

SEPARABLE FUNCTORS IN COALGEBRAS. APPLICATIONS

By

F. Castaño IGLESIAS, J. Gómez TORRECILLAS and C. NĂSTĂESCU*

Introduction

The notion of separable functor was introduced in [5], where some applications in the framework of group-graded rings were done. This notion fits satisfactorily to the classical notion of separable algebra over a commutative ring. The concept of coseparable coalgebra over a field appears in [1] to prove a result of Sullivan [7]. A more complete study of the separability of coalgebras was performed in [2]. In this last paper, an analysis of the relationship between coseparability and the cohomology theory for coalgebras is developed.

Our aim is to study the separability, in the sense of [5], of some canonical functors stemming from a morphism of coalgebras.

In Section 1 we fix some notation and we prove a preliminary characterization of the bicomodules.

The Section 2 contains the theoretical body of the paper. For a morphism of coalgebras $\varphi : C \rightarrow D$, we characterize the separability of the corestriction functor $(-)_\varphi$ (Theorem 2.4) and of the coinduction functor $(-)^{\varphi}$ (Theorem 2.7). The reader can find the definitions of these functors in Section 1. For the particular case of the coalgebra morphism $\varepsilon : C \rightarrow k$ given by the counit of the k -coalgebra C , the separability of the corestriction functor gives precisely the notion of coseparable coalgebra. We finish the section with Theorem 2.9, that entails that a coseparable coalgebra need not to be necessarily of finite dimension (Theorem 3.4).

Section 3 is devoted to study the relationship between coseparability and co-semisimplicity for coalgebras. As a consequence, we obtain that a k -coalgebra

* This paper was written while the third author was at the University of Almería as a Visiting Professor supported by the grant SAB94-0290 from DGICYT.

Received May 29, 1995.

Revised September 12, 1995.

C is coseparable if and only if the coalgebra induced by any field extension of k is co-semi-simple.

1. Notation and Preliminaries

Let k be a commutative field. Any tensor product \otimes_k over k will be simply denoted by \otimes . The identity map on a set X will be denoted by 1_X or even by 1 . A coalgebra over k is a k -vector space C together with two k -linear maps $\Delta_C : C \rightarrow C \otimes C$ and $\varepsilon_C : C \rightarrow k$ such that $(1 \otimes \Delta_C) \circ \Delta_C = (\Delta_C \otimes 1) \circ \Delta_C$ and $(\varepsilon_C \otimes 1) \circ \Delta_C = (1 \otimes \varepsilon_C) \circ \Delta_C = 1$. We shall refer to [8] for details. The dual space $C^* = \text{Hom}_k(C, k)$ can be canonically endowed with structure of k -algebra. A right C -comodule is a k -vector space M together with a structure k -linear map $\rho_M : M \rightarrow M \otimes C$ such that $(1_M \otimes \varepsilon) \circ \rho_M = 1_M$ and $(\rho_M \otimes 1_C) \circ \rho_M = (1_M \otimes \Delta_C) \circ \rho_M$. The coalgebra C can be considered as a right C -comodule with structure map $\rho_C = \Delta_C$. A k -linear map $f : M \rightarrow N$ between right C -comodules is said to be C -colinear or a *morphism of right C -comodules* if $(f \otimes 1) \circ \rho_M = \rho_N \circ f$. The right C -comodules together with the C -colinear maps between them form a Grothendieck category \mathcal{M}^C . In fact, \mathcal{M}^C is isomorphic to a closed subcategory of the category $C^* \text{-Mod}$ of all left modules over C^* . In particular, the C -colinear maps between C -comodules are precisely the C^* -linear maps between them. For the notion of closed subcategory we shall refer to [3, p. 395]. The notation $\text{Com}_C(M, N)$ stands for the k -vector space of all the C -colinear maps between two C -comodules M, N . The category of left C -comodules will be denoted by ${}^C\mathcal{M}$. We will use Sweedler's Σ -notation. For example, if M is a right C -comodule, then $\rho_M(m) = \sum_{(m)} m_0 \otimes m_1 \in M \otimes C$ for $m \in M$. The structure of left C^* -module is given by $fm = \sum_{(m)} m_0 f(m_1)$, for $f \in C^*$.

It is not difficult to see that if W is a k -vector space and X is a right C -comodule, then $W \otimes X$ is a right C -comodule with structure map $1_W \otimes \rho_X : W \otimes X \rightarrow W \otimes X \otimes C$. Moreover, if W is a right C -comodule, then the structure map $\rho_W : W \rightarrow W \otimes C$ becomes C -colinear. Consider coalgebras C and D . Following [9], a C - D -bicomodule M is a left C -comodule and a right D -comodule such that the C -comodule structure map $\rho_M^- : M \rightarrow C \otimes M$ is D -colinear or, equivalently, that the D -comodule structure map $\rho_M^+ : M \rightarrow M \otimes D$ is C -colinear. Equivalently, if $\rho_M^-(m) = \sum_{(m)} m_{-1} \otimes m_0$ and $\rho_M^+(m) = \sum_{(m)} m_0 \otimes m_1$, then

$$\sum_{(m)} m_{-1} \otimes (m_0)_0 \otimes (m_0)_1 = \sum_{(m)} (m_0)_{-1} \otimes (m_0)_0 \otimes m_1.$$

Furthermore, given a k -coalgebra D and a k -algebra R we can consider the

category M_R^D consisting of the right D -comodules and right R -modules M satisfying the compatibility condition

$$\sum_{(mr)} (mr)_0 \otimes (mr)_1 = \sum_{(m)} m_0 r \otimes m_1$$

for every $m \in M$ and $r \in R$ or, equivalently, that the homothety $h_r : M \rightarrow M$, $m \mapsto mr$ is a D -comodule map on M for every $r \in R$. The morphisms in this category are the right D -colinear and right R -linear maps.

