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MICROLOCAL COMPLEX FOLIATION OF $\boldsymbol{R}$-LAGRANGIAN CR SUBMANIFOLDS

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Abstract. Let X be a complex manifold, M a real analytic submanifold of $X^{\boldsymbol{R}}$, T^*X the cotangent bundle to X, T_M^*X the conormal bundle to M in X. Assume that T_M^*X is regular and CR in T^*X. We then show that T_M^*X is locally defined as the zero-set of the real and/or imaginary part of holomorphic symplectic coordinates of T^*X. It is well known that the similar description of M in local complex coordinates of X is true only if M is *Levi flat*. As an application we obtain a generalization of the celebrated *edge of the wedge* Theorem.

§1. Let X be a complex manifold of dimension n, $\pi: T^*X \to X$ the cotangent bundle to X, $\dot{T}^*X$ the bundle T^*X with the 0-section removed, $\alpha = \alpha^{\boldsymbol{R}} + \sqrt{-1}\alpha^{\boldsymbol{I}}$ (resp. $\sigma(= d\alpha) = \sigma^{\boldsymbol{R}} + \sqrt{-1}\sigma^{\boldsymbol{I}}$) the canonical 1-form (resp. 2-form) on T^*X. Let $X^{\boldsymbol{R}}$ (resp. $(T^*X)^{\boldsymbol{R}}$) be the real analytic manifold underlying to X (resp. T^*X); we have diagonal identifications:

$$(1.1)\quad X^{\boldsymbol{R}} \overset{j}{\rightarrow} X \times_X \overline{X}, \quad T(X^{\boldsymbol{R}}) \overset{j'}{\rightarrow} TX \times_{TX} T\overline{X} \simeq (TX)^{\boldsymbol{R}}, \quad T^*(X^{\boldsymbol{R}}) \overset{{}^t j'}{\leftarrow} (T^*X)^{\boldsymbol{R}}.$$

A complex analytic submanifold $V \subset \dot{T}^*X$ is $\boldsymbol{C}$-involutive (resp. Lagrangian, resp. isotropic) if at each $p \in V$ the tangent plane $v(p) = T_pV$ verifies $v^{\perp}(p) \subset v(p)$ (resp. $v^{\perp}(p) = v(p)$, resp. $v^{\perp}(p) \supset v(p)$). (The planes $v(p)$ themselves will be called in the corresponding manner.) V is called regular when $\alpha|_V \neq 0$. A real analytic submanifold $\Lambda \subset T^*X^{\boldsymbol{R}}$ is called $\boldsymbol{R}$-Lagrangian when $\lambda(p) :\overset{\text{def.}}{=} T_p\Lambda$ is Lagrangian for $\sigma^{\boldsymbol{R}}(p)$. Λ is called $\boldsymbol{I}$-symplectic when $\sigma^{\boldsymbol{I}}(p)$ is non-degenerate on $\lambda(p)$. All submanifolds of T^*X (resp. $T^*X^{\boldsymbol{R}}$) will be $\boldsymbol{C}^{\times}$-conic (resp. $\boldsymbol{R}^+$-conic).

Let M be a real analytic submanifold of $X^{\boldsymbol{R}}$ of codim l, and T_M^*X the conormal bundle to M in X identified, via the third of (1.1), to an $\boldsymbol{R}$-Lagrangian

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submanifold of $T^*X^{\boldsymbol{R}}$. We fix $p \in \dot{T}^*_M X, \pi(p) = z$, and define

$$\lambda_M(p) = T_p T^*_M X, \quad T^{\boldsymbol{C}}_z M = T_z M \cap \sqrt{-1} T_z M. \tag{1.2}$$

We define the Levi form $L_M(p)$ of M at p as the restriction to $T^{\boldsymbol{C}}_z M$ of the Hermitian form $\partial\bar{\partial} r_1(z)$, where r_1 is a function with $r_1|_M \equiv 0$ and $\partial r_1(z) = p$. We denote by $s^{+,-,0}_M(p)$ the numbers of respectively positive, negative, and null eigenvalues of $L_M(p)$.

