A_h(t,x) \text{ has real eigenvalues } \lambda_k(t,x,\xi) \text{ a.e. on } (0, \infty)\text{”}.$$

THEOREM 1. *Assume that the coefficients* $A_h(t,x)$ *and* $B(t,x)$ *satisfy* (2), (3). *Let* $T > 0$, $s \in \boldsymbol{R}$, $0 < \rho_0 < \min\{\rho_a/\sqrt{n}, \rho_b/\sqrt{n}\}$. $\rho(t)$ *is a function defined as* $\rho(t) = e^{-C\int_0^t M_a(\tau)d\tau}\{\rho_0 - \eta\int_0^t e^{C\int_0^\tau M_a(\sigma)d\sigma}(C'M_a(\tau)+1)d\tau\}$, *with* $^\exists C, ^\exists C' > 0$, $0 <^\exists \eta \le \rho_0(\int_0^T e^{C\int_0^T M_a(\sigma)d\sigma}(C'M_a(\tau)+1)d\tau)^{-1}$. *Then for any* $u_0(x) \in H^s_{\rho_0}$ *and,* $g(t,x)$ *satisfying* $e^{\rho(t)\langle D\rangle}g(t) \in L^1_{\text{loc}}((0, \in \infty), H^s)$, *the Cauchy Problem* (1) *has the unique* (*global*) *solution* $u(t)$ *satisfying*

(4) $$\begin{cases} e^{\rho(t)\langle D\rangle}u(t) \in C^0([0,T), H^s) \\ e^{\rho(t)\langle D\rangle}\partial_t u(t) \in L^1((0,T), H^{s-1}), \end{cases}$$

and it holds that

$$
(5)\quad \begin{cases} \|e^{\rho(t)\langle D\rangle}u(t)\|_{H^s} \le K(t)\left\{\|u_0\|_{H^s_{\rho_0}} + C''\int_0^t \|e^{\rho(t)\langle D\rangle}g(\tau)\|_{H^s}d\tau\right\}, \\ \|e^{\rho(t)\langle D\rangle}\partial_t u(t)\|_{H^{s-1}} \le G(t)\left\{\|u_0\|_{H^s_{\rho_0}} + C''\int_0^t \|e^{\rho(t)\langle D\rangle}g(\tau)\|_{H^s}d\tau\right\} \\ \qquad\qquad + \|e^{\rho(t)\langle D\rangle}g(t,x)\|_{H^{s-1}} \text{ a.e. on } [0,T), \end{cases}
$$

where $^\exists C'' > 0$, $K(t)$ *is a continuous function on* $[0, T)$, *and* $G(t)$ *is a integrable function on* $(0, T)$.

We rewrite the system (B) more precisely.

$$
(6)\quad \begin{cases} \partial_t u = f(t, x, u, \partial_1 u, \dots, \partial_n u) \\ u(0, x) = \varepsilon\phi(x), \end{cases}
$$

where

$f(t, x, y, z_1, \dots, z_n)$ is a $\boldsymbol{R}^N$-valued function which is analytic in $\boldsymbol{R}^n_x$ and a neighbourhood of 0 in $\boldsymbol{R}^N_y \times \boldsymbol{R}^n_{z_1} \times \cdots \times \boldsymbol{R}^N_{z_n}$, and satisfies,

$$
(7)\quad \begin{aligned} |\partial_x^\alpha \partial_y^\beta \partial_{z_1}^{\gamma_1} \cdots \partial_{z_n}^{\gamma_n} f(t, x, y, z_1, \dots, z_n)| &\le M(t)\rho_c^{-|\alpha|}\rho_d^{-|\beta|-|\gamma_1|-\cdots-|\gamma_n|} \\ &\quad \times \alpha!\beta!\gamma_1!\cdots\gamma_n! \text{ a.e. on } (0,\ \infty) \end{aligned}
$$

for $^\forall\alpha \in \boldsymbol{N}^n, {}^\forall\beta, {}^\forall\gamma_1 \cdots {}^\forall\gamma_n \in \boldsymbol{N}^N, {}^\forall(x, y, z_1, \dots, z_n) \in \boldsymbol{R}^n_x$

$\times$ "a neighbourhood of 0 in $\boldsymbol{R}^N_y \times \boldsymbol{R}^N_{z_1} \times \cdots \times \boldsymbol{R}^N_{z_n}$"

with $^\exists M(t) \in L^1_{\text{loc}}(0,\ \infty)$ such that $M(t) > 0$ a.e. on $(0,\ \infty)$, $^\exists\rho_c$, $^\exists\rho_d > 0$.

Besides we assume that (6) with $\phi = 0$ has the solution $u \equiv 0$, i.e.

$$
(8)\quad f(t, x, 0, 0, \dots, 0) = 0,
$$

and is weakly hyperbolic at $u = 0$, i.e.

$$
(9)\quad \text{“}\sum_{h=1}^{n} \xi_h \frac{\partial f}{\partial z_h}(t, x, 0, 0, \dots, 0) \text{ has real eigenvalues } \lambda_k(t, x, \xi) \text{ a.e. on } (0, \infty)\text{”}.
$$

THEOREM 2. *Assume that* f *satisfies* (7), (8), (9). *Let* $s > (n/2)$, $0 < \rho_0 < \rho_c/\sqrt{n}$, $\phi(x) \in H^s_{\rho_0}$. *Then there exists* $T > 0$ *such that the Cauchy Problem* (6) *has* *the solution* $u_\varepsilon(t)$ *satisfying*

$$(10) \qquad \begin{cases} e^{\rho(t)\langle D\rangle}u_\varepsilon(t) \in C^0([0,T),H^s) \\ e^{\rho(t)\langle D\rangle}\partial_t u_\varepsilon(t) \in L^1((0,T),H^{s-1}), \end{cases}$$

and it holds that

$$(11) \qquad \begin{cases} \text{i)} \quad T_\varepsilon \to \infty \text{ as } \varepsilon \to 0. \\ \qquad T_\varepsilon \text{ admits the asymptotic estimate} \\ \qquad T_\varepsilon \geq \psi^{-1}\left(\mu \log\log \frac{1}{\varepsilon}\right) \quad \text{where } \psi(t) = \int_0^T M(\tau)d\tau, \ \mu > 0. \\ \text{ii)} \quad u = 0 \text{ is a stable solution, i.e.} \\ \qquad \|e^{\rho(t)\langle D\rangle}u_\varepsilon\|_{H^s} \to 0 \text{ as } \varepsilon \to 0. \end{cases}$$

where $\rho(t)$ is a function defined as

$$\rho(t) = e^{-C\int_0^t M(\tau)d\tau}\left\{\tilde{\rho}_0 - C'\left\{\left(\int_0^t M(\tau)d\tau\right)\left(\log \frac{1}{\varepsilon}\right)^{-1}\right\}^{C''} \times \int_0^t e^{C\int_0^\tau M(\sigma)d\sigma}(C'''M(\tau)+1)d\tau\right\}$$

with $^\exists C, ^\exists C', ^\exists C'', ^\exists C''' > 0$, $\quad 0 <^\exists \tilde{\rho}_0 < \rho_0$, and T_ε is the positive number defined as $T_\varepsilon = \max\{T \geq 0; \rho(T) \geq 0\}$.

We remark that i) doesn't hold generally unless we assume (9). For example, for the Cauchy-Riemann system which is elliptic, T_ε is concerned with the radii of the analytic data, independently of ε. If we wouldn't stick at the type of equations, our theorem could be extended by using the methods of Kajitani [K] for linear systems, and we could know the relation between the domain of existence of analytic solutions and the imaginary part of the eigenvalues of the characteristic matrix. Further work will be required to get this fact.

The proofs of Theorem 1 and 2 rely on the following ideas. In the proof of Theorem 1, we give two transformations to the equation (1). One is due to decompose the Hermitian part and the small remainder part for the principal symbol of the equation, and another is due to change the type of the system into the parabolic system. Thanks to these we can get a priori estimate. In the proof of Theorem 2, we show that (6) can be reduce into a quasilinear system, and apply the estimate obtained in Theorem 1 to the quasilinear system whose analytic data is small. Then we can conclude Theorem 2.

Notations

$$x=(x_1,\dots,x_n)\in \boldsymbol{R}^n_x,\quad \alpha=(\alpha_1,\dots,\alpha_n)\in \boldsymbol{N}^n,$$

$$\langle x\rangle=\sqrt{1+|x|^2},\quad |\alpha|=\alpha_1+\cdots+\alpha_n,\quad \alpha!=\alpha_1!\cdots\alpha_n!,$$

$$\partial_j=\partial_{x_j}=\frac{\partial}{\partial_{x_j}},\quad D_{x_j}=-i\frac{\partial}{\partial_{x_j}},\quad \hat f(\xi)=\mathscr{F}[f](\xi)=\int e^{-ix\cdot\xi}f(x)dx,$$

$$H^s=\{u\in S';\langle\xi\rangle^s u\in L^2(\boldsymbol{R}^n_\xi)\},\quad H^s_\rho=\{u\in S';\langle\xi\rangle^s e^{\rho\langle\xi\rangle}\hat u\in L^2(\boldsymbol{R}^n_\xi)\},$$

"$f(x)$ is analytic in $\boldsymbol{R}^n_x$" means that $f(x)$ satisfies $|\partial^\alpha_x f(x)|\le C\rho^{-|\alpha|}\alpha!$ for $^\forall\alpha\in \boldsymbol{N}^n$, $^\forall x\in \boldsymbol{R}^n_x$.