The following characterization of the bicomodules will be useful in this paper.

PROPOSITION 1.1. *Let C, D be two coalgebras and consider a k -vector space M such that M is a left C -comodule and a right D -comodule. The following statements are equivalent*

- (i) M is a C - D -bicomodule.
- (ii) M is a D^* - C^* -bimodule.
- (iii) $M \in M_{C^*}^D$.
- (iv) $M \in {}_D^C M$.

PROOF. (i) \Rightarrow (ii) Let $f \in C^*$, $g \in D^*$ and $m \in M$.

$$(gm)f = \left(\sum_{(m)} m_0 g(m_1) \right) f = \left(\sum_{(m)} m_0 f \right) g(m_1) = \sum_{(m)} f((m_0)_{-1}) (m_0)_0 g(m_1)$$

Moreover

$$g(mf) = g \left(\sum_{(m)} f(m_{-1}) m_0 \right) = \sum_{(m)} f(m_{-1}) g m_0 = \sum_{(m)} f(m_{-1}) (m_0)_0 g((m_0)_1)$$

Since

$$\sum_{(m)} m_{-1} \otimes (m_0)_0 \otimes (m_0)_1 = \sum_{(m)} (m_0)_{-1} \otimes (m_0)_0 \otimes m_1$$

we conclude that $(gm)f = g(mf)$.

(ii) \Rightarrow (i) Let $m \in M$. The k -subspace mC^* of M is finite-dimensional and, thus, $D^*(mC^*)$ is finite-dimensional. Since M is a bimodule, $D^*(mC^*) = (D^*m)C^*$. Let

$\{e_1, \dots, e_n\}$ be a k -basis of this vector space. We will prove that

$$(1 \otimes \rho_M^+) \circ \rho_M^-(e_i) = (\rho_M^- \otimes 1) \circ \rho_M^+(e_i) \tag{I}$$

for every $i = 1, \dots, n$. Put

$$\rho_M^+(e_i) = \sum_j e_j \otimes d_i^j \quad \rho_M^-(e_i) = \sum_k c_i^k \otimes e_k$$

for $d_i^j \in D$ and $c_i^k \in C$. Choose a k -basis $\{c_1, \dots, c_r\}$ of the k -vector subspace of C spanned by the c_i^k 's, for $i, k = 1, \dots, n$. Analogously, let $\{d_1, \dots, d_s\}$ be a k -basis of the k -vector subspace of D spanned by the d_i^j 's. After some computations, we obtain

$$\begin{aligned} (\rho_M^- \otimes 1) \circ \rho_M^-(e_i) &= \sum_{h,l} c_h \otimes m_{h,l} \otimes d_l \\ (1 \otimes \rho_M^+) \circ \rho_M^-(e_i) &= \sum_{h,l} c_h \otimes m'_{h,l} \otimes d_l \end{aligned}$$

where $m_{h,l}, m'_{h,l} \in D^*mC^*$. Moreover, for $f \in C^*$ and $g \in D^*$, we can check that

$$\begin{aligned} (ge_i)f &= \sum_{h,l} f(c_h)m_{h,l}g(d_l) \\ g(e_if) &= \sum_{h,l} f(c_h)m'_{h,l}g(d_l) \end{aligned}$$

It is evident that certain particular choices of $f \in C^*$, $g \in D^*$ give rise to $m_{h,l} = m'_{h,l}$ for every $h = 1, \dots, s$; $l = 1, \dots, r$. Thus, the identity (I) holds.

(ii) \Rightarrow (iii). Let $f \in C^*$. We have to prove that the homothety $h_f : M \rightarrow M$ is a morphism of right D -comodules, i.e., that h_f is a morphism of left D^* -modules. But this is true because M is a D^*-C^* -bimodule.

(iii) \Rightarrow (ii) Since $h_f : M \rightarrow M$ is a D -comodule map we have

$$\sum_{(m)} f((m_0)_{-1})(m_0)_0 \otimes m_1 = \sum_{(m)} f(m_{-1})(m_0)_0 \otimes (m_0)_1$$

Therefore, for every $g \in D^*$, the equality

$$\sum_{(m)} f((m_0)_{-1})(m_0)_0 g(m_1) = \sum_{(m)} f(m_{-1})(m_0)_0 g((m_0)_1)$$

holds, that is, $(gm)f = g(mf)$.