We complete r_1 to a system of independent equations $(r_j)_{j=1,\dots,l} = 0$ for M, and give a parametric representation of $T^*_M X$:

$$\psi : M \times \boldsymbol{R}^l \xrightarrow{\sim} T^*_M X, \quad (z; (t_j)) \mapsto \left(z; \sum_j t_j \partial r_j(z)\right). \tag{1.3}$$

We take the composition $\psi \circ (j^{-1} \times \mathrm{id})$ where j is the map in (1.1). (This just means, for coordinates $z = x + \sqrt{-1} y \in X$, to consider ψ as a function of $(z, \bar{z})$ rather than (x,y).) By the aid of $\psi \circ (j^{-1} \times \mathrm{id})$, we get the identifications:

$$\begin{aligned} \lambda_M(p) = \Bigg\{ \left(u; \sum_j t_j \partial r_j + \partial\partial r_1(z) u + \partial\bar{\partial} r_1(z) \bar{u}\right); (t_j) \in \boldsymbol{R}^l, \\ \partial r_1(z) u + \bar{\partial} r_1(z) \bar{u} = 0 \Bigg\}, \end{aligned} \tag{1.4}$$

$$\begin{aligned} \lambda_M(p) \cap \sqrt{-1}\lambda_M(p) = \{(u; {}^t\partial\bar{\partial} r_1(z) u + \partial\partial r_1(z) u); \partial r_1(z) u = 0, \partial\bar{\partial} r_1(z) \bar{u} \in \\ T^*_S X_z + \sqrt{-1} T^*_S X_z\} \oplus \{(0; v); v \in T^*_M X_z \cap \sqrt{-1} T^*_M X_z\}, \end{aligned}$$

$(z = \pi(p))$. It follows

$$\lambda_M(p) \cap \sqrt{-1}\lambda_M(p) \simeq \mathrm{Ker}\, L_M(p) \oplus (T^*_M X_z \cap \sqrt{-1} T^*_M X_z) \tag{1.5}$$

Put $\gamma_M(z) = \dim_{\boldsymbol{C}}(T^*_M X_z \cap \sqrt{-1} T^*_M X_z)$; we get from (1.5)

$$\begin{aligned} \mathrm{rank}\, L_M(p) &= \dim T^{\boldsymbol{C}}_z M - \dim \mathrm{Ker}\, L_M(p) \\ &= (n - l - \dim_{\boldsymbol{C}}(\lambda_M(p) \cap \sqrt{-1}\lambda_M(p))) + 2\gamma_M(z). \end{aligned} \tag{1.6}$$

Let $M \subset X$ and $p \in \dot{T}^*_M X$.

THEOREM 1.1. *Assume that $\dot{T}^*_M X$ is regular at p and verifies*

$$\dim(\lambda_M(p) \cap \sqrt{-1}\lambda_M(p)) \equiv \text{const in a neighborhood of } p. \tag{1.7}$$

Then we may find local complex symplectic coordinates $(z;\zeta) = (z', z''; \zeta', \zeta'') \in \dot{T}^*X$, $z = x + \sqrt{-1}y, \zeta = \xi + \sqrt{-1}\eta$ *such that* $p = (0; i\,dy_1)$ *and*:

$$\dot{T}^*_M X = \{(z;\zeta) \in \dot{T}^*X; y' = \xi' = 0, \zeta'' = 0\}. \tag{1.8}$$