$S^m=\{p(x,\xi)\in C^\infty(\boldsymbol{R}^{2n});|p^{(\alpha)}_{(\beta)}(x,\xi)|\le C_{\alpha,\beta}\langle\xi\rangle^{m-|\alpha|}$ in $\boldsymbol{R}^{2n}$ for $^\forall\alpha,{}^\forall\beta\in \boldsymbol{N}^n\}$

"$p(x,D)\in OPS^m$" means that $p(x,D)$ is a pseudodifferental operator which has its symbol $p(x,\xi)\in S^m$.

§ 2. Preliminaries

We shall introduce some properties of the analytic norms defined as follows

$$\|u\|_{H^s_\rho}=\left\{\int_{\boldsymbol{R}^n_\xi}\langle\xi\rangle^{2s}e^{2\rho\langle\xi\rangle}|\hat u(\xi)|^2d\xi\right\}^{1/2},$$

which will be used in § 4.

i) $$\|u+v\|_{H^s_\rho}\le\|u\|_{H^s_\rho}+\|v\|_{H^s_\rho}$$

PROOF. It is easily proved by Schwarzian inequality.

ii) $$\|u_1\cdot u_2\|_{H^s_\rho}\le C_s\|u_1\|_{H^s_\rho}\|u_2\|_{H^r_\rho}\quad \text{for } r>\frac{n}{2}, 0\le s\le r$$

PROOF. See Theorem 4.4 in [CL].

iii) Let $a(x)$ be a real analytic function on $\boldsymbol{R}^n$ satisfying

$$|\partial_x^\alpha a(x)| \le C_{\rho_1}^{-|\alpha|}\alpha!.$$

If $|\rho| < \rho_1/n$, then it holds that

$$\|a \cdot u\|_{H_\rho^s} \le \tilde{C}_s \|u\|_{H_\rho^s}$$

PROOF. Using $e^{\rho\langle D\rangle} a(x) e^{-\rho\langle D\rangle} \in OPS^0$ (see Prop 2.3 in [KY]), we have for some non-negative integer $l = l(s)$ (see Th 2.7 in [KG]),

$$\begin{aligned}\|a \cdot u\|_{H_\rho^s} &= \|\langle D\rangle^s e^{\rho\langle D\rangle} a e^{-\rho\langle D\rangle} \langle D\rangle^{-s}(\langle D\rangle^s e^{\rho\langle D\rangle} u)\|_{L^2} \\ &\le C|\sigma(e^{\rho\langle D\rangle} a(x) e^{-\rho\langle D\rangle})|_l^{(0)} \|\langle D\rangle^s e^{\rho\langle D\rangle} u\|_{L^2} \\ &= \tilde{C}_s \|u\|_{H_\rho^s}.\end{aligned}$$

iv)
$$\|u\|_{H_{\rho'}^{s+1}} \le \frac{1}{\rho - \rho'} \|u\|_{H_\rho^s} \quad \text{for } \rho' < \rho$$

PROOF. We take $\rho'' = (\rho' + \rho)/2$. Since it generally holds that

$$e^{-x} \le \frac{1}{x} \quad \text{for } x > 0,$$

putting $x = 2(\rho'' - \rho')\langle\xi\rangle > 0$ and multiplying the both sides by $e^{2\rho''\langle\xi\rangle}$, we have

$$e^{2\rho'\langle\xi\rangle} \le \frac{e^{2\rho''\langle\xi\rangle}}{2(\rho'' - \rho')\langle\xi\rangle.}$$

Similarly,

$$e^{2\rho''\langle\xi\rangle} \le \frac{e^{2\rho\langle\xi\rangle}}{2(\rho - \rho'')\langle\xi\rangle}.$$

Noting that

$$\rho - \rho'' = \rho'' - \rho' = \frac{\rho - \rho'}{2},$$

we get

$$e^{2\rho'\langle\xi\rangle} \leq \frac{e^{2\rho\langle\xi\rangle}}{(\rho-\rho')^2\langle\xi\rangle^2}.$$

Hence it follows that

$$\begin{aligned}\|u\|^2_{H^{s+1}_{\rho'}} &= \int \langle\xi\rangle^{2(s+1)} e^{2\rho'\langle\xi\rangle}|\hat{u}(\xi)|^2 d\xi \\ &\leq \frac{1}{(\rho-\rho')^2}\int \langle\xi\rangle^{2s} e^{2\rho\langle\xi\rangle}|\hat{u}(\xi)|^2 d\xi \\ &= \frac{1}{(\rho-\rho')^2}\|u\|^2_{H^s_\rho}\end{aligned}$$

v) Let $f(x,y)$ be a real analytic function on $\boldsymbol{R}^n_x$ and a neighbourhood of 0 in $\boldsymbol{R}^N_y$ satisfying

$$|\partial_x^\alpha \partial_y^\beta f(x,y)| \leq C_{\rho_1}^{-|\alpha|}\rho_2^{-|\beta|}\alpha!\beta!$$

for $^\forall\alpha \in \boldsymbol{N}^n$, $^\forall\beta \in \boldsymbol{N}^N$, $^\forall(x,y) \in \boldsymbol{R}^n\times$ "a neighbourhood of 0 in $\boldsymbol{R}^N_y$". If $s > (n/2)$, $|\rho| < \rho_1/n$ and $f(x,0) = 0$ then it holds for $u \in H^s_\rho$ satisfying $1 - N(C_s/\rho_2)\|u\|_{H^s_\rho} > 0$ that

$$\|f(\cdot,u(\cdot))\|_{H^s_\rho} \leq \tilde{C}_s N\frac{C_s}{\rho_2}\left(1 - N\frac{C_s}{\rho_2}\|u\|_{H^s_\rho}\right)^{-1}\|u\|_{H^s_\rho}.$$

PROOF. Writing $u = (u_1, \dots, u_N)$, we have

$$f(x,u) = \sum_{\beta>0}\frac{(\partial_y^\beta f)(x,0)}{\beta!}u_1^{\beta_1}\cdots u_N^{\beta_N},$$

Putting $f_\beta(x) = (\rho_2^{|\beta|})/\beta!(\partial_y^\beta f)(x,0)$, we find

$$|\partial_x^\alpha f_\beta(x)| \leq C_{\rho_1}^{-|\alpha|}\alpha! \quad \text{for } ^\forall\beta \in N.$$

Thus by ii) and iii), it follows that

$$\begin{aligned}\|f(\cdot,u(\cdot))\|_{H^s_\rho} &\le \sum_{\beta>0}\rho_2^{-|\beta|}\|f_\beta\cdot(u_1^{\beta_1}\cdots u_N^{\beta_N})\|_{H^s_\rho}\\ &\le \tilde{C}_s\sum_{\beta>0}\left(\frac{C_s}{\rho_2}\right)^{|\beta|}\|u_1\|_{H^s_\rho}^{|\beta_1|}\cdots\|u_N\|_{H^s_\rho}^{|\beta_N|}\\ &= \tilde{C}_s\left\{\prod_{j=1}^{N}\left(1-\frac{C_s}{\rho_2}\|u_j\|_{H^s_\rho}\right)^{-1}-1\right\}.\end{aligned}$$

Using the inequality

$$\prod_{j=1}^{N}(1-a_j)^{-1}\le\left(1-N\left(\sum_{j=1}^{N}a_j^2\right)^{1/2}\right)^{-1}$$

and the relation between the H^s_ρ norms of vector valued functions and the ones of scalar functions

$$\sum_{j=1}^{N}\|u_j\|_{H^s_\rho}^2=\|u\|_{H^s_\rho}^2,$$

we have

$$\|f(\cdot,u(\cdot))\|_{H^s_\rho}\le\tilde{C}_sN\frac{C_s}{\rho_2}\left(1-N\frac{C_s}{\rho_2}\|u\|+H^s_\rho\right)^{-1}\|u\|_{H^s_\rho}.$$

vi) Under the same assumptions as v), it holds for $u,v\in H^s_\rho$ satisfying

$$1-2N\frac{C_s}{\rho_2}(\|u\|_{H^s_\rho}+\|v\|_{H^s_\rho})>0 \text{ that}$$

$$\|f(\cdot,u(\cdot))-f(\cdot,v(\cdot))\|_{H^s_\rho}\le\tilde{C}_sN\frac{C_s}{\rho_2}\left\{1-2N\frac{C_s}{\rho_2}(\|u\|_{H^s_\rho}+\|v\|_{H^s_\rho})\right\}^{-1}\|u-v\|_{H^s_\rho}.$$