The proof of (ii) \iff (iv) is similar to that of (ii) \iff (iii). □

We shall recall the concept of cotensor product from [9, 2]. If M is a right C -comodule and N is a left C -comodule, then the *cotensor product* $M \square_C N$ is the kernel of the k -linear map

$$\rho_M \otimes 1 - 1 \otimes \rho_N : M \otimes N \rightarrow M \otimes C \otimes N$$

Let C and D be two coalgebras. A *morphism of coalgebras* is a k -linear map $\varphi : C \rightarrow D$ such that $\Delta_D \circ \varphi = (\varphi \otimes \varphi) \circ \Delta_C$. The morphism of coalgebras φ induces a morphism of k -algebras $\varphi^* : D^* \rightarrow C^*$. Let $\varphi : C \rightarrow D$ be a morphism of k -coalgebras. Every right C -comodule M with structure map $\rho_M : M \rightarrow M \otimes C$ can be considered as a right D -comodule with structure map

$$M \xrightarrow{\rho_M} M \otimes C \xrightarrow{1_M \otimes \varphi} M \otimes D$$

This gives an exact functor $(-)_\varphi : M^C \rightarrow M^D$ called *co-restriction* functor. In particular, C can be viewed as D -bicomodule and we can also consider the coinduction functor $(-)^{\varphi} : M^D \rightarrow M^C$ where $N^{\varphi} = N \square_D C$ for every right D -comodule N . In fact, $N \square_D C$ is a D -subcomodule of the right D -comodule $N \otimes C$ whose structure map is

$$N \otimes C \xrightarrow{1_N \otimes \Delta_C} N \otimes C \otimes C \xrightarrow{1_N \otimes 1_C \otimes \varphi} N \otimes C \otimes D$$

It is proved in [4] that if M is a right C -comodule then the structure map ρ_M induces a C -colinear map $\overline{\rho}_M : M \rightarrow M_\varphi \square_D C$ such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{\rho_M} & M \otimes C \\ & \searrow \overline{\rho}_M & \nearrow i \\ & & M_\varphi \square_D C \end{array}$$

is commutative. Taking $M = C$ we obtain a C -bicomodule map $\overline{\Delta}_C : C \rightarrow C \square_D C$.

2. Separability of functors over comodules

DEFINITION 2.1. Consider abelian categories \mathcal{C} and \mathcal{D} . A covariant functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is said to be a *separable functor* (see [5]) if for all objects $M, N \in \mathcal{C}$ there are maps

$$v_{M,N}^F : \text{Hom}_{\mathcal{D}}(F(M), F(N)) \rightarrow \text{Hom}_{\mathcal{C}}(M, N)$$

satisfying the following *separability conditions*

1. For every $\alpha \in \text{Hom}_{\mathcal{C}}(M, N)$ we have $v_{M,N}^F(F(\alpha)) = \alpha$.
2. For $M', N' \in \mathcal{C}$, $f \in \text{Hom}_{\mathcal{D}}(F(M), F(N))$, $g \in \text{Hom}_{\mathcal{D}}(F(M'), F(N'))$, $\alpha \in \text{Hom}_{\mathcal{C}}(M, M')$ and $\beta \in \text{Hom}_{\mathcal{C}}(N, N')$, such that the following diagram is commutative

$$\begin{array}{ccc} F(M) & \xrightarrow{f} & F(N) \\ \downarrow F(\alpha) & & \downarrow F(\beta) \\ F(M') & \xrightarrow{g} & F(N') \end{array}$$

then the following diagram is also commutative

$$\begin{array}{ccc} M & \xrightarrow{v_{M,N}^F(f)} & N \\ \downarrow \alpha & & \downarrow \beta \\ M' & \xrightarrow{v_{M',N'}^F(g)} & N' \end{array}$$

In this section we will characterize the separability of the corestriction and coinduction functors defined by a morphism of coalgebras.

Let C, D be k -coalgebras. Let $F : M^C \rightarrow M^D$ be a k -functor, i.e., the induced map $\text{Com}_C(M, N) \rightarrow \text{Com}_D(F(M), F(N))$ is assumed to be a k -linear map. If M is a C -bicomodule, then $M \in M_{C^*}^C$ by Proposition 1.1. For $z \in F(M)$ and $f \in C^*$, define $z \cdot f = F(h_f)(z)$, where $h_f : M \rightarrow M$, $m \mapsto h_f(m) = mf$ is a morphism of C -comodules. This implies that $F(h_f)$ is a morphism of D -comodules. Thus, $F(M) \in M_{C^*}^D$.

PROPOSITION 2.2. *Let $F : M^C \rightarrow M^D$ be a separable k -functor. Assume that M, N are C -bicomodules and let*

$$v_{M,N} : \text{Com}_D(F(M), F(N)) \rightarrow \text{Com}_C(M, N)$$

be the map given by the separability conditions. If $\alpha \in \text{Com}_D(F(M), F(N))$ is also a morphism of right C^ -modules, then $v_{M,N}(\alpha)$ is a morphism of C -bicomodules.*

PROOF. Since α is C^* -linear, the following square is commutative

$$\begin{array}{ccc} F(M) & \xrightarrow{\alpha} & F(N) \\ \downarrow F(h_f) & & \downarrow F(h_f) \\ F(M) & \xrightarrow{\alpha} & F(N) \end{array}$$

The separability of F implies that the following diagram is commutative

$$\begin{array}{ccc}
 M & \xrightarrow{v_{M,N}(\alpha)} & N \\
 \downarrow h_f & & \downarrow h_f \\
 M & \xrightarrow{v_{M,N}(\alpha)} & N
 \end{array}$$

This means that $v_{N,M}(\alpha)$ is a morphism of right C^* -modules, that is, it is a morphism of left C -comodules and, thus, of C -bicomodules. \square

We will denote by $f - M^C$ the full subcategory of M^C consisting of the comodules of finite dimension.