Proof. (a) We put $\Lambda_M = \dot{T}^*_M X$. Regularity of Λ_M at p means that $\lambda_M(p)$ meets the complex plane spanned by the radial vector field at p along a real line. In this situation it is well known that Λ_M can be interchanged, by a complex symplectic transformation χ, with the conormal bundle to a hypersurface, and that $s^- = 0$ at $\chi(p)$ for such hypersurface. But we have indeed $s^- \equiv 0$ in a neighborhood of $\chi(p)$ by (1.7), because the constancy of $s^\pm - \gamma$ is a symplectic invariant due to (1.6). Thus this hypersurface is in fact the boundary of a pseudoconvex domain. By the same reason $s^+ \equiv$ const. Thus it is not restrictive to assume from the beginning M to be the boundary of a pseudoconvex domain with dim (Ker L_M) $\equiv$ const (say d). By [F], [R] (and [S]) M is locally foliated by the integral leaves of Ker L_M; these are complex manifolds of dim d (since they have complex tangent planes of the corresponding dim). (For a new proof with some improvements of the results on Levi foliations see also [Z].)

(b) There is a foliation of $\dot{T}^*_M X$ at p whose leaves are complex sections of $\dot{T}^*_M X$ over the leaves of M. In fact let Γ be a complex leaf of M defined, in complex coordinates $z = (z_1, z', z'') \in X$, by $z_1 = z' = 0$, and let $p = (0; i\,dy_1)$. One has

$$L_M(p')(w, \cdot) = 0 \quad \forall w \in \boldsymbol{C}^d_{z''} \quad \forall p' \in T^*_M X \cap \pi^{-1}(\Gamma) \text{ close to p.} \tag{1.9}$$

In fact if $r|_M \equiv 0$ with $\partial r(z) = p$, then clearly $\partial_{z''}\partial_{\bar{z}''} r \equiv 0$ on Γ and if by absurd $\partial_{z'}\partial_{\bar{z}''} r \neq 0$ at some point of Γ close to z, then the pseudoconvexity of M should be violated.

We denote by $g : M \to M' \simeq \boldsymbol{R}^{2n-l-2d}$ the foliation of M, and set $R = g^{-1}(M \cap \boldsymbol{C}_{z_1})$. We remark that R is a CR manifold (of CR dim d) due to dim $(TR \cap \sqrt{-1}TR) \equiv d$. Let $j : R \hookrightarrow X$, and let $Y = p_1 \circ j^{\boldsymbol{C}}(R^{\boldsymbol{C}})$ where $p_1 : X \times \bar{X} \to X$. Y is a complex manifold with $\dim(Y) = d + 1$ by the above remark. Moreover since $\bar{Z}g = 0 \forall$ antiholomorphic tangent vector field $\bar{Z} \in T^{0,1}R (= \{0\} \times T^{\boldsymbol{C}}R \hookrightarrow TX \times_X T\bar{X}|_M)$, then g extends to a holomorphic function $\tilde{g} : Y \to \boldsymbol{C}_{z_1}$. In complex coordinates in which $g : (z_1, z'') \to z_1$, we have $R = \boldsymbol{R}_{x_1} \times \{0\} \times \boldsymbol{C}^d_{z''}$. Since $S \supset R$, then we may write $r = y_1 + 0(|z'|)(0(|(z_1, z'')|) + O(|z'|)$. Thus for $\Gamma = \boldsymbol{C}^d_{z''}$, we have

$$\partial_{\bar{z}}(\partial_z r|_\Gamma)(\equiv (\partial_{\bar{z}''}\partial_z r)|_\Gamma) \equiv 0 \quad \text{(i.e. } \partial_z r|_\Gamma \text{ is holomorphic).} \tag{1.10}$$

In fact $\partial_{z_1} r|_\Gamma \equiv -\sqrt{-1}$ and $\partial_{\bar{z}''}\partial_{z_i} r|_\Gamma \equiv 0\ \forall i \neq 1$ by (1.9). Thus we have a foliation of $\dot{T}^*_M X$ by the complex leaves $\Gamma_t = \{(z; t\partial r(z)); z \in \Gamma\}$, $t \in \boldsymbol{R}$. This gives a projection

$$(1.11) \qquad \rho : \Lambda_M \to \Lambda',$$

with complex fibers.