PROOF. We can write

$$f(x,u)-f(x,v)=(\nabla_yf)(x,0)(u-v)+\int_0^1 g(x,(1-\theta)v+\theta u)d\theta\cdot(u-v),$$

where $g(x,y)=(\nabla_yf)(x,y)-(\nabla_yf)(x,0)$ satisfies $g(x,0)=0$ and

$$|\partial_x^\alpha\partial_y^\beta g(x,y)|\le CN\rho_2^{-1}\rho_1^{-|\alpha|}\left(\frac{\rho_2}{2}\right)^{-|\beta|}\alpha!\beta!.$$

Applying v) to $g(x,y)$ and noting that $|(\nabla_y f)(x,0)| \leq \tilde{C}_s N \rho_1^{-1}$, we have

$$\|f(\cdot,u(\cdot)) - f(\cdot,v(\cdot))\|_{H^s_\rho} \leq \tilde{C}_s N \rho_2^{-1}\|u-v\|_{H^s_\rho} + 2\tilde{C}_s N^2 \frac{C_s}{\rho_2^2}(\|v\|_{H^s_\rho} + \|u\|_{H^s_\rho})$$

$$\times \left\{1 - 2N\frac{C_s}{\rho_2}(\|v\|_{H^s_\rho} + \|u\|_{H^s_\rho})\right\}^{-1}\|u-v\|_{H^s_\rho}$$

$$\leq \tilde{C}_s N \frac{C_s}{\rho_2}\left\{1 - 2N\frac{C_s}{\rho_2}(\|u\|_{H^s_\rho} + \|v\|_{H^s_\rho})\right\}^{-1}\|u-v\|_{H^s_\rho}.$$

The following result of Jannelli is very useful to derive the estimate for the weakly hyperbolic system in § 3.

Lemma 1. *$A(t,\xi)$ is homogeneous of degree 1 in ξ. Let $\lambda_1(t,\xi),\dots,\lambda_N(t,\xi)$ be N functions which, for any fixed $\xi \in \boldsymbol{R}^n$, belong to $L^1_{\mathrm{loc}}(0,\infty)$ and coincide a.e. on $(0,\infty)$ with the eigenvalues (allowing multiplicity)* of *$A(t,\xi)$.*

Then, for $^\forall\eta \in (0,1]$, there exist a non-singular matrix $P_\eta(t,\xi) \in C^1(0,\infty)$ for $^\forall\xi \in \boldsymbol{R}^n\backslash\{0\}$, $\tilde{A}_\eta(t,\xi) = \mathrm{diag}\{\lambda_1^{(\eta)}(t,\xi),\dots,\lambda_N^{(\eta)}(t,\xi)\} \in L^1_{\mathrm{loc}}(0,\infty)$, and $R_\eta(t,\xi) \in L^1_{\mathrm{loc}}(0,\infty)$ such that:

i) *$P_\eta(t,\xi)$ is homogeneous of degree 0 in ξ, while $\tilde{A}_\eta(t,\xi)$ and $R_\eta(t,\xi)$ are homogeneous of degree 1 in ξ,*
ii) *$P_\eta(t,\xi)A(t,\xi)P_\eta^{-1}(t,\xi) = \tilde{A}_\eta(t,\xi) + R_\eta(t,\xi)$,*
iii) *$|P_\eta(t,\xi)|C_1, |P_\eta^{-1}(t,\xi)| \leq C_2\eta^{-C_3}$,*
iv) *$\int_0^t |R_\eta(s,\xi)|ds \leq C_4\eta|\xi| \int_0^t \sup_{|\xi|=1}|A(s,\xi)|ds$,*
v) *$|\frac{\partial}{\partial t}P_\eta(t,\xi)| \leq C_5$ for $\forall(t,\xi) \in [0,\infty) \times \boldsymbol{R}^n_\xi\backslash\{0\}$,*
vi) *$\int_0^t \sup_{|\xi|=1, 1\leq k\leq N}|\mathrm{Im}\,\lambda_k^{(\eta)}(s,\xi)|ds \leq \int_0^t \sup_{|\xi|=1,1\leq k\leq N}|\mathrm{Im}\,\lambda_k(s,\xi)|ds + \eta$,*

where constants C_1, C_2, C_3, C_4, depend on N and C_5 depends on η.

For the *proof*, refer to [J].

§ 3. Proof of Theorem 1

Assuming that u is the solution of (1), we shall derive the estimate (5). It is sufficient to show when $s = 0$.

Putting $v = e^{\rho(t)\langle D\rangle}u$ and operating $e^{\rho(t)\langle D\rangle}$ on the both sides of (1), we get the equation

$$\partial_t v = \rho'(t)\langle D\rangle v + \sum_h e^{\rho(t)\langle D\rangle} A_h(t,x) e^{-\rho(t)\langle D\rangle}\partial_h v \\ + e^{\rho(t)\langle D\rangle} B(t,x) e^{-\rho(t)\langle D\rangle} v + e^{\rho(t)\langle D\rangle} g(t,x). \tag{12}$$

Supposing $0 \le \rho(t) < min\{\rho_a/\sqrt{n}, \rho_b/\sqrt{n}\}$, from the analyticity of $A_h(t,x)$ and $B(t,x)$, we can write (see Prop 2.3 in [KY])

$$e^{\rho(t)\langle D\rangle} A_h(t,x) e^{-\rho(t)\langle D\rangle} = A_h(t, x + i\rho(t)\langle D\rangle^{-1}) + r_h(t,x,D),$$

$$e^{\rho(t)\langle D\rangle} B(t,x) e^{-\rho(t)\langle D\rangle} = B(t, x + i\rho(t)D\langle D\rangle^{-1}) + r_b(t,x,D),$$

where $A_h(t, x + i\rho(t)D\langle D\rangle^{-1}), B_h(t, x + i\rho(t)D\langle D\rangle^{-1} \in OPS^0$, and $r_h(t,x,D)$, $r_b(t,x,D) \in OPS^{-1}$ for a.e. $t \in (0, \infty)$. Hence we can arrange the equation (12) as follows,

$$\partial_t v = \rho'(t)\langle D\rangle v + A(t,x,D)v + B(t,x,D)v + e^{\rho(t)\langle D\rangle} g(t,x), \tag{13}$$

where $A(t,x,D) = \sum_h A_h(t, x + i\rho(t)D\langle D\rangle^{-1})\partial_h \in OPS^1$, and $B(t,x,D) = \sum_h r_h(t,x,D)\partial_h + B(t, x + i\rho(t)\langle D\rangle^{-1}) \in OPS^0$ for a.e. $t \in [0, \infty)$.

Let $B_\eta(x^{(j)})$ be a open sphere with center $x^{(j)}$, radius η^{C_3+1}, and $\varphi_j(x)$ be a function which belongs to $C_0^\infty(B_\eta(x^{(j)}))$ and satisfy

$$1 \equiv \sum_j \varphi_j(x)^2.$$

Then, multiplying the both sides of (13) by $\varphi_j(x)$, we obtain the equation

$$\partial_t \varphi_j(x) v = \rho'(t)\langle D\rangle \varphi_j(x) v + \rho'(t)\left[\varphi_j(x), \langle D\rangle\right] v + A(t,x,D)\varphi_j(x) v \\ + [\varphi_j(x), A(t,x,D)]v + \varphi_j(x) B(t,x,D)v + \varphi_j(x) e^{\rho(t)\langle D\rangle} g(t,x). \tag{14}$$

Writing $A(t,x,D) = A(t,x^{(j)},D) + (A(t,x,D) - A(t,x^{(j)},D))$ and putting $v_j = \varphi_j(x)v$, we make simple form as follows.

$$\partial_t v_j = \rho'(t)\langle D\rangle v_j + A(t,x^{(j)},D)v_j + (A(t,x,D) - A(t,x^{(j)},D))v_j \\ + \rho'(t)[\varphi_j(x), \langle D\rangle]v + [\varphi_j(x), A(t,x,D)]v \\ + \varphi_j(x) B(t,x,D)v + \varphi_j(x) e^{\rho(t)\langle D\rangle} g(t,x). \tag{15}$$

Furthermore we shall change the equation (15) by Fourier transform. Using $P_{\eta,j}(t,\xi)$, $P_{\eta,j}^{-1}(t,\xi)$ which are determined for each $A(t,x^{(j)},\xi)$ by Lemma 1, putting $w_j = P_{\eta,j}(t,\xi)\hat{v}_j$, and multiplying the both sides of the Fourier transform of the equation (15) by $P_{\eta,j}(t,\xi)$, we have the equation

$$(16)\qquad P_{\eta,j}(t,\xi)\partial_t P_{\eta,j}^{-1}(t,\xi)w_j = \rho'(t)\langle\xi\rangle w_j + \tilde{A}_{\eta,j}(t,\xi)w_j + R_{\eta,j}(t,\xi)w_j + P_{\eta,j}(t,\xi)\hat{h}_j(t,\xi),$$

where

$\tilde{A}_{\eta,j}(t,\xi)$, $R_{\eta,j}(t,\xi)$ are obtained in Lemma 1, and $h_j = (A(t,x,D) - A(t,x^{(j)},D))v_j + \rho'(t)[\varphi_j(x), \langle D\rangle]v + [\varphi_j(x), A(t,x,D)]v + \varphi_j(x)B(t,x,D)v + \varphi_j(x)e^{\rho(t)\langle D\rangle}g(t,x)$.