PROPOSITION 2.3. *Let $F : M^C \rightarrow M^D$ be a left exact k -functor that commutes with direct limits. Assume that $F(f - M^C) \subseteq f - M^D$. The functor F is separable if and only if its restriction $F' : f - M^C \rightarrow f - M^D$ is separable.*

PROOF. It is clear that if F is separable then its restriction $F' : f - M^C \rightarrow f - M^D$ is separable. Conversely, assume that this last functor is separable. Take $M, N \in M^C$ and $\alpha \in \text{Com}_D(F(M), F(N))$. Write $M = \cup_{i \in I} M_i$, $N = \cup_{j \in J} N_j$, as direct unions of finite-dimensional subcomodules. It is clear that $F(M) = \cup_{i \in I} F(M_i) = \cup_{i \in I} F'(M_i)$ and analogously $F(N) = \cup_{j \in J} F(N_j) = \cup_{j \in J} F'(N_j)$. For every $i \in I$, there is $j \in J$ such that $\alpha(F(M_i)) \subseteq F(N_j)$. Put $\alpha_i = \alpha|_{F(M_i)}$. Since F' is separable, there is a map

$$v_{M_i, N_j} : \text{Com}_D(F'(M_i), F'(N_j)) \rightarrow \text{Com}_C(M_i, N_j)$$

which satisfies the separability conditions. Put $\beta_i = v_{M_i, N_j}(\alpha_i) : M_i \rightarrow N_j$. Consider $i \leq i'$ with $i, i' \in I$ and let $\iota_{i, i'}$ denote the inclusion $M_i \leq M_{i'}$. There are $j, j' \in J$ with $j \leq j'$ such that $\alpha(F(M_i)) \subseteq F(N_j)$ and $\alpha(F(M_{i'})) \subseteq F(N_{j'})$. In other words, the diagram

$$\begin{array}{ccc}
 F'(M_i) & \xrightarrow{\alpha_i} & F'(N_j) \\
 \downarrow F'(\iota_{i, i'}) & & \downarrow F'(\iota_{j, j'}) \\
 F'(M_{i'}) & \xrightarrow{\alpha_{i'}} & F'(N_{j'})
 \end{array}$$

is commutative, where $\iota_{j, j'}$ denotes the inclusion $N_j \leq N_{j'}$. Since F' is separable,

we have that the following diagram is commutative

$$\begin{array}{ccc}
 M_i & \xrightarrow{\beta_i} & N_j \\
 \downarrow \iota_{i,i'} & & \downarrow \iota_{j,j'} \\
 M_{i'} & \xrightarrow{\beta_{i'}} & N_{j'}
 \end{array}$$

Therefore, we can define $v_{M,N}(\alpha) = \lim \beta_i$. Thus, we have defined a map $v_{M,N} : \text{Com}_D(F(M), F(N)) \rightarrow \text{Com}_C(M, \vec{N})$. Now it is a routine matter to check that these maps satisfy the separability conditions, i.e., F is separable. \square

Let $r : X \rightarrow Y, s : Y \rightarrow X$ be morphisms of bicomodules such that $r \circ s = 1_Y$. We will say that s is a *splitting monomorphism* of bicomodules and that r is a *splitting epimorphism* of bicomodules. The proof of the following Theorem was performed after [5, Proposition 1.3.(1)].

THEOREM 2.4. *Let $\varphi : C \rightarrow D$ be a morphism of coalgebras. The functor $(-)_\varphi : \mathbf{M}^C \rightarrow \mathbf{M}^D$ is separable if and only if the canonical morphism $\bar{\Delta}_C : C \rightarrow C \square_D C$ is a splitting monomorphism of C -bicomodules.*

PROOF. Assume that $(-)_\varphi$ is separable and consider the map $p : C \square_D C \rightarrow C$ defined as the restriction of the map $C \otimes C \rightarrow C, c_1 \otimes c_2 \mapsto c_1 \varepsilon_C(c_2)$. This p is a morphism of right D -comodules and of right C^* -modules. Let $\phi = v_{C \square_D C, C}(p)$, where the map

$$v_{C \square_D C, C} : \text{Com}_D((C \square_D C)_\varphi, C_\varphi) \rightarrow \text{Com}_C(C \square_D C, C)$$

is given by the separability of $(-)_\varphi$. By Proposition 2.2, ϕ is a morphism of C -bicomodules. Write $\bar{\Delta} = \bar{\Delta}_C$. Now, the diagram

$$\begin{array}{ccc}
 (C \square_D C)_\varphi & \xrightarrow{p} & C_\varphi \\
 \bar{\Delta}_\varphi \uparrow & \searrow 1 & \\
 C_\varphi & &
 \end{array}$$

is commutative. Since $(-)_\varphi$ is separable, the diagram

$$\begin{array}{ccc}
 C \square_D C_\varphi & \xrightarrow{\phi} & C \\
 \bar{\Delta} \uparrow & \searrow 1 & \\
 C & &
 \end{array}$$

is commutative. Thus, $\bar{\Delta}$ is a splitting monomorphism of C -bicomodules. Assume that there is a morphism of C -bicomodules $\phi : C \square_D C \rightarrow C$ such that $\phi \circ \bar{\Delta} = 1_C$. Let $M, N \in \mathcal{M}^C$ and $f \in \text{Com}_D(M_\varphi, N_\varphi)$. Define \tilde{f} by the following commutative diagram of right C -comodule maps:

$$\begin{array}{ccc}
 M \square_D C & \xrightarrow{f \square_D 1} & N \square_D C \\
 \cong \uparrow & & \downarrow \gamma \square_D 1 \\
 M \square_C C \square_D C & & N \square_C C \square_D C \\
 1 \square_D \bar{\Delta} \uparrow & & \downarrow 1 \square_D \phi \\
 M \square_C C & & N \square_C C \\
 \cong \uparrow & & \downarrow \cong \\
 M & \xrightarrow{\tilde{f}} & N
 \end{array}$$

where γ denotes the isomorphism $N \cong N \square_C C$. Let $u_M : M \rightarrow M \square_D C$, $v_N : N \square_D C \rightarrow N$ be the compositions of the vertical maps on the left and on the right in the diagram, respectively. From the condition $\phi \circ \bar{\Delta} = 1_C$ it follows easily that $v_M \circ u_M = 1_M$. Given C -comodules M', N' and $\alpha \in \text{Com}_C(M, M')$, $\beta \in \text{Com}_C(N, N')$ and $g \in \text{Com}_D(M'_\varphi, N'_\varphi)$, consider the diagram

$$\begin{array}{ccc}
 M \square_D C & \xrightarrow{f \square_D 1} & N \square_D C \\
 \swarrow u_M & & \swarrow u_N \\
 & M \xrightarrow{\tilde{f}} N & \\
 \searrow v_M & & \searrow v_N \\
 & M' \xrightarrow{\tilde{g}} N' & \\
 \swarrow u_{M'} & & \swarrow u_{N'} \\
 M' \square_D C & \xrightarrow{g \square_D 1} & N' \square_D C \\
 \downarrow \alpha \square_D 1 & & \downarrow \beta \square_D 1
 \end{array}$$

Now, it is easy to see that if the outer square is commutative then the inner square is commutative. Moreover, if f is a morphism of right C -comodules, then

$\tilde{f} = f$. Therefore, if we define

$$\nu_{M,N} : \text{Com}_D(M_\varphi, N_\varphi) \rightarrow \text{Com}_C(M, N)$$

by $\nu_{M,N}(f) = \tilde{f}$, then the functor $(-)_\varphi$ is separable. □

Following [4], we will say that a morphism of coalgebras φ is a *monomorphism of coalgebras* provided that $\varphi \circ u = \varphi \circ v$, for u, v morphisms of coalgebras, it follows that $u = v$. Although every injective morphism of coalgebras is a monomorphism of coalgebras, both notions are not equivalent.

COROLLARY 2.5. *If $\varphi : C \rightarrow D$ is a monomorphism of coalgebras, then the functor $(-)_\varphi : M^C \rightarrow M^D$ is separable.*

PROOF. By [4, Theorem 3.5], the map $\overline{\Delta}_C : C \rightarrow C \square_D C$ is an isomorphism. By Theorem 2.4, $(-)_\varphi$ is a separable functor. □

COROLLARY 2.6. *Let $\varphi : C \rightarrow D$ be a morphism of coalgebras. If the functor $(-)_\varphi : M^C \rightarrow M^D$ is separable and D is a co-semi-simple coalgebras, then C is a co-semi-simple coalgebra.*

PROOF. Let M be any right C -comodule. Since D is co-semi-simple, M_φ is completely reducible. By [5, Proposition 1.2.(2)], M is completely reducible and, thus, C is co-semi-simple. □

The proof of the following Theorem was performed after [5, Proposition 1.3.(2)].

THEOREM 2.7. *Let $\varphi : C \rightarrow D$ be a morphism of k -coalgebras. The functor $(-)_\varphi = -\square_D C : M^D \rightarrow M^C$ is separable if and only if φ is a splitting epimorphism of D -bicomodules.*

PROOF. Assume that $-\square_D C$ is separable. For $M, N \in M^D$, there exists the map

$$\nu_{M,N} : \text{Com}_C(M \square_D C, N \square_D C) \rightarrow \text{Com}_D(M, N)$$

satisfying the separability conditions. Taking $M = D$ and $N = C$, we have $\nu_{D,C} : \text{Com}_C(D \square_D C, C \square_D C) \rightarrow \text{Com}_D(D, C)$. Now, consider the canonical C -bilinear map $\overline{\Delta} : C \rightarrow C \square_D C$ and define $\overline{\Delta}' : D \square_D C \rightarrow C \square_D C$ as $\overline{\Delta}' = \overline{\delta} \circ (\delta_C)^{-1}$, where

$\delta_C : C \rightarrow D \square_D C$ is the canonical isomorphism. Put $\psi = v_{D,C}(\bar{\Delta}')$. Since $\bar{\Delta}'$ is a morphism of right D -comodules and of right D^* -comodules, we can apply Proposition 2.2 to obtain that ψ is a morphism of D -bicomodules. On the other hand, the following triangle is commutative

$$\begin{array}{ccc} C \square_D C & \xrightarrow{\varphi \square_D 1} & D \square_D C \\ \bar{\Delta}' \uparrow & \searrow 1 & \\ D \square_D C & & \end{array}$$