(c) We note that $\bar{Z}_e = 0\ \forall\, \bar{Z} \in T^{0,1}\Lambda_M$ (due to Ker $\rho' = \lambda_M \cap \sqrt{-1}\lambda_M$); thus e extends to a holomorphic map $\tilde{\rho} : V \to \Lambda'^{\boldsymbol{C}}$ where V is the partial complexification of V in T^*X, and $\Lambda'^{\boldsymbol{C}}$ a complexification of Λ'. Note here that such V exists because Λ_M is CR in $\dot{T}^*X$ by (1.7)

We claim that V is a regular involutive submanifold of $\dot{T}^*X$, and $\tilde{\rho}$ is the projection along the bicharacteristic leaves of V. In fact if $v = TV$ and $v^\perp$ is the symplectic orthogonal, then $v^\perp$ and Ker $\tilde{\rho}'$ are two complex bundles on V of dim d which verify $v^\perp|_{\Lambda_M} = \operatorname{Ker} \tilde{\rho}'|_{\Lambda_M} (= \lambda_M \cap \sqrt{-1}\lambda_M)$. Thus $v^\perp = \operatorname{Ker} \tilde{\rho}'$ which proves the claim. Let $V' = V/\sim$, where $\sim$ is the equivalence relation which identifies all points of V in the same bicharacteristic leaf; then $V' \equiv \Lambda'^{\boldsymbol{C}}$.

Clearly $v' = v/v^\perp$ and thus σ induces a non-degenerate form σ' on V'. We also have $\lambda' = \lambda_M/v^\perp = \lambda_M/(\lambda_M \cap \sqrt{-1}\lambda_M)$; thus Λ' is $\boldsymbol{R}$-Lagrangian and $\boldsymbol{I}$-symplectic in V'.

(d) We take complex symplectic coordinates $(z;\zeta) \in \dot{T}^*X, z = (z', z''), \zeta = (\zeta', \zeta')$, $z = x + \sqrt{-1}y, \zeta = \xi + \sqrt{-1}\eta$ s.t.:

$$V = \dot{T}^*X' \times \boldsymbol{C}^d, \quad V' = \dot{T}^*X', \quad \Lambda_M = \Lambda' \times \boldsymbol{C}^d, \quad X' = \boldsymbol{C}^{n-d}, \quad p = (0; i\,\mathrm{d}y_1).$$

We note that any $\boldsymbol{R}$-Lagrangian $\boldsymbol{I}$-symplectic submanifold of $\dot{T}^*\boldsymbol{C}^{n-d}$ can be transformed, by a complex symplectic transformation, into $\dot{T}^*_{\boldsymbol{R}^{n-d}}\boldsymbol{C}^{n-d}$; thus after this transformation $\Lambda_M = T^*_{\boldsymbol{R}^{n-d}}\boldsymbol{C}^{n-d} \times \boldsymbol{C}^d$. Q.E.D.

§2. We suppose in this section that M is a real analytic generic submanifold of $X^{\boldsymbol{R}}$ (i.e. $\gamma_M = 0$) of codim l, and that $\dot{T}^*_M X$ verifies (1.7) over an open cone $U \subset \dot{T}^*_M X$. ($\dot{T}^*_M X$ is automatically regular because $\gamma_M = 0$.) Let $\mathscr{C}_{M|X}$ and $\mathscr{B}_{M|X}$ be the sheaves of resp. CR microfunctions and CR hyperfunctions along M. These are concentrated in degree s^-_M and $s^-_M, 0$ respectively (cf. [K-S]). We recall that $\mathscr{B}_{M|X}$, (defined as $\boldsymbol{R}\Gamma_M(\mathcal{O}_X)[l]$ with $\mathcal{O}_X$ denoting the sheaf of holomorphic functions on X), turns out to coincide with the sheaf of the s^-_M-th cohomology of the tangential $\bar{\partial}$-complex over (usual) hyperfunctions $\mathscr{B}_M$. Let $\mathrm{sp} : H^{s^-_M}(\pi^{-1}(\mathscr{B}_{M|X})) \to H^{s^-_M}(\mathscr{C}_{M|X})$ be the spectral morphism, and define

$$(2.1) \qquad WF(f) = \operatorname{supp}(\operatorname{sp}(f)), f \in H^{s^-_M}(\mathscr{B}_{M|X}).$$