At length we are in position to estimate the solution. Noting that

$$\frac{\partial}{\partial t} w_j = \left\{\frac{\partial}{\partial t} P_{\eta,j}(t,\xi)\right\}\hat{v}_j + P_{\eta,j}(t,\xi)\frac{\partial}{\partial t}\hat{v}_j$$

$$= \left\{\frac{\partial}{\partial t} P_{\eta,j}(t,\xi)\right\}P_{\eta,j}^{-1}(t,\xi)w_j + P_{\eta,j}(t,\xi)\frac{\partial}{\partial t}P_{\eta,j}^{-1}(t,\xi)w_j,$$

and $\operatorname{Re}\tilde{A}_{\eta,j}(t,\xi)$ is Hermitian, it follows that

$$\begin{aligned}(17)\qquad \frac{d}{dt}\|w_j\|_{L^2_\xi}^2 &= 2\mathrm{Re}\left(\frac{\partial}{\partial t} w_j, w_j\right)\\
&= 2\mathrm{Re}\Big\{\Big(\Big\{\frac{\partial}{\partial t} P_{\eta,j}(t,\xi)\Big\}P_{\eta,j}^{-1}(t,\xi)w_j, w_j\Big)\\
&\quad + \rho'(t)(\langle\xi\rangle^{1/2}w_j, \langle\xi\rangle^{1/2}w_j)\\
&\quad + (\langle\xi\rangle^{-1/2}\mathrm{Im}\tilde{A}_{\eta,j}(t,\xi)\langle\xi\rangle^{-1/2}\langle\xi\rangle^{1/2}w_j, \langle\xi\rangle^{1/2}w_j)\\
&\quad + (\langle\xi\rangle^{-1/2}R_{\eta,j}(t,\xi)\langle\xi\rangle^{-1/2}\langle\xi\rangle^{1/2}w_j, \langle\xi\rangle^{1/2}w_j)\\
&\quad + (P_{\eta,j}(t,\xi)\hat{h}_j(t,\xi), w_j)\Big\}\\
&\le 2C_5C_2\eta^{-C_3}C_1^2\|v_j\|_{L^2_x}^2 + 2\rho'(t)\|\langle\cdot\rangle^{1/2}w_j\|_{L^2_\xi}^2\\
&\quad + 2\sup_\xi|\langle\xi\rangle^{-1}\mathrm{Im}\,\tilde{A}_{\eta,j}(t,\xi)|\|\langle\cdot\rangle^{1/2}w_j\|_{L^2_\xi}^2\\
&\quad + 2\sup_\xi|\langle\xi\rangle^{-1}R_{\eta,j}(t,\xi)|\|\langle\cdot\rangle^{1/2}w_j\|_{L^2_\xi}^2\\
&\quad + 2\,\mathrm{Re}(P_{\eta,j}(t,\xi)\hat{h}_j(t,\xi), w_j).\end{aligned}$$

Picking up the last term on the right side, we shall estimate as follows.

$$
\begin{aligned}
&2\mathrm{Re}(P_{\eta,j}(t,\xi)\hat{h}_j(t,\xi), w_j)\\
&\quad = 2\mathrm{Re}(P_{\eta,j}(t,\xi)\langle\xi\rangle^{-1/2}\mathscr{F}[(A(t,x,D)-A(t,x^{(j)},D))v_j], P_{\eta,j}(t,\xi)\langle\xi\rangle^{1/2}\hat{v}_j)\\
&\quad\quad + 2\mathrm{Re}(P_{\eta,j}(t,\xi)\mathscr{F}[\rho'(t)[\varphi_j(x),\langle D\rangle]v], P_{\eta,j}(t,\xi)\hat{v}_j)\\
&\quad\quad + 2\mathrm{Re}(P_{\eta,j}(t,\xi)\mathscr{F}[[\varphi_j(x), A(t,x,D)]v], P_{\eta,j}(t,\xi)\hat{v}_j)\\
&\quad\quad + 2\mathrm{Re}(P_{\eta,j}(t,\xi)\mathscr{F}[\varphi_j(x)B(t,x,D)v], P_{\eta,j}(t,\xi)\hat{v}_j)\\
&\quad\quad + 2\mathrm{Re}(P_{\eta,j}(t,\xi)\mathscr{F}[\varphi_j(x)e^{\rho(t)\langle D\rangle}g], P_{\eta,j}(t,\xi)\hat{v}_j),
\end{aligned}
$$

$$
\begin{aligned}
\text{The first term} &\le 2C_1\eta^{C_3+1}M_a(t)\|\langle\cdot\rangle^{1/2}\hat{v}_j\|L^2_\xi\|\langle\cdot\rangle^{1/2}w_j\|_{L^2_\xi}\\
&\le 2C_1\eta^{C_3+1}M_a(t)C_2\eta^{-C_3}\|\langle\cdot\rangle^{1/2}w_j\|^2_{L^2_\xi}\\
&\le 2C_1C_2M_a(t)\eta\|\langle\cdot\rangle^{1/2}w_j\|^2_{L^2_\xi}.
\end{aligned}
$$

$$
\begin{aligned}
\text{The second term} &\le 2C_1^2|\rho'(t)|\,\|[\varphi_j,\langle D\rangle]v\|_{L^2_x}\|v_j\|_{L^2_\xi}\\
&\le C_1^2|\rho'(t)|\,\|[\varphi_j,\langle D\rangle]v\|^2_{L^2_x} + C_1^2|\rho'(t)|\,\|v_j\|^2_{L^2_x},
\end{aligned}
$$

here we used $2ab \le a^2 + b^2$. Similarly, we get the followings.

The third term $\le C_1^2M_a(t)\|[\varphi_j,\langle D\rangle]v\|^2_{L^2_x} + C_1^2M_a(t)\|v_j\|^2_{L^2_x}$.

The fourth term $\le 2C_1^2(M_a(t)+M_b(t))\|v_j\|^2_{L^2_x}$.

Gathering these terms and throw into (17), we have

$$
\begin{aligned}
(18)\quad \frac{d}{dt}\|w_j(t)\|^2_{L^2_\xi} &\le 2\{\rho'(t) + \sup_\xi|\langle\xi\rangle^{-1}R_{\eta,j}(t,\xi)|\\
&\quad + \sup_\xi|\langle\xi\rangle^{-1}\mathrm{Im}\,\tilde{A}_{\eta,j}(t,\xi)| + C_1C_2M_a(t)\eta\}\|\langle\cdot\rangle^{1/2}w_j\|^2_{L^2_\xi}\\
&\quad + C_1^2\{2C_5C_2\eta^{-C_3} + |\rho'(t)| + M_a(t)\\
&\quad + 2(M_a(t)+M_b(t))\}\|v_j\|^2_{L^2_x}\\
&\quad + C_1^2\{|\rho'(t)| + M_a(t)\}\|[\varphi_j,\langle D\rangle]v\|^2_{L^2_x}\\
&\quad + 2\mathrm{Re}(P_{\eta,j}(t,\xi)\mathscr{F}[\varphi_j(x)e^{\rho(t)\langle D\rangle}g], P_{\eta,j}(t,\xi)\hat{v}_j).
\end{aligned}
$$

We also use the following facts.

$$\sum_j \|v_j\|_{L^2}^2 = \|v\|^2, \quad \sum_j \|[\varphi_j, \langle D\rangle]v\|_{L^2}^2 \le C_6\|v\|_{L^2}^2$$

and (see Lemma 2.3 in [M]),

(19)
$$\sum_j 2\mathrm{Re}(P_{\eta,j}(t,\xi)\mathscr{F}[\varphi_j(x)e^{\rho(t)\langle D\rangle}g], P_{\eta,j}(t,\xi)\hat{v}_j)$$
$$\le 2C_1^2 \sum_j \|\varphi_j e^{\rho(t)\langle D\rangle} g\|_{L_x^2}\|v_j\|_{L_x^2}$$
$$\le 2C_1^2\left(\sum_j \|\varphi_j e^{\rho(t)\langle D\rangle}g\|_{L_x^2}^2\right)^{1/2}\left(\sum_j \|v_j\|_{L_x^2}^2\right)^{1/2}$$
$$= 2C_1^2\|e^{\rho(t)\langle D\rangle}g\|_{L_x^2}\|v\|_{L_x^2}$$