By the separability conditions, we deduce that the diagram

$$\begin{array}{ccc} C & \xrightarrow{\varphi} & D \\ \psi \uparrow & \searrow 1 & \\ D & & \end{array}$$

is commutative. Hence, φ is a splitting epimorphism of D -bicomodules. Conversely, assume that there is a D -bilinear map $\vartheta : D \rightarrow C$ such that $\varphi \circ \vartheta = 1_D$. If $f \in \text{Com}_C(M \square_D C, N \square_D C)$, then we define them map $v_{M,N} : \text{Com}_C(M \square_D C, N \square_D C) \rightarrow \text{Com}_D(M, N)$ by putting $v_{M,N}(f) = \tilde{f}$, where \tilde{f} makes the following diagram commutative.

$$\begin{array}{ccc} M \square_D C & \xrightarrow{f} & N \square_D C \\ 1_M \square_D \vartheta \uparrow & & \downarrow 1_N \square_D \varphi \\ M \square_D D & & N \square_D D \\ \cong \uparrow & & \downarrow \cong \\ M & \xrightarrow{\tilde{f}} & N \end{array}$$

A verification shows (as in Theorem 2.4) that the functor $-\square_D C$ is then separable. □

LEMMA 2.8. *Let A, B be subcoalgebras of C . If $\varphi : C \rightarrow D$ is a morphism of coalgebras such that $\varphi(A) \cap \varphi(B) = 0$, then $A \square_D B = 0$.*

PROOF. We have the following equalizer

$$A \square_D B \rightarrow A \otimes B \xrightarrow[1_A \otimes \rho_B]{\rho_A \otimes 1_B} A \otimes D \otimes B$$

where $\rho_A = (1_A \otimes \varphi) \circ \Delta_A$ and $\rho_B = (\varphi \otimes 1_B) \circ \Delta_B$. If $z \in A \square_D B$, then $(\rho_A \otimes 1_B)(z) = (1_A \otimes \rho_B)(z)$. Since $\text{Im}(\rho_A \otimes 1_B) \subseteq A \otimes \varphi(A) \otimes B$ and $\text{Im}(1_A \otimes \rho_B) \subseteq A \otimes \varphi(B) \otimes B$, we have that $(\rho_A \otimes 1_B)(z) = (1_A \otimes \rho_B)(z) \subseteq (A \otimes \varphi(A) \otimes B) \cap (A \otimes \varphi(B) \otimes B) = 0$. Thus, $(\rho_A \otimes 1_B)(z) = 0$ and, since $\rho_A \otimes 1_B$ is a monomorphism, $z = 0$. \square

Let $\{C_i : i \in I\}$ be a set of coalgebras with structure maps Δ_i, ε_i . The vector space $\bigoplus C_i$ can be canonically endowed with structure of coalgebra (see e.g. [8, page 50]). Moreover, from a set of morphisms of coalgebras $\{\varphi_i : C_i \rightarrow D_i : i \in I\}$, we obtain the coalgebra morphism $\bigoplus \varphi_i : \bigoplus C_i \rightarrow \bigoplus D_i$.

THEOREM 2.9. *Let $\varphi_i : C_i \rightarrow D_i$ be morphisms of k -coalgebras, $i \in I$.*

1. *If the functor $(-)\varphi_i : \mathbf{M}^{C_i} \rightarrow \mathbf{M}^{D_i}$ is separable for every $i \in I$, then the functor $(-)\bigoplus \varphi_i : \mathbf{M}^{\bigoplus C_i} \rightarrow \mathbf{M}^{\bigoplus D_i}$ is separable.*
2. *If the functor $(-)\varphi_i : \mathbf{M}^{D_i} \rightarrow \mathbf{M}^{C_i}$ is separable for every $i \in I$, then the functor $(-)\bigoplus \varphi_i : \mathbf{M}^{\bigoplus D_i} \rightarrow \mathbf{M}^{\bigoplus C_i}$ is separable.*

PROOF. (1) By Theorem 2.4, for every $i \in I$, there is a morphism of C_i -bicomodules $\phi_i : C_i \square_{D_i} C_i \rightarrow C_i$ such that $\phi \circ \overline{\Delta}_i = 1_{C_i}$. The map $\bigoplus \phi_i : \bigoplus C_i \square_{\bigoplus D_i} \bigoplus C_i \rightarrow \bigoplus C_i$ is a morphism of $\bigoplus C_i$ -bicomodules and $\bigoplus \phi_i \circ \bigoplus \overline{\Delta}_i = 1_{\bigoplus C_i}$. If we prove that there is an isomorphism of $\bigoplus C_i$ -bicomodules

$$(\bigoplus C_i) \square_{\bigoplus D_i} (\bigoplus C_i) \cong \bigoplus (C_i \square_{D_i} C_i),$$

then we can deduce from Theorem 2.4 that the functor $(-)\bigoplus \varphi_i$ is separable. The Lemma 2.8 assures that $C_i \square_{\bigoplus D_i} C_j = 0$ if $i \neq j$. Therefore,

$$(\bigoplus C_i) \square_{\bigoplus D_i} (\bigoplus C_i) \cong \bigoplus_{i,j} (C_i \square_{\bigoplus D_i} C_j) \cong \bigoplus (C_i \square_{D_i} C_i)$$

(2) By Theorem 2.7, for every $i \in I$ there is a morphism of D_i -bicomodules $\psi_i : D_i \rightarrow C_i$ such that $\varphi_i \circ \psi_i = 1_{D_i}$. It is clear that $\bigoplus \varphi_i \circ \bigoplus \psi_i = 1_{\bigoplus D_i}$. Moreover, it is not difficult to see that $\bigoplus \psi_i$ is a morphism of $\bigoplus D_i$ -bicomodules. By Theorem 2.7, the functor $(-)\bigoplus \varphi_i$ is separable. \square

REMARK 2.10. Theorem 2.9 can be used to construct coalgebra morphisms between infinite-dimensional coalgebras such that the corestriction and the coinduction functors are separable.