WF coincides, at least for $s_M^- = 0$, with the usual analytic wave front set (cf. [B-C-T]). According to [S-K-K], [K-S], the symplectic transformation which gives (1.8) can be *quantized* to an isomorphism:

$$(2.2) \qquad \mathscr{C}_{M|X} \simeq \mathscr{C}_{\boldsymbol{R}^{n-d}\times \boldsymbol{C}^d|X}[-s_M^-].$$

Thus $\mathscr{C}_{M|X}$ is isomorphic, upto a shift $-s_M^-$, to the sheaf of usual microfunctions with holomorphic parameters. In particular, according to [S-K-K]:

$H^{s_M^-}(\mathscr{C}_{M|X})|_U$ *satisfies the principle of the analytic contination along the integral leaves of* $\lambda_M \cap \sqrt{-1}\lambda_M$.

Let δ be an open convex cone of $T_M X := M \times_X TX/TM$. We recall that a domain $W \subset X$ is said to be a *wedge* with *profile* δ when $C_M(X\backslash W) \cap \delta = \emptyset$ (where $C_M(\cdot)$ denotes the Whitney normal cone along M). Let η be a closed convex proper cone of $T_M^* X$ with $\eta \supset M$. We have

$$(2.3) \qquad H_\eta^j(T_M^* X, \mathscr{C}_{M|X}) \overset{b}{\leftarrow} \varinjlim_{W} H^j(W, \mathscr{O}_X),$$

where W ranges through the family of wedges with profile $\delta = \operatorname{int} \eta^{oa}$ the interior of the antipodal of the polar to η. (b is called the *boundary values* morphism.)

Fix $z \in M, z \in \pi(U)$, write $z = (z', z'')$, $M = M' \times Y$ (Y a polydisc with center z'').

Proposition 2.1. *Let M be real analytic generic and satisfy* (1.7) *in U. Let $\eta_j' = M' \times Z_j, j = 1, \dots, N$, be closed convex proper cones of U', and let $F_j \in H^{s_M^-}((W_j' \times Y) \cap B, \mathscr{O}_X)$ where B (resp. W_j') ranges through the family of neighborhoods of z (resp. of wedges of X' with profile $\delta_j' = M' \times \operatorname{int} Z_j^{oa}$). Assume $\sum_j b(F_j) = 0$. Then there exist $F_{ij} \in H^{s_M^-}((W'_{ij\times Y_1}) \cap B, \mathscr{O}_X)$ with $Y_1 \subset Y$ and with W_{ij}' wedges with profile and proper subcone of the convex hull δ_{ij}' of δ_i', δ_j':*

$$F_{ij} = -F_{ji} \qquad F_j = \sum_i F_{ij} \ \forall j.$$

Proof. Let $f_j = b(F_j)|_U$. Then supp $(f_j) \subset (\cup_{i\neq j}(\eta_i' \cap \eta_j')) \times Y = (M \times \cup_{i\neq j}(Z_i \cap Z_j)) \times Y$.

Observe that $H^{s_M^-}(C_{\text{mix}})|_U$ satisfies a kind of "transveral softness" with respect to the complex foliation of $T_M^* X$; this follows easily from (2.2). Thus we can decompose $f_j = \sum_i f_{ij}$ with $WF(f_{ij}) \subset (\tilde{\eta}_{ij}' \times Y_1) = (\tilde{\eta}_i' \cap \tilde{\eta}_j') \times Y_1$ for $\tilde{\eta}_i' \supset \eta_i'$ and over a (possibly smaller) neighborhood of z. If we observe that $\operatorname{int}(\eta_i' \cap \eta_j')^{oa}$ equals the convex hull of $\operatorname{int} \eta_i'^{oa}$, $\operatorname{int} \eta_j'^{oa}$ and use (2.3), we get the conclusion. Q.E.D.

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