Then it holds that

(20)
$$\sum_j \frac{d}{dt}\|w_j(t)\|_{L_\xi^2}^2$$
$$\le 2\left\{\rho'(t) + \sup_{\xi,j}|\langle\xi\rangle^{-1}R_{\eta,j}(t,\xi)| + \sup_{\xi,j}|\langle\xi\rangle^{-1}\mathrm{Im}\,\tilde{A}_{\eta,j}(t,\xi)| + C_1C_2M_a(t)\eta\right\}$$
$$\times \sum_j \|\langle\cdot\rangle^{1/2}w_j\|_{L_\xi^2}^2 + 2C_2\eta^{-C_3}\mu(t)\|v\|_{L_x^2}^2 + C_1^2\|e^{\rho(t)\langle D\rangle}g\|_{L_x^2}\|v\|_{L_x^2},$$

where $\mu(t)$ is a locally integrable function on $(0, \infty)$

$$\mu(t) = \frac{1}{2}C_2^{-1}\eta^{C_3}C_1^2\{2C_5C_2\eta^{-C_3} + |\rho'(t)| + M_a(t)$$
$$+ 2(M_a(t) + M_b(t)) + C_6(|\rho'(t)| + M_a(t))\}$$
$$\equiv Const\{\eta^{C_3}(M_a(t) + M_b(t) + 1\}$$

Moreover, noting that

(21)
$$\|w_j\|_{L_\xi^2}^2 \ge (C_2\eta^{-C_3})^{-2}\|v_j\|_{L_x^2}^2,$$

the left side of (20) is estimated as follows

$$\sum_j \frac{d}{dt}\|w_j\|_{L_\xi^2}^2 = \frac{d}{dt}\left\{\left(\sum_j \|w_j\|_{L_\xi^2}^2\right)^{1/2}\right\}^2$$

$$= 2\left(\sum_j \|w_j\|_{L_\xi^2}^2\right)^{1/2} \frac{d}{dt}\left(\sum_j \|w_j\|_{L_\xi^2}^2\right)^{1/2}$$

$$\geq 2(C_2\eta^{-C_3})^{-1}\|v\|_{L_x^2}\frac{d}{dt}\left(\sum_j \|w_j\|_{L_\xi^2}^2\right)^{1/2}$$

Hence, integrating from 0 to t and using again (21), we get

$$\begin{aligned}\|v(t)\|_{L_x^2} \leq C_\eta\|v(0)\|_{L_x^2} &+ C'_\eta \int_0^t \{\rho'(\tau) + \sup_{\xi,j}|\langle\xi\rangle^{-1}R_{\eta,j}(\tau,\xi)| \\ &+ \sup_{\xi,j}|\langle\xi\rangle^{-1}\mathrm{Im}\,\tilde{A}_{\eta,j}(\tau,\xi) + C_1C_2M_a(\tau)\eta\}\left(\sum_j \|\langle\cdot\rangle^{1/2}w_j\|_{L_\xi^2}^2/\|v\|_{L_x^2}\right)d\tau \\ &+ C''_\eta\int_0^t \mu(\tau)\|v(\tau)\|_{L_x^2}d\tau + C'''_\eta\int_0^t \|e^{\rho(\tau)\langle D\rangle}g(\tau)\|_{L_x^2}d\tau,\end{aligned}$$

Where $C_\eta = C_1C_2\eta^{-C_3}$, $C'_\eta = (C_2\eta^{-C_3})^2$, $C''_\eta = (C_2\eta^{-C_3})^3$, $C'''_\eta = (C_2\eta^{-C_3})^2C_1^2$. From Lemma 1 we remark that

$$\begin{aligned}\int_0^t \sup_{\xi,j}|\langle\xi\rangle^{-1}R_{\eta,j}(\tau,\xi)|d\tau &\leq \sup_{\xi,j}\langle\xi\rangle^{-1}C_4\eta|\xi|\int_0^t \sup_{|\xi|=1}|A(\tau,x^{(j)},\xi)|d\tau \\ &\leq \int_0^t C_4\eta M_a(\tau)d\tau,\end{aligned}$$

and

$$\begin{aligned}&\int_0^t \sup_{\xi,j}|\langle\xi\rangle^{-1}\mathrm{Im}\,\tilde{A}_{\eta,j}(\tau,\xi)|d\tau \\ &\leq \sup_{\xi,j}\langle\xi\rangle^{-1}|\xi|\int_0^t \sup_{|\xi|=1,1\leq k\leq N}|\mathrm{Im}\,\lambda_{k,j}^{(\eta)}(\tau,x^{(j)} + i\rho(\tau)\xi\langle\xi\rangle^{-1},\xi)|d\tau \\ &\leq \sup_j \int_0^t \sup_{|\xi|=1,1\leq k\leq N}|\mathrm{Im}\,\lambda_{k,j}(\tau,x^{(j)} + i\rho(\tau)\xi\langle\xi\rangle^{-1},\xi)|d\tau + \eta\end{aligned}$$

$$\leq \int_0^t \sup_{|\xi|=1} C_7 M_a(\tau)|\rho(\tau)\xi\langle\xi\rangle^{-1}\||\xi| d\tau + \eta \quad \text{(see [}K\text{] p1491)}$$

$$\leq \int_0^t C_7 M_a(\tau)\rho(\tau) d\tau + \eta.$$

Supposing $\{\rho'(t) + C_7 M_a(t)\rho(t) + \eta(C_4 M_a(t) + 1 + C_1 C_2 M_a(t))\} \geq 0$, since $\sum_j \|\langle\cdot\rangle^{1/2} w_j\|^2_{L^2_\xi} / \|v\|_{L^2_x} \leq C_8 \|\langle D\rangle v\|_{L^2_x}$, we get

$$(22) \quad \|v(t)\|_{L^2_x} \leq C_\eta \|v(0)\|_{L^2_x} + C'_\eta \int_0^t \{\rho'(\tau) + C_7 M_a(\tau)\rho(\tau) + \eta(C_9 M_a(\tau) + 1 + C_1 C_2 M_a(\tau))\} C_8 \|\langle D\rangle v\|_{L^2_x} d\tau + C''_\eta \int_0^t \mu(\tau)\|v(\tau)\|_{L^2_x} d\tau + C'''_\eta \int_0^t \|e^{\rho(\tau)\langle D\rangle} g(\tau)\|_{L^2_x} d\tau,$$

where $C_9 = C_4 + C_1 C_2$.

Then we can choose $0 \leq \rho(t) < min\{\rho_a/\sqrt{n}, \rho_b/\sqrt{n}\}$ such that

$$\begin{cases} \rho'(t) + C_7 M_a(t)\rho(t) + \eta(C_9 M_a(t) + 1) = 0 & \text{a.e. on } [0, T), \\ \rho(0) = \rho_0 \quad (0 \leq \rho_0 < min\{\rho_a/\sqrt{n}, \rho_b/\sqrt{n}\}), & \text{i.e.} \end{cases}$$

$$(23) \qquad \rho(t) = e^{-C_7 \int_0^t M_a(\tau) d\tau} \left\{ \rho_0 - \eta \int_0^t e^{C_7 \int_0^\tau M_a(\sigma) d\sigma} (C_9 M_a(\tau) + 1) d\tau \right\},$$

where T implies the maximum of t satisfying $\rho(t) \geq 0$ and is defined as

$$(24) \qquad \rho_0 = \eta \int_0^T e^{C_7 \int_0^\tau M_a(\sigma) d\sigma} (C_9 M_a(\tau) + 1) d\tau.$$

Thus if we take $\eta > 0$ satisfying (24) for any given $T > 0$ and choose $\rho(t) \geq 0$ satisfying (23), we can eliminate the second term on the right side of (22). And then, returning to the original functions, we get

$$(25) \quad \|e^{\rho(t)\langle D\rangle} u(t)\|_{L^2} \leq C_\eta \|u_0\|_{L^2_{\rho_0}} + C''_\eta \int_0^t \mu(\tau)\|e^{\rho(\tau)\langle D\rangle} u(\tau)\|_{L^2} d\tau + C'''_\eta \int_0^t \|e^{\rho(\tau)\langle D\rangle} g(\tau)\|_{L_2} d\tau \quad \text{for } \forall t \in [0, T).$$

Finally from (25), Gronwall's inequality yields the estimate

$$\|e^{\rho(t)\langle D\rangle}u(t)\|_{L_2} \le K_\eta(t)\left\{\|u_0\|_{L^2_{\rho_0}} + \tilde{C}_\eta\int_0^t \|e^{\rho(\tau)\langle D\rangle}g(\tau)\|_{L^2}d\tau\right\} \tag{26}$$

for $\forall t \in [0, T)$,

where $\tilde{C}_\eta = C'''_\eta C_\eta^{-1}$, $K_\eta(t) = C_\eta e^{C'_\eta\int_0^t \mu(\tau)d\tau}$ has continuity on $[0, T)$.

Similarly, we can also get the estimate which is the general case for s

$$\|e^{\rho(t)\langle D\rangle}u(t)\|_{H_s} \le K_\eta(t)\left\{\|u_0\|_{H^s_{\rho_0}} + \tilde{C}_\eta\int_0^t \|e^{\rho(\tau)\langle D\rangle}g(\tau)\|_{H^s}d\tau\right\} \tag{27}$$

for $\forall t \in [0, T)$.