3. Applications

Recall that a k -algebra A is said to be separable if the canonical map $A \otimes A \rightarrow A$ is a splitting epimorphism of A -bimodules. By [5, Proposition 1.3] this is equivalent to say that the restriction functor $A - \text{Mod} \rightarrow k - \text{Mod}$ is separable. In this section we investigate the coseparable coalgebras.

A morphism of k -coalgebras $\varphi : C \rightarrow D$ induces a morphism of k -algebras $\varphi^* : D^* \rightarrow C^*$. Let us denote by $(-)_\varphi : C^* - \text{Mod} \rightarrow D^* - \text{Mod}$ the functor restriction of scalars. Recall that if C is a finite-dimensional coalgebra, then there is an isomorphism of categories $M^C \cong C^* - \text{Mod}$.

PROPOSITION 3.1. *Let $\varphi : C \rightarrow D$ be a morphism of coalgebras. Assume that C and D are finite-dimensional. The following statements are equivalent.*

- (i) *The functor $(-)_\varphi : M^C \rightarrow M^D$ is separable.*
- (ii) *The functor $(-)_\varphi^* : C^* - \text{Mod} \rightarrow D^* - \text{Mod}$ is separable.*

PROOF. (i) \Leftrightarrow (ii) The functorial diagram

$$\begin{array}{ccc}
 M^C & \xrightarrow{(-)_\varphi} & M^D \\
 \downarrow & & \downarrow \\
 C^* - \text{Mod} & \xrightarrow{(-)_\varphi^*} & D^* - \text{Mod}
 \end{array}$$

where the vertical arrows represent canonical isomorphisms of categories, commutes. This entails that $(-)_\varphi$ is separable if and only if $(-)_\varphi^*$ is separable. \square

PROPOSITION 3.2. *Let $\varphi : C \rightarrow D$ be a morphism of coalgebras.*

1. *The functor $(-)_\varphi : M^C \rightarrow M^D$ is separable if and only if the restriction $(-)_\varphi : f - M^C \rightarrow f - M^D$ is separable.*
2. *Let $C' \leq C$, $D' \leq D$ be subcoalgebras such that $\varphi(C') \leq D'$, and let us denote by $\varphi' : C' \rightarrow D'$ the induced coalgebra map. If the functor $(-)_\varphi$ is separable then the functor $(-)_\varphi' : M^{C'} \rightarrow M^{D'}$ is separable.*
3. *If C is cosemisimple then the functor $(-)_\varphi$ is separable if and only if for any finite-dimensional subcoalgebras $C' \leq C$ and $D' \leq D$ such that $\varphi(C') \leq D'$, the functor $(-)_\varphi'$ is separable.*

PROOF. (1) This follows from Proposition 2.3.

(2) Let C' be a subcoalgebra of C and let us denote by $i : C' \rightarrow C$ the inclusion coalgebra map. By [4, Theorem 3.5] the functor $(-)_i : M^{C'} \rightarrow M^C$ is separable. For any subcoalgebra D' of D with $\varphi(C') \subseteq D'$ we can consider the commutative diagram

$$\begin{array}{ccc}
 M^C & \xrightarrow{(-)_\varphi} & M^D \\
 (-)_i \uparrow & & \uparrow (-)_j \\
 M^{C'} & \xrightarrow{(-)_{\varphi'}} & M^{D'}
 \end{array}$$

where $j : D' \rightarrow D$ is the inclusion map. If $(-)_\varphi$ is separable, then $(-)_\varphi \circ (-)_i = (-)_j \circ (-)_{\varphi'}$ is separable. By [5, Lemma 1.1.(3)], $(-)_{\varphi'}$ is a separable functor.

(3) Assume that C is co-semi-simple. Then $C = \bigoplus C_i$, where the C_i 's are simple subcoalgebras. If we put $D_i = \varphi(C_i)$, and we denote by $\varphi_i : C_i \rightarrow D_i$ the induced coalgebra morphism, we have by hypothesis that $(-)_{\varphi_i}$ is a separable functor. For every i , there is a D_i -bicomodule morphism $\psi_i : C_i \square_{D_i} C_i \rightarrow C_i$ such that $\psi_i \circ \overline{\Delta_{C_i}} = 1_{C_i}$. We have that $C_i \square_{D_i} C_i = C_i \square_D C_i$ and

$$C \square_D C = \bigoplus_i (C_i \square_D C_i) \oplus \bigoplus_{i \neq j} (C_i \square_D C_j)$$

Then the maps $\{\psi_i\}$ give a bicomodule map $\psi : C \square_D C \rightarrow C$ if we put $\psi = \bigoplus \psi_i$ on $\bigoplus_i (C_i \square_D C_i)$ and zero on $\bigoplus_{i \neq j} (C_i \square_D C_j)$. Clearly, $\psi \circ \overline{\Delta_C} = 1_C$. By Theorem 2.4, $(-)_\varphi$ is a separable functor. □