In order to prove the existence of solutions for system (1), we consider the following system

$$\begin{cases} \partial_t u_l = \sum_h A_h(t,x) il\sin(D_h/l)u_l + B(t,x)u_l + g(t,x) \\ u_l(0,x) = u_0(x), \end{cases} \tag{28}$$

Here remark that $\zeta_l(\xi) = (l\sin(\xi_1/l), \dots, l\sin(\xi_n/l))$ satisfies

$$\begin{cases} \text{i)} \quad \zeta_l(\xi) \to \xi \quad (l \to \infty) \\ \text{ii)} \quad |\zeta_l(\xi)| \le |\xi| \\ \text{iii)} \quad |\zeta_l^{(\alpha)}(\xi)| \le C_\alpha\langle\xi\rangle_l^{1-|\alpha|} \quad (\langle\xi\rangle_l = \langle\zeta_l(\xi)\rangle) \end{cases}$$

Since $il\sin(D_h/l)(h = 1, \dots, n)$ belongs to OPS^0 for any fixed l, $\sum_h A_h(t,x)il\sin(D_h/l)$ is a bounded linear operator on H^s. Thus the integral equation

$$\begin{aligned} u_l(t) = u_0 &+ \int_0^t \sum_h A_h(\tau,x)il\sin(D_h/l)u_l(\tau)d\tau \\ &+ \int_0^t B(\tau,x)u_l(\tau)d\tau + \int_0^t g(\tau,x)d\tau \end{aligned} \tag{29}$$

is solvable by successive approximations.

With the same methods, we can get the analogous estimate

$$\|e^{\rho(t)\langle D\rangle_l}\langle D\rangle_l^s u_l(t)\|_{L^2} \le K_\eta(t)\left\{\|u_0\|_{H^s_{\rho_0}} + \tilde{C}_\eta\int_0^t \|e^{\rho(\tau)\langle D\rangle}g(\tau)\|_{H^s}d\tau\right\} \tag{30}$$

for $\forall t \in [0, T)$.

Here, we used

$$\|e^{\rho_0\langle D\rangle_l}\langle D\rangle_l^s u_0\|_{L^2} \le \|u_0\|_{H^s_{\rho_0}}, \quad \|e^{\rho(t)\langle D\rangle_l}\langle D\rangle_l^s g(\tau)\|_{L^2} \le \|e^{\rho(\tau)\langle D\rangle} g(\tau)\|_{H^s}.$$

Furthermore it holds that

$$(31)\quad \|e^{\rho(t)\langle D\rangle_l}\langle D\rangle_l^s \partial_t u_l(t)\|_{L_2}$$

$$\le \|e^{\rho(t)\langle D\rangle_l}\langle D\rangle_l^s \left\{\sum_h A_h(t,x) il \sin\left(\frac{D_h}{l}\right) + B(t,x)\right\}$$

$$\times \langle D\rangle_l^{-s-1} e^{-\rho(t)\langle D\rangle_l} e^{\rho(t)\langle D\rangle_l}\langle D\rangle_l^{s+1} u_l\|_{L^2} + \|e^{\rho(t)\langle D\rangle_l}\langle D\rangle_l^s g(t)\|_{L^2}$$

$$\le C(M_a(t) + M_b(t))\|e^{\rho(t)\langle D\rangle_l}\langle D\rangle_l^{s+1} u_l\|_{L^2} + \|e^{\rho(t)\langle D\rangle_l}\langle D\rangle_l^s g(t)\|_{L^2}$$

$$\le G_\eta(t)\left\{\|u_0\|_{H^{s+1}_{\rho_0}} + \tilde{C}_\eta \int_0^t \|e^{\rho(\tau)\langle D\rangle} g(\tau)\|_{H^{s+1}} d\tau\right\} + \|e^{\rho(t)\langle D\rangle} g(t)\|_{H^s}$$

a.e. on $(0,T)$,

where $G_\eta(t)$ is a integrable function on $(0,T)$.

While, writing

$$(32)\quad e^{\rho(t)\langle D\rangle_l}\langle D\rangle_l^s u_l(t) - e^{\rho(t')\langle D\rangle_l}\langle D\rangle_l^s u_l(t') = \int_{t'}^t \partial_t\{e^{\rho(\tau)\langle D\rangle_l}\langle D\rangle_l^s u_l(\tau)\} d\tau$$

$$= \int_{t'}^t \{\rho'(\tau) e^{\rho(\tau)\langle D\rangle_l}\langle D\rangle_l^{s+1} u_l(\tau)$$

$$+ e^{\rho(\tau)\langle D\rangle_l}\langle D\rangle_l^s \partial_t u(\tau)\} d\tau,$$

it also holds

$$\|e^{\rho(t)\langle D\rangle_l}\langle D\rangle_l^s u_l(t) - e^{\rho(t')\langle D\rangle_l}\langle D\rangle_l^s u_l(t')\|_{L^2} \le \int_{t'}^t H(\tau) d\tau,$$

where

$$H(t) = \left\{\max_{0\le\tau\le T} \rho'(\tau) K_\eta(t) + G(t)\right\}\left\{\|u_0\|_{H^{s+1}_{\rho_0}} + \tilde{C}_\eta \int_0^t \|e^{\rho(t)\langle D\rangle} g(\tau)\| H^{s+1} d\tau\right\}$$

$$+ \|e^{\rho(t)\langle D\rangle} g(t)\|_{H^s}$$

is a integrable function on $(0,T)$.

From (30) and (32), we find than the sequence $\{e^{\rho(t)\langle D\rangle_l}\langle D\rangle_l^s u_l(t)\}_{l=1}^\infty$ is bounded in L^2 and has a week limit $e^{\rho(t)\langle D\rangle}\langle D\rangle^s u(t)$ satisfying

$$\|e^{\rho(t)\langle D\rangle}u(t) - e^{\rho(t')\langle D\rangle}u(t')\|_{H^s} \leq \int_{t'}^{t} H(\tau)d\tau, \tag{33}$$

and

$$\|e^{\rho(t)\langle D\rangle}\partial_t u(t)\|_{H^{s-1}} \leq G(t)\left\{\|u_0\|_{H^s_{\rho_0}} + \tilde{C}_\eta \int_0^t \|e^{\rho(\tau)\langle D\rangle}g(\tau)\|_{H^s}d\tau\right\} \tag{34}$$
$$+ \|e^{\rho(t)\langle D\rangle}g(t)\|H^{s-1} \qquad \text{a.e. on } (0,T),$$

Considering $\int_0^T \int_{\boldsymbol{R}^n} u_l(t,x)\overline{\psi(t,x)}dt\,dx$ for $\psi(t,x) = \psi_1(t)\psi_2(x) \in C_0^\infty((0,T) \times \boldsymbol{R}^n_x)$, we get the following as the limit of (29)

$$u(t) = u_0 + \int_0^t \sum_h A_h(\tau,x)\partial_h u(\tau)d\tau$$
$$+ \int_0^t B(\tau,x)u(\tau)d\tau + \int_0^t g(\tau)d\tau.$$

Then for $J_\varepsilon u_0(x) \in H^\infty_{\rho_0}$ and $J_\varepsilon g(t,x)$ satisfying $e^{\rho(t)\langle D\rangle}J_\varepsilon g(t) \in L^1_{\mathrm{loc}}((0, \infty), H^\infty)$ with then Friedrichs mollifier $\{J_\varepsilon\}_{0\leq\varepsilon\leq 1}$, it holds that

$$\begin{cases} e^{\rho(t)\langle D\rangle}u(t) \in C^0([0,T), H^s) \\ e^{\rho(t)\langle D\rangle}\partial_t u(t) \in L^1((0,T), H^{s-1}). \end{cases} \tag{35}$$

As $\varepsilon \to 0$, (35) also holds for $u_0(x) \in H^s_{\rho_0}$ and $g(t,x)$ satisfying $e^{\rho(t)\langle D\rangle}g(t) \in L^1_{\mathrm{loc}}((0, \infty), H^s)$.

§ 4. Proof of Theorem 2

It is sufficient to prove Theorem 2 for quasilinear systems. In fact we can easily show that the fully nonlinear Cauchy Problem (6) is equivalent to a quasilinear system as below.

If $u = {}^t(u_1, \ldots, u_n)$ is a solution to (6) on $[0,T) \times \boldsymbol{R}^n$, by differentiating (6) we see that the $N(n+1)$-vector $U = {}^t(u, \partial_1 u, \ldots, \partial_n u)$ is a solution of the quasilinear system

$$\begin{cases} \partial_t U^j = \sum_{h=1}^{n} \dfrac{\partial f}{\partial z_h}(t,x,U)\partial_h U^j + \dfrac{\partial f}{\partial y}(t,x,U)U^j + g_j(t,x,U) \quad (0 \leq j \leq n) \\ U^0(0,x) = \varepsilon\phi(x), \quad u^j(0,x) = \varepsilon\partial_j\phi(x), \end{cases} \tag{36}$$

where $g_0(t,x,U) = f(t,x,U) - \sum_{h-1}^{n}(\partial f/\partial z_h)(t,x,U)U^h - (\partial f/\partial y)(t,x,U)U^0$, $g_j(t,x,U) = (\partial f/\partial x_j)(t,x,U)$ $(1 \le j \le n)$.

Now it seems that apparently the dimension of the system has increased from N to $N(n+1)$. But, considering that (36) is the system of the $(n+1)$-vector equations which have same principal part, we can see that (36) corresponds actually to a system of order N. Moreover if $u \equiv 0$ is a solution to (6), then $U \equiv 0$ is also a solution to (36). And then the characteristic roots of (36) are exactly same with those of (6), since the characteristic equation of (36) at $u = 0$ is the form that $\{\det(\lambda - \sum_h \xi_h(\partial f/\partial z_h)(t,x,0))\}^{n+1} = 0$. Hence the hyperbolicity also holds for (36).

Conversely if $U = {}^t(U^0, \dots, U^n)$ is a solution to (36), the Nn-vector $V = {}^t(V^1, \dots, V^n) = {}^t(U^1 - \partial_1 U^0, \dots, U^n - \partial_n U^0)$ is a solution of the another quasilinear system

$$\begin{cases} \partial_t V^j = \displaystyle\sum_{h=1}^{n} \frac{\partial f}{\partial z_h}(t,x,U)\partial_h V^j + \frac{\partial f}{\partial y}(t,x,U)V^j + \sum_{h=1}^{n}\left\{\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial z_h}(t,x,U)\right)\right\}V^h \\ \qquad (1 \le j \le n) \\ V(0,x) = 0. \end{cases}$$

Noting that the initial datum and inhomogeneous part are zero, we have $V \equiv 0$, i.e. $U^j \equiv \partial_j U^0$. Hence returning to (36), we find that $u = U^0$ is a solution to (6). This shows the equivalence of (6) with the quasilinear system (36).

Taking account of this fact, we can reduce (6) to the following quasilinear system

$$(37) \qquad \begin{cases} \partial_t u - \displaystyle\sum_{h=1}^{n} A_h(t,x)\partial_h u - B(t,x)u = \sum_{h=1}^{n} F_h(t,x,u)\partial_h u + F_0(t,x,u)u \\ u(0,x) = \varepsilon\phi(x) \end{cases}$$

where

- $A_h(t,x) = (\partial f/\partial z_h)(t,x,0)$, $B(t,x) = (\partial f/\partial y)(t,x,0)$ are $N \times N$ matrices which are analytic in $\boldsymbol{R}_x^n$ and satisfy $|\partial_x^\alpha A_h(t,x)| \le M(t)\rho_c^{-|\alpha|}\alpha!$, $|\partial_x^\alpha B(t,x)| \le M(t)\rho_c^{-|\alpha|}\alpha!$ a.e. on $(0, \infty)$ for $^\forall\alpha \in \boldsymbol{N}^n$, $^\forall x \in \boldsymbol{R}^n$,
- $F_h(t,x,y)(0 \le h \le n)$ are $N \times N$ matrices which are analytic in $\boldsymbol{R}_x^n$ and a neighbourhood of 0 in $\boldsymbol{R}_y^N$, and satisfy $|\partial_x^\alpha \partial_y^\beta F_h(t,x,y)| \le M(t)\rho_c^{-|\alpha|}\rho_d^{-|\beta|}\alpha!\beta!$ a.e. on $(0, \infty)$ for $^\forall\alpha$, $^\forall\beta \in \boldsymbol{N}^N$, $\forall(x,y) \in \boldsymbol{R}_x^n \times$ "a nighbouhood of 0 in $\boldsymbol{R}_y^N$", and satisfy $F_h(t,x,0) = 0$.

Moreover the assumption (3) also holds for the system (37).

We next consider the equation

$$(38)\qquad \begin{cases} L(\tilde{u})u = l(t,x) \\ u(0,x) = u_0(x), \end{cases}$$

where the linearized operator $L(\tilde{u})$ is defined as

$$\begin{aligned} L(\tilde{u})u = \partial_t u - \sum_{h=1}^{n} A_h(t,x)\partial_h u - B(t,x)u \\ - \sum_{h=1}^{n} F_h(t,x,\tilde{u})\partial_h u - F_0(t,x,\tilde{u})u. \end{aligned}$$

In order to estimate solutions of (38), we modify the estimate (5). Putting $k(t,x) = \sum_{h=1}^{n} F_h(t,x,\tilde{u})\partial_h u + F_0(t,x,\tilde{u})u$ and $g(t,x) = l(t,x) + k(t,x)$, the sixth term on the right side in (19) is changed into

$$\begin{aligned} &\sum_j 2\,\mathrm{Re}(P_{\eta,j}(t,\xi)\langle\xi\rangle^s \mathscr{F}[\varphi_j(x)e^{\rho(t)\langle D\rangle}l], P_{\eta,j}(t,\xi)\langle\xi\rangle^s \hat{v}_j) \\ &\quad + \sum_j 2\,\mathrm{Re}(P_{\eta,j}(t,\xi)\langle\xi\rangle^{s-1/2} \mathscr{F}[\varphi_j(x)e^{\rho(t)\langle D\rangle}k], P_{\eta,j}(t,\xi)\langle\xi\rangle^{s+1/2} \hat{v}_j) \\ &\le C_1^2\|e^{\rho\langle D\rangle}l\|_{H^s}\|v\|_{H^s} + 2C_1^2\|e^{\rho\langle D\rangle}k\|_{H^{s-1/2}}\|v\|_{H^{s+1/2}} \\ &\le C_1^2\|e^{\rho\langle D\rangle}l\|_{H^s}\|v\|_{H^s} + 2C_1^2\sum_{h=0}^{n}\|F_h(\tilde{u})\langle D\rangle u\|_{H_\rho^{s-1/2}}\|v\|_{H^{s+1/2}} \\ &\le C_1^2\|e^{\rho\langle D\rangle}l\|_{H^s}\|v\|_{H^s} + 2C_1^2\sum_{h=0}^{n}\|F_h(\tilde{u})\|_{H_\rho^s}\|\langle D\rangle u\|_{H_\rho^{s-1/2}}\|v\|_{H^{s+1/2}} \\ &= C_1^2\|e^{\rho\langle D\rangle}l\|_{H^s}\|v\|_{H^s} + 2C_1^2\sum_{h=0}^{n}\|F_h(\tilde{u})\|_{H_\rho^s}\left(\sum_j \varphi_j^2\|v\|_{H_\rho^{s+1/2}}^2\right) \\ &\le C_1^2\|e^{\rho\langle D\rangle}l\|_{H^s}\|v\|_{H^s} + 2C_1^2\sum_{h=0}^{n}\|F_h(\tilde{u})\|_{H_\rho^s}\left(\sum_j \|v_j\|_{H^{s+1/2}}^2 + C_6\|v\|_{H^{s-1/2}}^2\right) \\ &\le C_1^2\|e^{\rho\langle D\rangle}l\|_{H^s}\|v\|_{H^s} + 2C_1^2C_6\sum_{h=0}^{n}\|F_h(\tilde{u})\|_{H_\rho^s}\|v\|_{H^s}^2 \\ &\quad + 2C_1^2C_2\eta^{-C_3}\sum_{h=0}^{n}\|F_h(\tilde{u})\|H_\rho^s\left(\sum_j \|\langle\cdot\rangle^{s+1/2}w_j\|_{L_\xi^2}^2\right) \end{aligned}$$

In the third inequality we used ii) in § 2. Similarly we get the following estimate corresponding to (22)

(39) $$\|v(t)\|_{H^s} \leq C_\eta\|v(0)\|_{H^s} + C'_\eta\int_0^t \{\rho'(\tau) + C_7M(\tau)\rho(\tau) + \eta(C_9M_a(\tau)+1)$$

$$+ C_1^2C_2\eta^{-C_3}\sum_{h=0}^n \|F_h(\tilde{u})\|H_\rho^s\}\|v(\tau)\|H^{s+1}d\tau$$

$$+ C''_\eta\int_0^t \mu(\tau)\|v(\tau)\|_{H^s}d\tau + C'''_\eta\int_0^t \|e^{\rho(\tau)\langle D\rangle}l(\tau)\|_{H^s}d\tau,$$

where $\mu(t)$ includes $2C_1^2C_6\sum_{h=0}^n\|F_h(\tilde{u})\|_{H_\rho^s}$.