A k -coalgebra C is *coseparable* (see [2]) if there exists k -linear map $\tau : C \otimes C \rightarrow k$ such that $(I \otimes \tau)(\Delta \otimes I) = (\tau \otimes I)(I \otimes \Delta)$ and $\tau \Delta = \varepsilon$. As it was observed in [2, p. 41], C is coseparable if and only if there exists a C -bicomodule map $\pi : C \otimes C \rightarrow C$ such that $\pi \Delta = I$. Now, the counit $\varepsilon : C \rightarrow k$ is a morphism of coalgebras and, in this case, $C \square_k C = C \otimes C$. It follows from Theorem 2.4 that C is coseparable if and only if the corestriction functor $(-)_\varepsilon : M^C \rightarrow k\text{-Mod}$ is separable. From Corollary 2.6 every coseparable coalgebra is co-semi-simple. This result is also given in [2]. Moreover, from Proposition 3.2.3, if C is co-semi-simple then C is coseparable if and only if any finite-dimensional subcoalgebra of C is coseparable.

If C is a k -coalgebra and $k \subseteq K$ is any field extension, then we can define the K -coalgebra $C \otimes K = C \otimes_k K$, with comultiplication given by

$$\Delta_{C \otimes K} = \Delta_C \otimes 1_K : C \otimes K \rightarrow (C \otimes C) \otimes K \cong (C \otimes K) \otimes_K (C \otimes K)$$

and counit given by

$$\varepsilon_{C \otimes K} = \varepsilon \otimes 1_K : C \otimes K \rightarrow k \otimes K \cong K$$

With this notation, we can prove the following result.

PROPOSITION 3.3. *Let $\varphi : C \rightarrow D$ be a morphism of k -coalgebras. The following statements are equivalent.*

- (i) *The functor $(-)_\varphi : M^C \rightarrow M^D$ is separable.*
- (ii) *For any field extension $k \subseteq K$ the functor $(-)_\varphi \otimes 1_K : M^{C \otimes K} \rightarrow M^{D \otimes K}$ is separable.*

PROOF. (i) \Rightarrow (ii) By Theorem 2.4, $\overline{\Delta_C}$ is a splitting monomorphism of C -bicomodules. It is not difficult to see that $(C \otimes K) \square_{D \otimes K} (C \otimes K) \cong (C \square_D C) \otimes K$. Therefore, $\overline{\Delta_{C \otimes K}} = \overline{\Delta_C} \otimes 1_K$ is a splitting monomorphism of $C \otimes K$ -bicomodules.

(ii) \Rightarrow (i) This is clear. □

The following theorem gives a “classical” interpretation of the notion of coseparable coalgebra.

THEOREM 3.4. *A k -coalgebra C is coseparable if and only if $C \otimes K$ is cosemi-simple for every field extension $k \subseteq K$.*

PROOF. Assume that C is coseparable. By Proposition 3.3, $C \otimes K$ is a K -coalgebra coseparable for every field extension $k \subseteq K$. By Corollary 2.6, $C \otimes K$ is a co-semi-simple coalgebra.

Conversely, assume that $C \otimes K$ is co-semi-simple for every field extension $k \subseteq K$. By Proposition 3.2 we have only to prove that every finite-dimensional subcoalgebra of C is coseparable. Indeed, if $E \leq C$ is a finite-dimensional subcoalgebra then $E \otimes K$ is co-semi-simple by Corollary 2.6. Therefore,

$$\text{Hom}_K(E \otimes K, K) = (E \otimes K)^* \cong E^* \otimes K$$

is a semisimple K -algebra for every field extension $k \subseteq K$. This entails that E^* is a separable k -algebra. By Proposition 3.1, E is a coseparable k -coalgebra. □

REMARK 3.5. If H is a Hopf k -algebra then H is coseparable as k -coalgebra if and only if H is co-semi-simple. This fact follows from Theorem 3.4 and [1, Theorem 3.3.2].

References

- [1] Abe, E., "Hopf Algebras", Cambridge University Press, 1977.
- [2] Doi, Y., Homological coalgebra, *J. Math. Soc. Japan* **33** (1981), 31–50.
- [3] Gabriel, P., Des catégories abeliennes, *Bull. Soc. Math. France* **90** (1962), 323–448.
- [4] Năstăsescu, C. and Torrecillas, B., Torsion Theories for Coalgebras, *J. Pure Appl. Algebra* **97** (1994), 203–220.
- [5] Năstăsescu, C. Van Oystaeyen, F. and Van den Bergh, M., Separable functors applied to graded rings, *J. Algebra* **123** (1989), 397–413.
- [6] Rafael, M. D., Separable functors revisited, *Comm. Algebra* **18** (1990), 1445–1459.
- [7] Sullivan, J. B., The uniqueness of integrals for Hopf algebras and some existence theorems of integrals for commutative Hopf algebras, *J. Algebra* **19** (1971), 426–440.
- [8] Sweedler, M. E., "Hopf Algebras", Benjamin, New York, 1969.
- [9] Takeuchi, M., Morita theorems for categories of comodules, *J. Fac. Sci. Univ. Tokyo* **24** (1977), 629–644.

F. Castaño Iglesias:

Departamento de Matemática Aplicada,
Universidad de Almería, E04120-Almería. Spain.

J. Gómez Torrecillas:

Departamento de Algebra, Facultad de Ciencias,
Universidad de Granada, E18071-Granada. Spain.

C. Năstăsescu:

Facultatea de Matematică,
Str. Academiei 14, R70109-Bucharest. Romania.