When $\|\tilde{u}\|_{H_\rho^s} < \eta^{C_3+1}$, by v) in § 2 it holds that

$$\sum_{h=0}^n \|F_h(\tilde{u})\|_{H_\rho^s} \leq n(CM(t))N\frac{C_s}{\rho_d}\left(1 - N\frac{C_s}{\rho_d}\|\tilde{u}\|_{H_\rho^s}\right)^{-1}\|\tilde{u}\|_{H_\rho^s}$$

$$\leq n(CM(t))N\frac{C_s}{\rho_d}\left(1 - N\frac{C_s}{\rho_d}\eta^{C_3+1}\right)^{-1}\eta^{C_3+1}$$

$$\leq C'M(t)\eta^{C_3+1} \quad \text{(for small } \eta > 0)$$

Hence if $\rho(t)$ is given as

$$\begin{cases} \rho'(t) + C_7M(t)\rho(t) + \eta((C_9 + C_1^2C_2C')M(t) + 1) = 0 \quad \text{a.e. on } (0,T) \\ \rho(0) = \rho_0 \quad (0 < \rho_0 < \rho_c/\sqrt{n}) \quad \text{i.e.} \end{cases}$$

(40) $$\rho(t) = e^{-C_7\int_0^t M(\tau)d\tau}\left\{\rho_0 - \eta\int_0^t e^{C_7\int_0^\tau M(\sigma)d\sigma}(C_{10}M(\tau)+1)d\tau\right\},$$

where $C_{10} = C_9 + C_1^2C_2C'$ and T is defined as

(41) $$\rho_0 = \eta\int_0^T e^{C_7\int_0^T M(\sigma)d\sigma}(C_{10}M(\tau)+1)d\tau,$$

then we can eliminate the last term on the right side of (39). Thus by the proof of Theorem 1, the problem (38) has a unique solution satisfying

(42) $$\|e^{\rho(t)\langle D\rangle}u(t)\|_{H^s} \leq K_\eta(t)\left\{\|u_0\|_{H_{\rho_0}^s} + \tilde{C}_\eta\int_0^t \|e^{\rho(\tau)\langle D\rangle}l(\tau)\|_{H^s}d\tau\right\}.$$

where $K_\tau(t) = C_\eta e^{C''_\eta\int_0^t\mu(\tau)d\tau}$ has continuity on $[0,T)$. Since we can take the small η to satisfy $K_\eta(t) \geq 1$, we suppose $K_\eta(t) \geq 1$ without loss of generality.

In order to solve the system (37), writing (37) as

(43) $$\begin{cases} L(u)u = 0 \\ u(0,x) = u_0(x) = \varepsilon\psi(x), \end{cases}$$

we define recursively the sequence $\{u_i\}_{i\geq 0}$ as

$$u_0(t,x) = u_0(x) \quad \text{for } i = 0$$

$$(44) \qquad \begin{cases} L(u_{i-1})u_i = 0 \\ u_i(0,x) = u_0(x) = \varepsilon\phi(x) \quad \text{for } i \leq 1. \end{cases}$$

LEMMA 2. *Assuming that*

$$(45) \qquad \|u_0\|_{H^s_{\rho_0}} (= \varepsilon\|\phi\|_{H^s_{\rho_0}}) \leq \frac{\eta^{C_3+1}}{K_\eta(t)},$$

then the function $u_i(t,x)$ *are well defined on* $[0,T_\varepsilon)\times \boldsymbol{R}^n_x$ *and satisfy the estimate*

$$(46) \qquad \|e^{\rho(t)\langle D\rangle}u_i(t)\|_{H^s} \leq K_\eta(t)\|u_0\|_{H^s_{\rho_0}} \quad \textit{for } {}^\forall t\in[0,T)$$

PROOF. Since $K_\eta(t)\geq 1$, it follows that

$$\|e^{\rho(t)\langle D\rangle}u_0\|_{H^s} \leq \|u_0\|_{H^s_{\rho_0}} \leq K_\eta(t)\|u_0\|_{H^s_{\rho_0}}.$$

Hence we find (46) holds for $i=0$.

Assuming that (46) holds for some $i\geq 0$, by (45) we obtain

$$\|e^{\rho(t)\langle D\rangle}u_i(t)\|^2_{H^s} \leq \eta^{C_3+1} \quad \text{a.e. on } (0,T).$$

Thus by (42) we have the estimate with $l(t,x)=0$

$$\|e^{\rho(t)\langle D\rangle}u_{i+1}\|_{H^s} \leq K_\eta(t)\|u_0\|_{H^s_{\rho_0}}.$$

This concludes the proof of Lemma 2.

LEMMA 3. *Under the assumption of Lemma* 1, *it holds that*

$$(47) \qquad \sum_{i=1}^{\infty}\|e^{\tilde{\rho}(t)\langle D\rangle}(u_i(t)-u_{i-1}(t))\|_{H^s} < \infty \quad \textit{for } \forall t\in[0,T),$$

where $\tilde{\rho}(t) = e^{-C_7\int_0^t M(\tau)d\tau}\{\tilde{\rho}_0 - \eta\int_0^t e^{C_7\int_0^\tau M(\sigma)d\sigma}(C_{10}M(\tau)+1)d\tau\}$ *with* ${}^\forall\tilde{\rho}_0 < \rho_0$.

PROOF. Putting $w_i = u_i - u_{i-1}, i = 1,2,\ldots$, we can see that w_{i+1} satisfies the problem

$$(48) \qquad \begin{cases} L(u_i)w_{i+1} = l_i(t,x) \\ w_{i+1}(0,x) = 0, \end{cases}$$

where $l_i(t,x) = \sum_{h=1}^{n}(F_h(u_i) - F_h(u_{i-1}))\partial_h u_i + (F_0(u_i) - F_0(u_{i-1})u_i$. By vi) in § 2 it holds that

$$\|l_i\|_{H^s_{\tilde{\rho}}} \leq (n+1)\tilde{C}_s N \frac{C_s}{\rho_d}\left\{1 - 2N\frac{C_s}{\rho_d}(\|u_i\|_{H^s_{\tilde{\rho}}} + \|u_{i-1}\|_{H^s_{\tilde{\rho}}})\right\}^{-1}\|w_i\|_{H^s_{\tilde{\rho}}}\|u_i\|_{H^{s+1}_{\tilde{\rho}}}.$$

Consequently, by (46) and iv) in § 2, we see that

$$\|e^{\tilde{\rho}(t)\langle D\rangle}l_i(t)\|_{H^s} \leq \frac{C}{\rho_0 - \tilde{\rho}_0}\|e^{\tilde{\rho}(t)\langle D\rangle}w_i(t)\|_{H^s} \quad \text{for } {}^\forall t \in [0,T).$$

Now using again (42) with $u_0(x) = 0$, $l(t,x) = l_i(t,x)$, we get

$$\|e^{\tilde{\rho}(t)\langle D\rangle}w_{i+1}(t)\|_{H^s} \leq \frac{C'}{\rho_0 - \tilde{\rho}_0}\int_0^t \|e^{\tilde{\rho}(t)\langle D\rangle}w_i(\tau)\|_{H^s}d\tau \quad \text{for } {}^\forall t \in [0,T).$$

Hence (47) easily follows.

The sequence $\{u_i\}$ converges to some function $u_\varepsilon(t,x)$, and passing to the limit in (44), (46), we see that u_ε is a solution of (43) and holds that for ${}^\forall t \in [0,T)$

$$\|e^{\rho(t)\langle D\rangle}u_\varepsilon(t)\|_{H^s} \leq K_\eta(t)\|u_0\|_{H^s_{\rho_0}} = \varepsilon K_\eta(t)\|\phi\|_{H^s_{\rho_0}}. \tag{49}$$

In conclusion, if $\varepsilon \to 0$, then by (49) $\|e^{\rho(t)\langle D\rangle}u_\varepsilon\|_{H^s} \to 0$. While from the condition (45), we find the relation

$$\|\phi\|_{H^s_{\rho_0}} C_\eta e^{C''_\eta \int_0^t \mu(\tau)d\tau}\eta^{-C_3-1} \leq \frac{1}{\varepsilon}$$

$$C_\phi C_1 C_2 \eta^{-C_3} e^{(C_2\eta^{-C_3})^3 \int_0^t C(\eta^{C_3}M(\tau)+1)d\tau}\eta^{-C_3-1} \leq \frac{1}{\varepsilon}$$

$$\eta \geq C\left\{\left(\int_0^t M(\tau)d\tau\right)\left(\log\frac{1}{\varepsilon}\right)^{-1}\right\}^{C'}$$

$$({}^\exists C, {}^\exists C' > 0)$$

Thus by (41) we can find the relation between ε and T_ε

$$\rho_0 \geq C\left\{\left(\int_0^{T_\varepsilon} M(\tau)d\tau\right)\left(\log\frac{1}{\varepsilon}\right)^{-1}\right\}^{C'}\int_0^{T_\varepsilon} e^{C_7\int_0^\tau M(\sigma)d\sigma}(C_{10}M(\tau)+1)d\tau$$

Putting $\psi(t)=\int_0^t M(\tau)d\tau$,

$$\leq C\left\{\psi(T_\varepsilon)\left(\log\frac{1}{\varepsilon}\right)^{-1}\right\}^C\int_0^{T_\varepsilon}e^{C\psi(\tau)}C\frac{\partial}{\partial\tau}\psi(\tau)d\tau$$

$$=C\left\{\psi(T_\varepsilon)\left(\log\frac{1}{\varepsilon}\right)^{-1}\right\}^C\int_0^{T_\varepsilon}\frac{\partial}{\partial\tau}(e^{C\psi(\tau)})d\tau$$

$$=C\left\{\psi(T_\varepsilon)\left(\log\frac{1}{\varepsilon}\right)^{-1}\right\}^C(e^{C\psi(T_\varepsilon)}-1).$$

Since $\psi(t)$ is a increasing function for t, we can see $T_\varepsilon\to\infty$ as $\varepsilon\to 0$. Hence we can also get the asymptotic estimate

$$T_\varepsilon\geq\psi^{-1}\left(\mu\log\log\frac{1}{\varepsilon}\right)\quad(\mu>0).$$

This concludes our proof.

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